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Almost Linear Complexity Methods for Delay-Doppler Channel Estimation

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***Abstract*—A fundamental task in wireless communication is *channel estimation*: Compute the channel parameters of a medium between a transmitter and a receiver. In the case of delay-Doppler channel, i.e., a signal undergoes only delay and Doppler shifts, a widely used method to compute delay-Doppler parameters is the *pseudo-random* method. It uses a pseudo-random sequence of length N, and, in case of non-trivial relative velocity between transmitter and receiver, its computational complexity is $O(N^2 \log N)$ arithmetic operations. In [1] the flag method was introduced to provide a faster algorithm for delay-Doppler channel estimation. It uses specially designed flag sequences and its complexity is $O(rN \log N)$ for channels of *sparsity* r. In these notes, we introduce the *incidence* and *cross* methods for channel estimation. They use triple-chirp and double-chirp sequences of length N, correspondingly. These sequences are closely related to chirp sequences widely used in radar systems. The arithmetic complexity of the incidence and cross methods is $O(N \log N + r^3)$, and $O(N \log N + r^2)$, respectively.**

I. Introduction

A BASIC building block in many wireless communication protocols is *channel estimation*: learning the channel parameters of the medium between a transmitter and a receiver [6]. In these notes we develop efficient algorithms for delay-Doppler (also called time-frequency) channel estimation. Throughout these notes we denote by $\mathbb{Z}_N$ the set of integers $\{0, 1, ..., N-1\}$ equipped with addition and multiplication modulo N. We will assume, for simplicity, that N is an odd prime. We denote by $\mathcal{H} = \mathbb{C}(\mathbb{Z}_N)$ the vector space of complex valued functions on $\mathbb{Z}_N$, and refer to it as the *Hilbert space of sequences*.

A. Channel Model

We describe the discrete channel model which was derived in [1]. We assume that a transmitter uses a sequence $S \in \mathcal{H}$ to generate an analog waveform $S_A \in L^2(\mathbb{R})$ with bandwidth W and a carrier frequency $f_c \gg W$. Transmitting S_A, the receiver obtains the analog waveform $R_A \in L^2(\mathbb{R})$. We make the sparsity assumption on the number of paths for propagation of the waveform S_A. As a result, we have[1]

$$R_A(t) = \sum_{k=1}^{r} \beta_k \cdot \exp(2\pi i f_k t) \cdot S_A(t - t_k) + \mathcal{W}(t), \qquad \text{(I-A.1)}$$

where r—called the *sparsity* of the channel—denotes the number of paths, $\beta_k \in \mathbb{C}$ is the *attenuation coefficient*, $f_k \in \mathbb{R}$ is the *Doppler shift* along the k-th path, $t_k \in \mathbb{R}_+$ is the *delay* associated with the k-th path, and $\mathcal{W}$ denotes a random white noise. We assume the normalization $\sum_{k=1}^{r} |\beta_k|^2 \leq 1$. The Doppler shift depends on the relative velocity, and the delay encodes the distance along a path, between the transmitter and the receiver. We will call

$$(\beta_k, t_k, f_k), \ k = 1, ..., r, \qquad \text{(I-A.2)}$$

channel parameters, and the main objective of channel detection is to estimate them.

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[1]In these notes i denotes $\sqrt{-1}$.

B. Channel Estimation Problem

Sampling the waveform R_A at the receiver side, with sampling rate $1/W$, we obtain a sequence $R \in \mathcal{H}$. It satisfies

$$R[n] = H(S)[n] + \mathcal{W}[n], \qquad \text{(I-B.1)}$$

where H, called the *channel operator*, acts on $S \in \mathcal{H}$ by[2]

$$H(S)[n] = \sum_{k=1}^{r} \alpha_k e(\omega_k n) S[n - \tau_k], \ n \in \mathbb{Z}_N, \qquad \text{(I-B.2)}$$

with α_k's are the complex-valued (digital) attenuation coefficients, $\sum_k |\alpha_k|^2 \leq 1$, $\tau_k \in \mathbb{Z}_N$ is the (digital) delay associated with the path k, $\omega_k \in \mathbb{Z}_N$ is the (digital) Doppler shift associated with path k, and $\mathcal{W}$ denotes the random white noise. We will assume that all the coordinates of $\mathcal{W}$ are independent identically distributed random variables of expectation zero.

