

Delocalisation of one-dimensional marginals of product measures and the capacity of LTI discrete channels

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Abstract—We consider discrete linear time invariant (LTI) channels satisfying the phase independence (PI) assumption. We show that under the PI assumption the capacity of LTI channels is positive. The main technical tool that we use to establish the positivity of the capacity is the delocalisation theorem for one-dimensional marginals of the product measure due to Ball and Nazarov. We also prove two delocalisation results that can be seen as extensions of Ball-Nazarov Theorem.

Index Terms—LTI channels, PI assumption, Shannon capacity, delocalisation.

I. INTRODUCTION

In this work we consider linear time-invariant (LTI) discrete channels. Given a sequence $\mathbf{x} \in \mathbb{C}^N$ at the transmitter side, we assume that at the receiver we obtain the sequence $\mathbf{y} \in \mathbb{C}^N$ given by:

$$\mathbf{y}[n] = \sum_{m=0}^{N-1} a_m \mathbf{x}[n - m \bmod N] + \omega[n], \quad (1)$$

$$n = 1, \dots, N,$$

where $\{a_m\}$'s are the attenuation coefficients, and ω is the additive white Gaussian noise (AWGN). The parameter N will be the total length of the sequence. We will see in Section II that in the case of wireless communication we have $N = TW$, where W is the bandwidth and T is the time of the modulation. We will assume that the signal to

noise ratio (SNR) is fixed and equal to P . Let us denote by $\alpha_m = |a_m|$, and $X_m = \arg(a_m)$, $m = 0, \dots, N-1$. Then we rewrite equation (1) by use of the operator notation:

$$\mathbf{y} = H_N(\mathbf{x}) + \omega. \quad (2)$$

The operator $H_N : \mathbb{C}^N \rightarrow \mathbb{C}^N$ is given by:

$$H_N = \sum_{m=0}^{N-1} \alpha_m X_m \pi_m, \quad (3)$$

where π_m is the circular shift by m on \mathbb{C}^N , i.e., $\pi_m(x)[n] = x[n - m \bmod N]$, $n = 0, \dots, N-1$.

We introduce the following assumption for requirements on the channel operator.

Definition I.1 (PI assumption). We will say that the channel operator H_N given by Equation (3) satisfies the phase independence (PI) assumption, if all X_m 's are independent identically distributed random variables having uniform distribution on the unit circle.

Remark I.2. In the case of a channel appearing in wireless communication under assumption that the carrier frequency f_c is much higher than the bandwidth W , it is natural to assume that $\arg(a_m)$'s are independent identically distributed random variables having uniform distribution on the unit circle (see Section II).

We further assume that at the receiver the channel estimation is performed, and we know the attenuation coefficients $\{a_m\}$'s. We will say that an N -dimensional probability distribution \mathbf{x} satisfies

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A_{PN} , if $\mathbb{E}(\text{tr}(\mathbf{x}\mathbf{x}^t)) = PN$. We will denote the latter by $\mathbf{x} \in A_{PN}$. The main result of the paper is the following.

Theorem I.3. The asymptotic Shannon capacity of the channel given by (2) and (3), and satisfying the PI assumption is positive:

$$\liminf_{N \rightarrow \infty} \frac{1}{N} \sup_{\mathbf{x} \in A_{PN}} I(\mathbf{x}; (\mathbf{y}, H_N)) > 0.$$

Here $I(\mathbf{x}, \mathbf{y})$ denotes the mutual information of the vectors \mathbf{x} and \mathbf{y} , see [9].

In order to prove Theorem I.3 we use the probabilistic result of Ball and Nazarov [3] on uniform delocalisation of one-dimensional marginals of product measure for a given delocalised probability distribution in \mathbb{R}^d . Let us recall that for a \mathbb{R}^d -valued random variable X , the Levy concentration function of X is defined for every $\varepsilon > 0$ by

$$C_\varepsilon(X) = \sup_{z \in \mathbb{R}^d} \mathbb{P}(\|X - z\| \leq \varepsilon).$$

To state Ball-Nazarov theorem, we will also have to define the class \mathcal{F} of the random variables:

$$\mathcal{F} = \left\{ \sum_{n=1}^N a_n X_n \mid N \geq 1, \sum_{n=1}^N |a_n|^2 = 1, \right. \\ \left. X_1, \dots, X_N \text{ are i.i.d. as } X \right\}.$$

Then the following theorem holds true:

Theorem I.4 (Ball-Nazarov [3]). Let X be a \mathbb{R}^d -valued random variable. Then there exists a universal constant $A > 0$ such that for every $\varepsilon > 0$ we have

$$\sup_{S \in \mathcal{F}} C_\varepsilon(S) \leq AC_\varepsilon(X).$$

Remark I.5. The quantitative bound $A = \sqrt{2}$ in the case where X has a bounded density in \mathbb{R} has been recently obtained by Rudelson and Vershynin (Theorem 1.2 in [8]). Moreover, Theorem of Rudelson-Vershynin does not require from the random variables to be identically distributed, but only independent. Their theorem is a corollary of K. Ball Theorem, see [2], on the slice of a maximal area of the unit cube in \mathbb{R}^N .

In this paper, we prove a stronger version of Theorem I.4 in \mathbb{R}^2 under more restrictive conditions. We will identify \mathbb{R}^2 with the complex plane \mathbb{C} . We assume that X is uniformly distributed on the unit circle in \mathbb{C} , and there exists $\varepsilon_0 > 0$ such that a projection of the unit weight vector on any 4 coordinates has ℓ_2 -norm at most $1 - \varepsilon_0$. More precisely, let us denote by $\mathcal{S}_{4,N}$ the collection of all 4-dimensional subspaces in \mathbb{R}^N generated by any 4 vectors from the standard basis $\{e_1, \dots, e_N\}$ of \mathbb{R}^N . We define the class

$$S_{4,\varepsilon_0} = \{(a_1, \dots, a_N) \in S^{N-1} \mid \|\pi_E(a)\|_2^2 \leq 1 - \varepsilon_0, \\ \forall E \in \mathcal{S}_{4,N}, N \geq 5\},$$

where π_E denotes the orthogonal projection on the subspace E , and S^{N-1} is the unit sphere in \mathbb{R}^N defined by

$$S^{N-1} = \{(a_1, \dots, a_N) \in \mathbb{R}^N \mid \sum_{k=1}^N a_k^2 = 1\}.$$

We think on the set S_{4,ε_0} as the set of non too sparse power profiles of the channel operator H_N . We define the collection of \mathbb{C} -valued random variables $\mathcal{F}_{4,\varepsilon_0}$ to be:

$$\mathcal{F}_{4,\varepsilon_0} = \{a_1 X_1 + \dots + a_N X_N \in \mathcal{F} \mid (a_1, \dots, a_N) \in S_{4,\varepsilon_0}\}.$$

We prove the following result¹ which is slightly better than the bound that we can deduce from Theorem I.4.

Theorem I.6. Let X be a random variable uniformly distributed on the unit circle in \mathbb{C} , and $\varepsilon_0 > 0$. Then there exists a constant $B > 0$ such that for every $\varepsilon > 0$ we have

$$\sup_{S \in \mathcal{F}_{4,\varepsilon_0}} C_\varepsilon(S) \leq B\varepsilon^2.$$

Theorem I.6 is sharp. We know that the 3-fold convolution of the uniform probability measure on the unit circle has density function $f(x)$ in the disk

¹Our method can be also used to provide an alternative proof of Theorem I.4 in the special case, where X is uniformly distributed on the unit circle in \mathbb{C} .

of radius 3 in \mathbb{C} with singularities on the unit circle. More precisely, there are constants $C, c > 0$ such that for $x \in \mathbb{C}$ close to the unit circle we have [4]:

$$c |\log(1 - |x|)| \leq f(x) \leq C |\log(1 - |x|)|.$$

The latter implies that if $S = \frac{1}{2}(X_1 + X_2 + X_3 + X_4)$, where X_1, \dots, X_4 are independent uniformly distributed on the unit circle, then there exists $K > 0$ such that

$$\mathbb{P}(|S| \leq \varepsilon) \geq K\varepsilon.$$

Another consequence of our approach to the delocalisation is a multi-dimensional analog of Rudelson-Vershynin's result (Theorem 1.2 in [8]).

Theorem 1.7. Let $K, L, M, \delta, \eta > 0$. Let (X_n) be \mathbb{R}^d -valued independent random variables, having densities bounded by K , and Γ_n be the covariance matrices² of X_n 's. Assume that for all n , all the entries of Γ_n 's are bounded by L , $\det \Gamma_n \geq \delta$, and $\mathbb{E}[\|X_n\|^{2+\eta}] \leq M$. Then there exists $C > 0$ such that for every $N \geq 1$, the density function of $a_1 X_1 + \dots + a_N X_N$ is bounded by CK , provided that $\sum_{n=1}^N a_n^2 = 1$.

II. LINEAR TIME INVARIANT (LTI) DISCRETE CHANNELS

In this Section we will justify the channel model given by the Equations (2) and (3) in the scenario of wireless communication in urban environment. Our assumptions are the following:

- The communication is done between a transmitter and a receiver by use the frequency band $[f_c - \frac{W}{2}, f_c + \frac{W}{2}]$, where f_c is the carrier frequency and W is the bandwidth. Also, we assume $f_c \gg W$.
- The environment, the transmitter and the receiver are almost static (have very low speed). As a result the signal experiences a negligible Doppler effect. The environment has many obstacles, and there is no line of sight between the transmitter and the receiver.

²Let $\mu_n := \mathbb{E}(X_n)$ be the expectation of X_n . Then $\Gamma_n := \mathbb{E}((X_n - \mu_n)(X_n - \mu_n)^t)$ is the covariance matrix of X_n .

- The processing is performed at the baseband $[-\frac{W}{2}, \frac{W}{2}]$.