Remark I-B.1: The relation between the physical (I-A.2) and the discrete channel parameters is as follows (see Section I.A. in [1] and references therein): If a standard method suggested by sampling theorem is used for the discretization, and S_A has bandwidth W, then $\tau_k = t_k W$ modulo N, and $\omega_k = N f_k / W$ modulo N, provided that $t_k \in \frac{1}{W}\mathbb{Z}$, and $f_k \in \frac{W}{N}\mathbb{Z}$, $k = 1, ..., r$. In particular, we note that the integer N determines the frequency resolution of the channel detection, i.e., the resolution is of order W/N.

The objective of delay-Doppler channel estimation is:

*Problem I-B.2 (**Channel Estimation**):* Design $S \in \mathcal{H}$, and an effective method for extracting the channel parameters $(\alpha_k, \tau_k, \omega_k)$, $k = 1, ..., r$, using S and R satisfying (I-B.1).

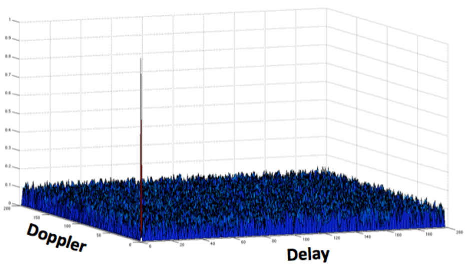

Fig. 1. Profile of $\mathcal{A}(\varphi, \varphi)$ for φ pseudo-random sequence.

C. Ambiguity Function and Pseudo-Random Method

A classical method to estimate the channel parameters in (I-B.1) is the *pseudo-random method* [2], [3], [4], [6], [7]. It uses two ingredients - the ambiguity function, and a pseudo-random sequence.

[2]We denote $e(t) = \exp(2\pi i t / N)$.

*1) **Ambiguity Function**:* In order to reduce the noise component in (I-B.1), it is common to use the ambiguity function that we are going to describe now. We consider the *Heisenberg operators* $\pi(\tau,\omega)$, $\tau,\omega \in \mathbb{Z}_N$, which act on $f \in \mathcal{H}$ by

$$[\pi(\tau,\omega)f]\,[n] = e(-2^{-1}\tau\omega)\cdot e(\omega n)\cdot f[n-\tau], \qquad \text{(I-C.1)}$$

where 2^{-1} denotes $(N+1)/2$, the inverse of $2 \bmod N$. Finally, the *ambiguity function* of two sequences $f,g \in \mathcal{H}$ is defined[3] as the $N\times N$ matrix

$$\mathcal{A}(f,g)[\tau,\omega] = \langle \pi(\tau,\omega)f, g\rangle\,, \quad \tau,\omega \in \mathbb{Z}_N, \qquad \text{(I-C.2)}$$

where $\langle\ ,\ \rangle$ denotes the standard inner product on $\mathcal{H}$.

*Remark I-C.1 (**Fast Computation of Ambiguity Function**):* The restriction of the ambiguity function to a line in the delay-Doppler plane, can be computed in $O(N\log N)$ arithmetic operations using fast Fourier transform [5]. For more details, including explicit formulas, see Section V of [1]. Overall, we can compute the entire ambiguity function in $O(N^2\log N)$ operations.

For R and S satisfying (I-B.1), the law of the iterated logarithm implies that, with probability going to one, as N goes to infinity, we have

$$\mathcal{A}(S,R)[\tau,\omega] = \mathcal{A}(S,H(S))[\tau,\omega] + \varepsilon_N, \qquad \text{(I-C.3)}$$

where $|\varepsilon_N| \le \sqrt{2\log\log N}/\sqrt{N\cdot SNR}$, with SNR denotes the *signal-to-noise ratio*[4].