Let us assume that at the transmitter side we have a sequence $S[n] = b_n$, $n = 1, 2, \dots$, of real (complex) numbers that we would like to send to a receiver. It is a standard procedure in wireless communication, to use some Digital-to-Analog transform, in order to generate a function $s(t)$ with a spectral profile in a given band which encompasses the information. We will stick in this paper to the Shannon-Nyquist transform given by

$$s(t) = \exp(ift) \left(\sum_m b_m \operatorname{sinc}(-Wt + m) \right),$$

where $\operatorname{sinc}(t) = \frac{\sin \pi t}{\pi t}$. It follows from the properties of the sinc function that:

- The signal $s(t)$ satisfies that $\hat{s} \subset [f_c - \frac{W}{2}, f_c + \frac{W}{2}]$.
- The Inverse (Analog-to-Digital) operator is given by the sampling $s(t)$ at the time slots $\{\frac{1}{W}, \frac{2}{W}, \dots\}$. In other words, we have

$$s\left(\frac{n}{W}\right) = b_n = S[n], \text{ for } n \geq 1.$$

We use the model of multi-path propagation of a signal [12]. Assume that at the transmitter side we generate the signal $s(t)$ as above. Then for any path \mathcal{L} of length ℓ we obtain at the receiver the signal:

$$r_{\mathcal{L}}(t) = \alpha_{\mathcal{L}} s(t - \ell/c) + \omega_{\mathcal{L}}(t),$$

where c denotes the speed of light, $\alpha_{\mathcal{L}} \in \mathbb{R}$ is the attenuation coefficient which depends on the path length (decays as $\frac{1}{\ell^2}$) and on the environment, and $\omega_{\mathcal{L}}$ denotes the additive noise along the path \mathcal{L} .

We assume that the received signal $r_{\mathcal{L}}(t)$ is shifted to the baseband first, and then, we sample $\exp(-ift)r_{\mathcal{L}}(t)$ at the time slots $\{\frac{1}{W}, \frac{2}{W}, \dots\}$. We also assume, as in [6], that the time shift $\frac{\ell}{c}$ lies on the lattice $\frac{1}{W}\mathbb{Z}$, i.e., there exists $n_{\mathcal{L}} \in \mathbb{Z}$ such that $\ell/c = \frac{n_{\mathcal{L}}}{W}$. Then, at the receiver side, we obtain the following sequence:

$$\begin{aligned}
R_{\mathcal{L}}[n] &= \exp(-if_c n_{\mathcal{L}}/W) r_{\mathcal{L}} \left(\frac{n}{W} \right) \\
&= \exp(-if_c n_{\mathcal{L}}/W) \alpha_{\mathcal{L}} b_{n-n_{\mathcal{L}}} + \omega_{\mathcal{L}}(n/W), \\
& \quad n = 1, 2, \dots
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
R_{\mathcal{L}}[n] &= \exp(-if_c n_{\mathcal{L}}/W) \alpha_{\mathcal{L}} S[n - n_{\mathcal{L}}] \\
& \quad + \omega_{\mathcal{L}}(n/W).
\end{aligned}$$

By our assumptions, $f_c \gg W$, and therefore, the phase $f_c n_{\mathcal{L}}/W$ will change drastically if we replace $n_{\mathcal{L}}$ by another close to it integer. So, we approximate $R_{\mathcal{L}}$ by the following:

$$R_{\mathcal{L}}[n] \approx \alpha_{\mathcal{L}} X_{\mathcal{L}} S[n - n_{\mathcal{L}}] + \omega_{\mathcal{L}}(n/W),$$

where $X_{\mathcal{L}}$ is uniformly distributed random variable on the unit circle in \mathbb{C} .

Finally, taking all the paths from the transmitter to the receiver, by the law of superposition, we obtain the sequence R given by:

$$R[n] = \sum_{k \in \mathbb{N}} \alpha_k X_k S[n - k] + \omega[n], \quad n = 1, 2, \dots$$

where $\omega[n] = \sum_{\mathcal{L}} \omega_{\mathcal{L}}(n/W)$. We will assume that the phases X_k are independent uniformly distributed on the unit circle in \mathbb{C} , since they belong to different paths, and therefore are not related one to each other. It is a standard assumption that all $\omega[n]$'s are independent complex Gaussian random variables with zero mean and variance one.

By use of the trick of periodic prefix, like in [6], using a transmission of a finite duration time T , we can assume that the transmitted and received sequences, S and R respectively, have length $N = TW$, and the relation between S and R is described by equation:

$$R[n] = \sum_{k=0}^{N-1} \alpha_k X_k S[n - k \bmod N] + \omega[n], \quad (4)$$

where $n = 0, 1, \dots, N-1$. Notice, that the Equation (4) coincides with the model Equation (3) of the channel given in Section I.

III. ON THE CAPACITY OF LTI CHANNELS UNDER THE PI ASSUMPTION

For each N , let H_N be the channel operator defined by Equation (3). It is a standard fact that H_N is diagonalisable and its eigenvectors are the exponential functions $e_k^{(N)} \in \mathbb{C}^N$ given by

$$e_k^{(N)}[m] = e^{\frac{2\pi i k m}{N}}, \quad m \in \{0, 1, \dots, N-1\}.$$

Therefore, the matrix of the operator H_N in the

basis of the exponentials is $\begin{bmatrix} \lambda_1^{(N)} & \dots & 0 \\ & \ddots & \\ 0 & \dots & \lambda_N^{(N)} \end{bmatrix}$

where the $\lambda_k^{(N)}$ are the eigenvalues of H_N given by:

$$\lambda_k^{(N)} = \sum_{m=0}^{N-1} \alpha_m X_m e^{-\frac{2\pi i k m}{N}}, \quad k = 1, 2, \dots, N. \quad (5)$$

Remark III.1. Since the uniform distribution on the circle is invariant by rotations, $(\lambda_k^{(N)})$ is a sequence of identically distributed random variables (but **not** independent) for each N .