*2) **Pseudo-Random Sequences**:* We will say that a norm-one sequence $\varphi \in \mathcal{H}$ is *B-pseudo-random,* $B \in \mathbb{R}$—see Figure 1 for illustration—if for every $(\tau,\omega) \neq (0,0)$ we have

$$|\mathcal{A}(\varphi,\varphi)[\tau,\omega]| \le B/\sqrt{N}. \qquad \text{(I-C.4)}$$

There are several constructions of families of pseudo-random (PR) sequences in the literature (see [2], [3] and references therein).

*3) **Pseudo-Random Method**:* Consider a pseudo-random sequence φ, and assume for simplicity that $B=1$ in (I-C.4). Then we have

$$\mathcal{A}(\varphi,H(\varphi))[\tau,\omega] \qquad \text{(I-C.5)}$$
$$= \begin{cases} \widetilde{\alpha}_k + \sum\limits_{j\neq k} \widetilde{\alpha}_j/\sqrt{N}, & \text{if } (\tau,\omega) = (\tau_k,\omega_k)\,,\ 1\le k\le r; \\ \sum\limits_j \widehat{\alpha}_j/\sqrt{N}, & \text{otherwise,} \end{cases}$$

where $\widetilde{\alpha}_j,\ \widehat{\alpha}_j,\ 1\le j\le r$, are certain multiples of the α_j's by complex numbers of absolute value less or equal to one. In particular, we can compute the delay-Doppler parameter (τ_k,ω_k) if the associated attenuation coefficient α_k is sufficiently large. It appears as a peak of $\mathcal{A}(\varphi,H(\varphi))$. Finding the peaks of $\mathcal{A}(\varphi,H(\varphi))$ constitutes the pseudo-random method. Notice that the arithmetic complexity of the pseudo-random method is $O(N^2\log N)$, using Remark I-C.1. For applications to sensing, that require sufficiently high frequency resolution, we will need to use sequences of large length N. Hence, the following is a natural problem.

*Problem I-C.2 (**Arithmetic Complexity**):* Solve Problem I-B.2, with method for extracting the channel parameters which requires almost linear arithmetic complexity.

[3]For our purposes it will be convenient to use this definition of the ambiguity function. The standard definition appearing in the literature is $A(f,g)[\tau,\omega] = \langle e(\omega n)f[n-\tau], g[n]\rangle$.

[4]We define $SNR = \langle S,S\rangle\,/\,\langle \mathcal{W},\mathcal{W}\rangle$.

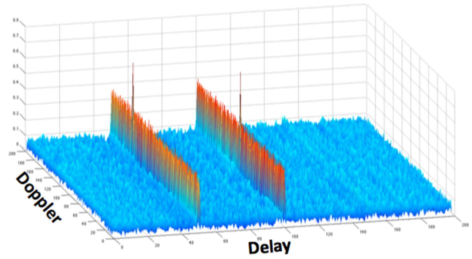

Fig. 2. Profile of $\mathcal{A}(f_L,H(f_L))$ for flag f_L, $L=\{(0,\omega)\}$, $N=199$, and channel parameters $(0.7,50,150)$, $(0.7,100,100)$.

D. Flag Method

In [1] the flag method was introduced in order to deal with the complexity problem. It computes the r channel parameters in $O(rN\log N)$ arithmetic operations. For a given line L in the plane $\mathbb{Z}_N\times\mathbb{Z}_N$, one construct a sequence f_L—called flag—with ambiguity function $\mathcal{A}(f_L,H(f_L))$ having special profile—see Figure 2 for illustration. It is essentially supported on shifted lines parallel to L, that pass through the delay-Doppler shifts of H, and have peaks there. This suggests a simple algorithm to extract the channel parameters. First compute $\mathcal{A}(f_L,H(f_L))$ on a line M transversal to L, and find the shifted lines on which $\mathcal{A}(f_L,H(f_L))$ is supported. Then compute $\mathcal{A}(f_L,H(f_L))$ on each of the shifted lines and find the peaks. The overall complexity of the flag algorithm is therefore $O(rN\log N)$, using Remark I-C.1. If r is large, it might be computationally insufficient.