We express $\omega := \omega_1 e_1 + \dots + \omega_N e_N$, where e_1, \dots, e_N is the basis of the exponentials. Since ω is a white Gaussian noise of total power N , the ω_k are distributed according to the complex symmetric Gaussian distribution, with 0 mean and variance 1. We assume that the noise affects each coordinate independently, ie $\mathbb{E}[\omega \omega^t] = I_N$.

A. Proof of Theorem I.3

We denote by $C_N = \sup_{\mathbf{x} \in \mathcal{A}_{PN}} I(\mathbf{x}; (\mathbf{y}, H_N))$. In this Section we prove the lower bound for C_N which implies Theorem I.3.

In [11], E. Telatar studied AWGN linear channels³ with random attenuation coefficient matrix and proved that if \mathbf{x} is constrained to have covariance matrix Q then the choice of \mathbf{x} that maximises

³the noise is assumed to be an additive white Gaussian noise and the operator acts linearly on the signals

$I(\mathbf{x}; (\mathbf{y}, H_N))$, where H_N is random and known at the receiver, is the circularly symmetric complex Gaussian of covariance Q and that

$$\begin{aligned} & \sup_{\mathbf{x} \in A_{PN}} I(\mathbf{x}; (\mathbf{y}, H_N)) \\ &= \sup_Q \mathbb{E} [\log \det(I_N + H_N Q H_N^\dagger)], \end{aligned}$$

where the supremum is taken over the choices of non-negative definite Q subject to $\text{tr}(Q) \leq PN$, and H_N^\dagger denotes the adjoint operator to H_N .

Since for any matrices A, B with suitable sizes $\det(I + AB) = \det(I + BA)$, we can rewrite C_N as follows:

$$\begin{aligned} C_N &= \sup_Q \mathbb{E} \left[\log \det(I_N + Q H_N^\dagger H_N) \right] \\ &= \sup_Q \mathbb{E} \left[\log \det(I_N + Q |H_N|^2) \right], \quad (6) \end{aligned}$$

where $|H_N|^2$ denotes the diagonal matrix whose coefficients are $|\lambda_i|^2$'s. Now, taking $Q = P I_N$ we have:

$$\begin{aligned} C_N &\geq \mathbb{E} \left[\log \det(I_N + P |H_N|^2) \right] \\ &= \sum_{m=1}^N \mathbb{E} \left[\log(1 + P |\lambda_m|^2) \right] \\ &= N \mathbb{E} \left[\log(1 + P |\lambda_1|^2) \right]. \end{aligned}$$

In the last transition we used Remark III.1. Then for any $\varepsilon > 0$ we have:

$$\begin{aligned} \log(1 + P |\lambda_1|^2) &\geq \log(1 + P |\lambda_1|^2) \mathbb{1}_{\{|\lambda_1| \geq \varepsilon\}} \\ &\geq \log(1 + P \varepsilon^2) \mathbb{1}_{\{|\lambda_1| \geq \varepsilon\}} \end{aligned}$$

The linearity and monotonicity of the expectation imply:

$$\mathbb{E} \left[\log(1 + P |\lambda_1|^2) \right] \geq \log(1 + P \varepsilon^2) \mathbb{P}\{|\lambda_1| \geq \varepsilon\}.$$

By Theorem⁴ I.4, there exists⁵ $B > 0$ such that for all $\varepsilon > 0$ we have:

$$\mathbb{P}\{|\lambda_1| < \varepsilon\} \leq B\varepsilon.$$

Finally, taking $\varepsilon = \frac{1}{2B}$ we obtain:

$$\frac{1}{N} C_N \geq \frac{1}{2} \log \left(1 + \frac{P}{4B^2} \right) > 0.$$

Taking the lim inf of the left hand side as $N \rightarrow \infty$ concludes the proof of Theorem I.3. \square

IV. ON DELOCALISATION OF ONE-DIMENSIONAL MARGINALS OF THE PRODUCT MEASURE

A. Preliminaries

The uniform measure on a circle is not absolutely continuous with respect to Lebesgue measure. However, we can compute the 2-fold convolution of this measure to see that it is absolutely continuous (see [4]). By Lemma IV.1, a linear combination of random variables uniformly distributed on the unit circle also has a density. In Appendix A we will prove the following lemmata:

Lemma IV.1. Let X, Y be \mathbb{R}^d -valued independent random variables, such that Y has density. Then for every real number a , the random variable $aX + Y$ has density. If, moreover, the density of Y is bounded by M , then the density of $aX + Y$ is also bounded by M .

Remark IV.2. If both X and Y have a density, then this result directly follows from Young inequality.

Lemma IV.3. Let $a_1, \dots, a_5 \in \mathbb{C}$, $b_1, \dots, b_5 > 0$ such that for every k , $|a_k| \geq b_k$. Let X_1, \dots, X_5 be independent random variables uniformly distributed on the unit circle. Then there exists a function φ

⁴At this point of the proof, we could, alternatively, use Theorem I.6 instead of Ball-Nazarov Theorem I.4. Our approach would require the delocalisation estimates for 2, 3, and 4-fold convolutions of the uniform measure on the unit circle.

⁵Indeed, if X is uniformly distributed on the unit circle in \mathbb{C} , then there exists $B > 0$ such that for every $\varepsilon > 0$ we have $C_\varepsilon(X) \leq B\varepsilon$.

which only depends on the b_i 's such that the density f of $a_1X_1 + \dots + a_5X_5$ satisfies:

$$\|f\|_\infty \leq \varphi(b_1, \dots, b_5).$$

Lemma IV.4. Let X be an \mathbb{R}^d -valued random variable having density f , $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{R}^d$. Then $aX + b$ has density at $x \in \mathbb{R}^d$ equal to $\frac{1}{|a|^d} f\left(\frac{x-b}{a}\right)$.

In our work, we use a local \mathbb{R}^d -valued Central Limit Theorem that follows from Theorem 1 in [10], see Appendix B.

Theorem IV.5 (Local \mathbb{R}^d -valued CLT). For each n , let $X_{n,m}$, $1 \leq m \leq n$ be a triangular array of independent \mathbb{R}^d -valued, centered, random variables with covariance matrices $C_{n,m} := \mathbb{E}[X_{n,m}X_{n,m}^t]$. Let $\sigma_{n,m}^2 := \text{tr}(C_{n,m}) = \mathbb{E}[\|X_{n,m}\|_2^2]$. If

- i) There exists a symmetric, positive-definite $d \times d$ matrix C such that:

$$\sum_{m=1}^n C_{n,m} \rightarrow C$$

- ii) For any $\varepsilon > 0$, for any $\theta \in \mathbb{R}^d$,

$$\sum_{m=1}^n \mathbb{E} [|\langle \theta, X_{n,m} \rangle|^2 \mathbf{1}_{\{|\langle \theta, X_{n,m} \rangle| > \varepsilon\}}] \rightarrow 0$$

then, the sum $S_n := X_{n,1} + \dots + X_{n,n}$ converges in distribution towards the d -variate Gaussian with 0 mean and covariance C .

Moreover, if one of the following conditions holds true:

- 1) Starting from some n , S_n has a density, and there exists an integer $p > 1$ such that $\|\phi_{n,m}\|_{L^p}^{\frac{2p}{d}} \sigma_{n,m}^2$ is uniformly bounded, where $\phi_{n,m}$ denotes the Fourier transform of $X_{n,m}$.

- 2) All the $X_{n,m}$'s have uniformly bounded density.

then

$$\|f_n - g_C\|_\infty \rightarrow 0,$$

where f_n and g_C denote the densities of S_n and of the d -variate Gaussian with 0 mean and covariance matrix C , respectively.

B. Proof of Theorem 1.6

Assume that, contrary to the assertion of Theorem 1.6, for each n there exist $z \in \mathbb{C}$, $(a^{(n)}) \in S^{(n-1)}$, $\varepsilon_n \rightarrow 0$ such that

$$\frac{\mathbb{P}\{|S_n - z| < \varepsilon_n\}}{\varepsilon_n^2} \rightarrow +\infty,$$

where $S_n = \sum_{k=1}^n a_k^{(n)} X_k$. In the following, f_n will denote the density function of S_n (for $n \geq 2$) and g_{σ^2} will be the density function of a circular symmetric Gaussian with 0 mean and variance σ^2 .

1) Case 1: Uniform decay of the coefficients:

If we have

$$\max_{1 \leq k \leq n} |a_k^{(n)}| \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (7)$$

then by Theorem IV.5 (see Appendix C),

$$\|f_n - g_1\|_\infty \rightarrow 0.$$

Therefore, $\|f_n\|_\infty \leq \frac{1}{2} + \|g_1\|_\infty$ for n sufficiently large. Then

$$\begin{aligned} \mathbb{P}\{|S_n - z| < \varepsilon_n\} &= \int_{B(z, \varepsilon_n)} f_n(x) dx \\ &\leq M \varepsilon_n^2 \end{aligned}$$

for some constant M independent of n which leads to a contradiction.

2) Case 2: Up to 4 bounded away from zero coefficients:

There exist $k \in \{1, \dots, 4\}$ and $\delta_1, \dots, \delta_k > 0$, and there exist a sequence $\varphi(n)$ and $i_1(n) < \dots < i_k(n)$ such that for all $j \in \{1, \dots, k\}$ we have

$$|a_{i_j(n)}^{(\varphi(n))}| \geq \delta_j,$$

and

$$\max_{\ell \notin \{i_1(n), \dots, i_k(n)\}} |a_\ell^{(\varphi(n))}| \rightarrow 0$$

By the compactness argument, we can assume without loss of generality that:

$$a_{i_j(n)}^{(\varphi(n))} \xrightarrow{k \rightarrow \infty} \gamma_{i_j} \geq \delta_j.$$

By the symmetry, we will assume that $i_j(n) = j$ for every n . Since the coefficient vectors are in $\mathcal{F}_{4,\varepsilon_0}$, it follows that

$$\sum_{i=1}^k \gamma_i^2 \leq 1 - \varepsilon_0.$$

By Theorem IV.5 the density of

$$\sum_{i=k+1}^{\varphi(n)} a_i^{(\varphi(n))} X_i$$

is uniformly bounded. Hence by Lemma IV.1, $S_{\varphi(n)}$ has uniformly bounded density. This contradicts the initial assumption.