E. Incidence and Cross Methods

In these notes we suggest two new schemes for channel estimation that have much better arithmetic complexity than previously known methods. The schemes are based on the use of double and triple chirp sequences.

*1) **Incidence Method**:* We propose to use triple-chirp sequences for channel estimation. We associate with three distinct lines L, M, and M° in $\mathbb{Z}_N\times\mathbb{Z}_N$, passing through the origin, a sequence $C_{L,M,M^\circ} \in \mathcal{H}$. This sequence has ambiguity function essentially supported on the union of L, M, and M°. As a consequence—see Figure 3 for illustration—the ambiguity function $\mathcal{A}(C_{L,M,M^\circ}, H(C_{L,M,M^\circ}))$ is essentially supported on the shifted lines $\{(\tau_k,\omega_k) + (L\cup M\cup M^\circ)\,|\,k = 1,\ldots,r\}$. This observation, which constitutes the bulk of the incidence method, enables a computation in $O(N\log N + r^3)$ arithmetic operations of all the time-frequency shifts (see Section III). In addition, the estimation of the corresponding r attenuation coefficients takes $O(r)$ operations. Hence, the overall complexity of incidence method is $O(N\log N + r^3)$ operations.

*2) **Cross Method**:* We propose to use double-chirp sequences for channel estimation. For two distinct lines L and M in $\mathbb{Z}_N\times\mathbb{Z}_N$, passing through the origin, we introduce a sequence $C_{L,M} \in \mathcal{H}$ with ambiguity function supported on L, and M. Under genericity assumptions—see Figure 4 for illustration—the essential support of $\mathcal{A}(C_{L,M}, H(C_{L,M}))$ lies on $r\times r$ grid generated by shifts of the lines L, and M. Denote by $v_{ij} = l_i + m_j,\ l_i \in L, m_j \in M;\ 1\le i,j\le r$, the intersection points of the lines in the grid. Using Remark I-C.1 we find all the points $v_{ij}, 1\le i,j\le r$, in $O(N\log N)$ operations. The following matching problem arises: Find the r points from v_{ij}, $1\le i,j\le r$, which belong to the support of H. To suggest a solution,

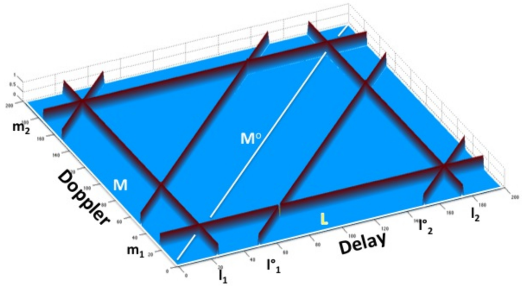

Fig. 3. Essential support of the ambiguity function $\mathcal{A}(C_{L,M,M^\circ}, H(C_{L,M,M^\circ}))$, where L is the delay line, M is the Doppler line, and M° is a diagonal line, and the support of H consists two parameters. Points of $\mathbb{Z}_N \times \mathbb{Z}_N$ through them pass three lines are the true delay-Doppler parameters of H.

we use the values of the ambiguity function to define a certain simple hypothesis function $h : L \times M \to \mathbb{C}$ (see Section IV). We obtain:

*Theorem I-E.1 (**Matching**):* Suppose $v_{ij} = l_i + m_j$ is a delay-Doppler shift of H, then $h(l_i, m_j) = 0$.