3) Case 3: At least 5 bounded away from zero coefficients: There exist i_1, \dots, i_5 and $\delta_1, \dots, \delta_5 > 0$ such that for $j \in \{1, \dots, 5\}$, $|a_{i_j}^{(n)}| \geq \delta_j$ for infinitely many n . Without loss of generality, we can assume that $i_j = j$ and the inequalities hold for all n . Then, by Lemma IV.3, the density of $a_1^{(n)} X_1 + \dots + a_5^{(n)} X_5$ is bounded by $\varphi(\delta_1, \dots, \delta_5)$. Hence, by Lemma IV.1, S_n has density bounded by $\varphi(\delta_1, \dots, \delta_5)$ for all n . This leads to a contradiction, and completes the proof. \square

C. Proof of Theorem 1.7

The approach is similar to the proof of Theorem I.6. Assume that contrary to the assertion of Theorem I.7, there exists a sequence of $S_n = \sum_{k=1}^n a_k^{(n)} X_k$ whose density functions f_n are unbounded. By Lemma IV.4 we may assume that $\mu_k = \mathbb{E}(X_k) = 0$ for all k .

In the following we denote by D_δ the set of all symmetric, positive-definite $d \times d$ matrices with determinant at least δ . It is a closed and convex set (see proof of Theorem 1 in [7]).

Let

$$C_n = \sum_{k=1}^n |a_k^{(n)}|^2 \Gamma_k$$

be the covariance matrix of S_n .

By convexity of D_δ we have $C_n \in D_\delta$ for every n . Moreover, all entries of C_n are bounded by L . By the compactness argument we can assume without loss of generality that:

$$C_n \rightarrow C \in D_\delta.$$

In particular, the matrix C is positive-definite.

1) Uniform decay of the coefficients:

Let's assume that

$$\max_{1 \leq k \leq n} |a_k^{(n)}| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (8)$$

Then, by Theorem IV.5 (see Appendix C)

$$\|f_n - g_C\|_\infty \rightarrow 0.$$

Therefore, we have $\|f_n\|_\infty \leq \frac{1}{2} + \|g_C\|_\infty$ for n sufficiently large which leads to a contradiction.

2) At least 1 bounded away from 0 coefficient: If condition (8) is not true, then there exist $\gamma > 0$, and sequences $\varphi(n)$ and $i(n)$ such that

$$|a_{i(n)}^{(\varphi(n))}| \geq \gamma$$

Without loss of generality, we can assume that $i(n) = 1$ for every n . It follows from Lemma IV.1 and IV.4 that for every n the density of $S_{\varphi(n)}$ is bounded by $\gamma^{-d} K$ which leads to a contradiction. \square

V. CONCLUDING REMARKS

The main difference between our work and known results for LTI discrete channels is in the following:

- We argue in Section II that it is reasonable, especially, in the scenario of wireless communication in urban environment, to assume that the channel operator H_N satisfies the PI assumption. We have no assumptions on the power profile of the channel operator, i.e., no assumptions on the vector of (α_m) 's.
- The exact variational formula for the capacity is well known in the literature [1], [11], [13]. See also Formula (6) in Section III, and Formula 1.4 in [13]. We provide a positive lower

bound on the capacity under minor assumptions on the channel operators H_N .

- We relate the information theoretical problem at the hand to well established Delocalisation problem in Probability, and provide a new resolution of the latter through Lindeberg-Feller Local Central Limit Theorem IV.5. We also establish a completely new multi-dimensional delocalisation result, see Theorem I.7, which might have further applications in Information Theory.

APPENDIX A
PROOF OF LEMMATA IV.1 AND IV.3

Proof (Lemma IV.1).

- **Absolute continuity of $aX + Y$.** First we need to show that \mathbb{P}_{aX+Y} is absolutely continuous. Let f denote the density of Y and let A be a Borel set with Lebesgue measure 0. Then by independence we have:

$$\begin{aligned}\mathbb{P}_{aX+Y}(A) &= \int_{\mathbb{R}^d} \mathbb{P}(Y \in A - x) d\mathbb{P}_{aX}(x) \\ &= \int_{\mathbb{R}^d} \int_{A-x} f(y) dy d\mathbb{P}_{aX}(x) \\ &= 0\end{aligned}$$

In the last transition we used the fact that the Lebesgue measure is invariant by translation, so $A - x$ has Lebesgue measure 0 for all $x \in \mathbb{R}^d$.

- **Bound on the density of $aX + Y$.** Again, using the independence of X and Y and Fubini theorem, the following holds for any Borel set A :

$$\begin{aligned}\mathbb{P}_{aX+Y}(A) &= \int_{\mathbb{R}^d} \mathbb{P}(Y \in A - x) d\mathbb{P}_{aX}(x) \\ &= \int_{\mathbb{R}^d} \int_{A-x} f(y) dy d\mathbb{P}_{aX}(x) \\ &= \int_{\mathbb{R}^d} \int_A f(u - x) du d\mathbb{P}_{aX}(x) \\ &= \int_A \int_{\mathbb{R}^d} f(u - x) d\mathbb{P}_{aX}(x) du\end{aligned}$$

By uniqueness of the Radon-Nikodym derivative for almost every $u \in \mathbb{R}^d$, the density of the sum is defined by

$$\begin{aligned}g(u) &:= \int_{\mathbb{R}^d} f(u - x) d\mathbb{P}_{aX}(x) \\ &= \mathbb{E}(f(u - aX))\end{aligned}$$

Hence we have $\|g\|_\infty \leq \mathbb{E}(M) = M$. □

To prove Lemma IV.3 we will need some estimates on Bessel function of the first kind. Denote by μ_r the uniform measure on the circle centered at 0 of radius r , and by $\widehat{\mu}_r$ its Fourier transform defined for all $x \in \mathbb{R}^2$ as

$$\widehat{\mu}_r(x) = \int_{rS^1} e^{-i\langle y, x \rangle} d\mu_r(y)$$

Lemma A.1. $\widehat{\mu}_r$ is radial and for all $x \in \mathbb{R}^2$:

$$\widehat{\mu}_r(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ir|x|\cos(\theta)} d\theta$$

The latter is also known as $J_0(r|x|)$ where J_0 is a Bessel function of the first kind.

Proof (Lemma A.1). It follows directly by passing to polar coordinates:

$$\begin{aligned}\widehat{\mu}_r(x) &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ir|x|\cos(\theta - \text{Arg}(x))} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-ir|x|\cos(\theta)} d\theta,\end{aligned}$$

where the last transition is made by rotation invariance of the uniform measure on the unit circle. □

Lemma A.2. Let X be an \mathbb{R}^d -valued random variable and let φ denote its Fourier transform defined as $\varphi(y) := \mathbb{E}(e^{-i\langle y, X \rangle})$. If $\varphi \in L^1(\mathbb{R}^d)$ then X has a density, bounded by $(2\pi)^{-d} \|\varphi\|_{L^1}$.

Proof (Lemma A.2). This is a standard fact about \mathbb{R}^d -valued random variables. □

Lemma A.3. There exists K such that for all $r > 0$, $|J_0(r)| \leq \min\left(1, \frac{K}{\sqrt{r}}\right)$.

Proof (Lemma A.3).

$$\begin{aligned} J_0(r) &= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-ircos(\theta)} d\theta \\ &+ \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} e^{-ircos(\theta)} d\theta \\ &= \frac{1}{2\pi} (I_1 + I_2) \end{aligned}$$

By the Principle of Stationary Phase the following estimate holds as $r \rightarrow \infty$,

$$I_k = \sqrt{\frac{2\pi}{r}} \cos\left(r - \frac{\pi}{4}\right) + O\left(\frac{1}{r}\right)$$

Hence $J_0(r) = \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi}{4}\right) + O\left(\frac{1}{r}\right)$. Therefore, $\sqrt{r}|J_0(r)|$ is bounded. We can take $K := \sup_{r>0} \sqrt{r}|J_0(r)|$. \square

Lemma A.4. $\widehat{\mu}_a \in L^5(\mathbb{R}^2)$ for any $a > 0$.

Proof (Lemma A.4). $\widehat{\mu}_a$ is radial, continuous on \mathbb{R}^2 and by Lemma A.3, $|\widehat{\mu}_a(x)|^5 = O\left(\frac{1}{|x|^{5/2}}\right)$ as $|x| \rightarrow \infty$, which is integrable on \mathbb{R}^2 . Therefore, so is $|\widehat{\mu}_a|^5$. \square

We are now able to prove Lemma IV.3:

Proof (Lemma IV.3). Let μ_a denote the probability measure of aX . By independence, the probability measure μ of the sum $a_1X_1 + \dots + a_5X_5$ is the convolution $\mu_{a_1} * \dots * \mu_{a_5}$. Then, $\widehat{\mu} = \widehat{\mu}_{a_1} \dots \widehat{\mu}_{a_5}$. It follows from Lemmata A.1 and A.3 that

$$\begin{aligned} |\widehat{\mu}(x)| &= |J_0(|a_1x|) \dots J_0(|a_5x|)| \\ &\leq \prod_{k=1}^5 \min\left(1, \frac{K}{\sqrt{a_k|x|}}\right) \\ &\leq \prod_{k=1}^5 \min\left(1, \frac{K}{\sqrt{b_k|x|}}\right) \end{aligned}$$

Hence,

$$\|\widehat{\mu}\|_{L^1} \leq \left\| \prod_{k=1}^5 \min\left(1, \frac{K}{\sqrt{b_k|x|}}\right) \right\|_{L^1}$$

Therefore, the result follows from Lemma A.2. \square

APPENDIX B PROOF OF THEOREM IV.5

In order to prove Theorem IV.5, it will suffice to prove that (S_n) converges in distribution towards the Gaussian. This will hold true using an \mathbb{R}^d version of Lindeberg-Feller Theorem that we recall hereafter. We will denote by \implies the convergence in distribution.

Theorem B.1 (Lindeberg-Feller CLT). For each n let $X_{n,m}$, $1 \leq m \leq n$ be independent real-valued random variables with $\mathbb{E}[X_{n,m}] = 0$. If

- 1) $\sum_{m=1}^n \mathbb{E}[X_{n,m}^2] \rightarrow \sigma^2$ as $n \rightarrow \infty$
- 2) For each $\varepsilon > 0$,

$$\sum_{m=1}^n \mathbb{E}\left[|X_{n,m}|^2 \mathbb{1}_{\{|X_{n,m}|>\varepsilon\}}\right] \rightarrow 0,$$

then $S_n := X_{n,1} + \dots + X_{n,n} \implies \mathcal{N}(0, \sigma^2)$.