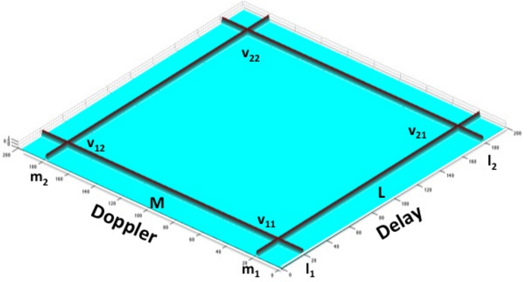

Fig. 4. Essential support of the ambiguity function $\mathcal{A}(C_{L,M}, H(C_{L,M}))$, where L is the delay line, M is the Doppler line, and the support of H consists two parameters.

The cross method makes use of Theorem I-E.1 and checks the values $h(l_i, m_j)$, $1 \leq i, j \leq r$. If a value is less than a priori chosen threshold, then the algorithm returns $v_{ij} = l_i + m_j$ as one of the delay-Doppler parameters. To estimate the attenuation coefficient corresponding to v_{ij} takes $O(1)$ arithmetic operations (see details in Section IV). Overall, the cross method enables channel estimation in $O(N \log N + r^2)$ arithmetic operations.

II. Chirp, Double-Chirp, and Triple-Chirp Sequences

In this section we introduce the chirp, double-chirp, and triple-chirp sequences, and discuss their correlation properties.

A. Definition of the Chirp Sequences

We have $N + 1$ lines[5] in the *discrete delay-Doppler plane* $V = \mathbb{Z}_N \times \mathbb{Z}_N$. For each $a \in \mathbb{Z}_N$ we denote by $L_a = \{(\tau, a\tau); \tau \in \mathbb{Z}_N\}$ the line of finite slope a, and we denote by $L_\infty = \{(0, \omega);\ \omega \in \mathbb{Z}_N\}$ the line of infinite slope. To every line L_a, it corresponds the orthonormal basis for $\mathcal{H}$:

$$\mathcal{B}_{L_a} = \left\{C_{L_{a,b}};\ b\ \in \mathbb{Z}_N\right\},$$

of *chirp sequences* associated with L_a, where

$$C_{L_{a,b}}[n] = e(2^{-1}an^2 - bn)/\sqrt{N}, n \in \mathbb{Z}_N.$$

[5]In these notes by a *line* $L \subset V$, we mean a line through $(0,0)$.

To the line L_∞ it corresponds the orthonormal basis

$$\mathcal{B}_{L_\infty} = \left\{C_{L_{\infty,b}}\ ;\ b\ \in \mathbb{Z}_N\right\},$$

of chirp sequences associated with L_∞, where

$$C_{L_{\infty,b}} = \delta_b,$$

denotes the Dirac delta sequence supported at b.

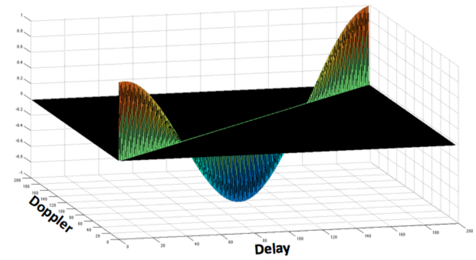

Fig. 5. Plot (real part) of $\mathcal{A}(C_{L_{1,1}}, C_{L_{1,1}})$, for chirp $C_{L_{1,1}}[n] = e[2^{-1}n^2 - n]$, associated with the line $L_1 = \{(\tau,\tau)\}$.