Proof (Theorem B.1). The proof can be found in Section 2.4 of [5]. \square

Using Cramér-Wold's characterisation of convergence in distribution in \mathbb{R}^d , we can easily generalise Theorem B.1 to random vectors.

Theorem B.2 (Cramér-Wold). Let (X_n) be a sequence of random vectors in \mathbb{R}^d . We have $X_n \implies X_\infty$ if and only if $\langle \theta, X_n \rangle \implies \langle \theta, X_\infty \rangle$ for all $\theta \in \mathbb{R}^d$.

Theorem B.3 (\mathbb{R}^d -valued Lindeberg-Feller CLT). For each n let $X_{n,m}$, $1 \leq m \leq n$, be independent \mathbb{R}^d -valued random vectors with $\mathbb{E}[X_{n,m}] = 0$. Let $C_{n,m} := \mathbb{E}[X_{n,m}X_{n,m}^t]$ be the covariance matrix of $X_{n,m}$.

If we have

- i) There exists C a symmetric, positive-definite, $d \times d$ matrix such that :

$$\sum_{m=1}^n C_{n,m} \rightarrow C$$

- ii) For any $\varepsilon > 0$, for any $\theta \in \mathbb{R}^d$,

$$\sum_{m=1}^n \mathbb{E}\left[|\langle \theta, X_{n,m} \rangle|^2 \mathbb{1}_{\{|\langle \theta, X_{n,m} \rangle|>\varepsilon\}}\right] \rightarrow 0$$

then $S_n := X_{n,1} + \dots + X_{n,n}$ converges in distribution towards a Gaussian with 0 mean and covariance matrix C .

Theorem IV.5 is now a particular case of Theorem 1 of [10].

APPENDIX C

In what follows, we will check that all the requirements of Theorem IV.5 are fulfilled.

A. In the proof of Theorem 1.6:

For the uniform distribution on the unit circle, the real and imaginary parts are uncorrelated. Moreover, their first moment is 0 and their variance are equal. In what follows, let us denote by $X_{n,m} = a_m^{(n)} X_m$, where X_n 's are independent uniformly distributed random variables on the unit circle in \mathbb{C} .

For each m , and n we have $C_{n,m} = \mathbb{E}(X_{n,m} X_{n,m}^t) = \frac{1}{2} |a_m^{(n)}|^2 I_2$, and therefore $\sigma_{n,m}^2 = \text{tr}(C_{n,m}) = |a_m^{(n)}|^2$. Hence,

$$\sum_{m=1}^n C_{n,m} = \frac{1}{2} I_2. \quad (9)$$

If we have

$$\max_{1 \leq m \leq n} |a_m^{(n)}| \rightarrow 0, \text{ as } n \rightarrow \infty,$$

then for every $\varepsilon > 0$ the event

$$\left\{ |a_m^{(n)} X_m| > \varepsilon \right\}$$

is empty starting from some n . Then, Lindeberg-Feller's condition **ii**) in Theorem IV.5 holds true.

Moreover, by the results of Appendix A, the assumption on the Fourier transforms is satisfied for $p = 5$, so the requirements of Theorem IV.5 hold true.

B. In the proof of Theorem 1.7:

Let $\theta \in \mathbb{R}^d$ and $\varepsilon > 0$. Then we have the following:

$$\begin{aligned} & |\langle \theta, a_k^{(n)} X_k \rangle|^2 \mathbb{1}_{\{|\langle \theta, a_k^{(n)} X_k \rangle| > \varepsilon\}} \\ &= |a_k^{(n)}|^2 |\langle \theta, X_k \rangle|^2 \mathbb{1}_{\left\{ \left| \frac{a_k^{(n)} \langle \theta, X_k \rangle}{\varepsilon} \right| > 1 \right\}} \\ &\leq \frac{1}{\varepsilon^\eta} |a_k^{(n)}|^{2+\eta} |\langle \theta, X_k \rangle|^{2+\eta}. \end{aligned}$$

Denote by $\gamma_n = \max_{1 \leq k \leq n} |a_k^{(n)}|$ and assume that $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$. Then, by taking the expectation and using Cauchy-Schwarz inequality we have:

$$\begin{aligned} & \sum_{k=1}^n \mathbb{E} [|\langle \theta, a_k^{(n)} X_k \rangle|^2 \mathbb{1}_{\{|\langle \theta, a_k^{(n)} X_k \rangle| > \varepsilon\}}] \\ &\leq \frac{1}{\varepsilon^\eta} \sum_{k=1}^n |a_k^{(n)}|^{2+\eta} \mathbb{E} [|\langle \theta, X_k \rangle|^{2+\eta}] \\ &\leq \frac{\|\theta\|^{2+\eta}}{\varepsilon^\eta} \gamma_n^\eta \sum_{k=1}^n |a_k^{(n)}|^2 \mathbb{E} [\|X_k\|^{2+\eta}] \\ &\leq \frac{M \|\theta\|^{2+\eta}}{\varepsilon^\eta} \gamma_n^\eta \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Recall, we derived in the Proof of Theorem 1.7 that the covariance matrices of the sums S_n converge to a positive-definite matrix C . Therefore, all the requirements of Theorem IV.5 are satisfied.

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