B. Chirps as Eigenfunctions of Heisenberg Operators

The Heisenberg operators (I-C.1) satisfy the commutation relations

$$\pi(\tau,\omega)\pi(\tau',\omega') = e(\omega\tau' - \tau\omega') \cdot \pi(\tau',\omega')\pi(\tau,\omega), \qquad \text{(II-B.1)}$$

for every $(\tau,\omega), (\tau',\omega') \in V$. In particular, for a given line $L \subset V$, we have the family of commuting operators $\pi(l)$, $l \in L$. Hence they admit an orthonormal basis $\mathcal{B}_L$ for $\mathcal{H}$ of common eigenfunctions. Important property of the chirp sequences is that for every chirp sequence $C_L \in \mathcal{B}_L$, there exists a character[6] $\psi_L : L \to \mathbb{C}^*$, i.e. $\psi_L(l + l') = \psi_L(l)\psi_L(l')$, $l, l' \in L$, such that

$$\pi_L(l)C_L = \psi_L(l)C_L, \text{ for every } l \in L.$$

This implies—see Figure 5—that for every $C_L \in \mathcal{B}_L$ we have

$$\mathcal{A}(C_L, C_L)[v] = \begin{cases} \psi_L(v) & \text{if } v \in L; \\ 0 & \text{if } v \notin L. \end{cases} \qquad \text{(II-B.2)}$$

It is not hard to see [4] that for distinct lines L, and M, and two chirps $C_L \in \mathcal{B}_L, C_M \in \mathcal{B}_M$ we have

$$|\mathcal{A}(C_L, C_M)[v]| = 1/\sqrt{N}, \qquad \text{for every } v \in V. \qquad \text{(II-B.3)}$$

C. Double-Chirp Sequences

For any two distinct lines $L, M \in V$, and two characters ψ_L, ψ_M on them, respectively, denote by C_L the chirp corresponding to L and ψ_L, and by C_M the chirp corresponding to M, and ψ_M. We define the *double-chirp* sequence

$$C_{L,M} = (C_L + C_M)/\sqrt{2}.$$

It follows from (II-B.2) and (II-B.3) that for the line $K = L$, or M, we have

$$\mathcal{A}(C_K, C_{L,M})[v] \approx \begin{cases} \psi_K(v)/\sqrt{2} & \text{if } v \in K; \\ 0 & \text{if } v \notin K. \end{cases}$$

[6]We denote by $\mathbb{C}^*$ the set of non-zero complex numbers

D. *Triple-Chirp Sequences*

Consider three distinct lines $L, M, M^{\circ} \in V$, and three characters $\psi_L, \psi_M, \psi_{M^{\circ}}$ on them, respectively. Denote by C_L, C_M and $C_{M^{\circ}}$ the chirps corresponding to L, M and M°, and ψ_L, ψ_M, and $\psi_{M^{\circ}}$, respectively. We define the *triple-chirp* sequence

$$C_{L,M,M^{\circ}} = (C_L + C_M + C_{M^{\circ}})/\sqrt{3}.$$

It follows from (II-B.2) and (II-B.3) that for the line $K = L, M$ or M°, we have

$$\mathcal{A}(C_K, C_{L,M,M^{\circ}})[v] \approx \begin{cases} \psi_K(v)/\sqrt{3} & \text{if } v \in K; \\ 0 & \text{if } v \notin K. \end{cases}$$

III. Incidence Method

We describe—see Figure 3 for illustration—the incidence algorithm.

Incidence Algorithm

Input: Randomly chosen lines L, M, and M°, and characters $\psi_L, \psi_M, \psi_{M^{\circ}}$ on them, respectively. Echo $R_{L,M,M^{\circ}}$ of the triple-chirp $C_{L,M,M^{\circ}}$, threshold $T > 0$, and value of SNR.

Output: Channel parameters.

1) Compute $\mathcal{A}(C_M, R_{L,M,M^{\circ}})$ on L, obtain peaks[7] at $l_1, ..., l_{r_1}$.
2) Compute $\mathcal{A}(C_L, R_{L,M,M^{\circ}})$ on M, obtain peaks at $m_1, ..., m_{r_2}$.
3) Compute $\mathcal{A}(C_{M^{\circ}}, R_{L,M,M^{\circ}})$ on L, obtain peaks at $l_1^{\circ}, ..., l_{r_3}^{\circ}$.
4) Find $v_{ij} = l_i + m_j$ which solve $l_i + m_j \in M^{\circ} + l_k^{\circ}$, $1 \leq i \leq r_1$, $1 \leq j \leq r_2$, $1 \leq k \leq r_3$.
5) For every delay-Doppler parameter $v_{ij} = l_i + m_j$ found in the previous step, compute $\alpha_{v_{ij}} = \sqrt{3}\mathcal{A}(C_L, R_{L,M,M^{\circ}})[m_j]\psi_L(l_i)$. Return the parameter $(\alpha_{v_{ij}}, v_{ij})$.

IV. Cross Method

Let $C_{L,M}$ be the double-chirp sequence associated with the lines $L, M \subset V$, and the characters ψ_L, and ψ_M, on L, and M, correspondingly. We define *hypothesis* function $h : L \times M \to \mathbb{C}$ by

$$\begin{aligned} h(l, m) &= \mathcal{A}(C_L, R_{L,M})[m] \cdot \psi_L[l] \qquad \text{(IV-.1)} \\ &\quad -\mathcal{A}(C_M, R_{L,M})[l] \cdot e(\Omega[l, m]) \cdot \psi_M[m], \end{aligned}$$

where[8] $\Omega : V \times V \to \mathbb{Z}_N$ is given by $\Omega[(\tau, \omega), (\tau', \omega')] = \tau\omega' - \omega\tau'$. Below we describe—see Figure 4—the Cross Algorithm.

V. Conclusions

In these notes we present the incidence and cross methods for efficient channel estimation. These methods, in particular, suggest solutions to the arithmetic complexity problem. Low arithmetic complexity enables working with sequences of larger length N, and hence higher velocity resolution of channel parameters is plausible. We summarize these important features in Figure 6, and putting them in comparison with the pseudo-random (PR) and Flag methods.

[7] We say that at $v \in V$ the ambiguity function of f and g has *peak,* if $|\mathcal{A}(f, g)[v]| > T\sqrt{2 \log \log N}/\sqrt{N \cdot SNR}$.

[8] In linear algebra Ω is called *symplectic form.*

[9] We say that at $v \in V$ the ambiguity function of f and g has *peak*, if $|\mathcal{A}(f, g)[v]| > T_1\sqrt{2 \log \log N}/\sqrt{N \cdot SNR}$.

Cross Algorithm

Input: Randomly chosen lines L, M, and characters ψ_L, ψ_M on them, respectively. Echo $R_{L,M}$ of the double-chirp $C_{L,M}$; thresholds $T_1, T_2 > 0$, and the value of SNR.

Output: Channel parameters.

1) Compute $\mathcal{A}(C_M, R_{L,M})$ on L, and take the r_1 peaks[9] located at points l_i, $1 \leq i \leq r_1$.
2) Compute $\mathcal{A}(C_L, R_{L,M})$ on M, and take the r_2 peaks located at the points $m_j, 1 \leq j \leq r_2$.
3) Find $v_{ij} = l_i + m_j$ which solve $|h(l_i, m_j)| \leq T_2\sqrt{2 \log \log(N)}/\sqrt{N \cdot SNR}$, where $1 \leq i \leq r_1$, $1 \leq j \leq r_2$.
4) For every delay-Doppler parameter $v_{ij} = l_i + m_j$ found in the previous step, compute $\alpha_{v_{ij}} = \sqrt{2}\mathcal{A}(C_L, R_{L,M})[m_j]\psi_L(l_i)$. Return the parameter $(\alpha_{v_{ij}}, v_{ij})$.

Method	**Complexity**
PR	*O(N^2logN)*
Flag	*O(rNlogN)*
Incidence	*O(NlogN+r^3)*
Cross	*O(NlogN+r^2)*

Fig. 6. Comparing methods, with respect to arithmetic complexity, for channels with r parameters.

Remark V-.1: Both new methods are robust to a certain degree of noise since they use the values of the ambiguity functions, which is a sort of averaging.

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