Discrete Discreet Asymptotics

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@monsoon0

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Divergent Series

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- From Littlewood's preface to GH Hardy's *Divergent Series*

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"Then came a time when it was found that something after all could be done about them. This is now a matter of course, but in the early years of the century the subject, while in no way mystical or unrigorous, was regarded as sensational, and about the present title, now colourless, there hung an aroma of paradox and audacity."

Series

$$1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots$$

$$1 - \epsilon + \epsilon^2 - \epsilon^3 + \epsilon^4 + \ldots$$

$$1 + \epsilon + 2!\epsilon^2 + 3!\epsilon^3 + \ldots$$

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Interpretation?

$$\frac{1}{1-2\,x} = 1 + 2x + 4x^2 + 8x^3 + \ldots$$

$$x = 1$$

$$\Downarrow$$

$$-1 = 1 + 2 + 4 + 8 + \ldots$$

GH Hardy, *Divergent Series, 1949*

Asymptotics

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$$f(x) = \mathcal{O}(g(x)) \iff \exists\ M \text{s.t.} \left|\frac{f(x)}{g(x)}\right| < M$$

$$\forall x \in \gamma$$

Asymptotic Series

- f is said to be asymptotic to a series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$

iff $\forall N \in \mathbb{N}$

$$\lim_{x \to x_0} \frac{f(x) - \sum_{n=0}^{N} a_n (x - x_0)^n}{(x - x_0)^N} = 0$$

and we write

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n, \text{ as } x \to x_0$$

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- The series is not required to converge.

An example

$$u'' - u' \left(1 - \frac{1}{x}\right) = 0$$

- One solution is

$$Ei(x) = \int_{-\infty}^{x} \frac{e^t}{t} dt$$

$$\sim \frac{e^x}{x} \sum_{n=0}^{\infty} \frac{n!}{x^n}, \quad x \to +\infty$$

- Other solutions differ from this one by a constant.

$$Ei(x) + C$$

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- *Divergent*

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Approximation

- Consider $F(x) = x\, e^{-x} Ei(x)$

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$$\sum_{n=0}^{10} \frac{n!}{100^n} = 1.0102062527748311680\ldots$$

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$$\sum_{n=0}^{10} \frac{n!}{100^n} = 1.010206252774831168 0\ldots$$

$$\sum_{n=0}^{100} \frac{n!}{100^n} = 1.0102062527748357112\ldots$$

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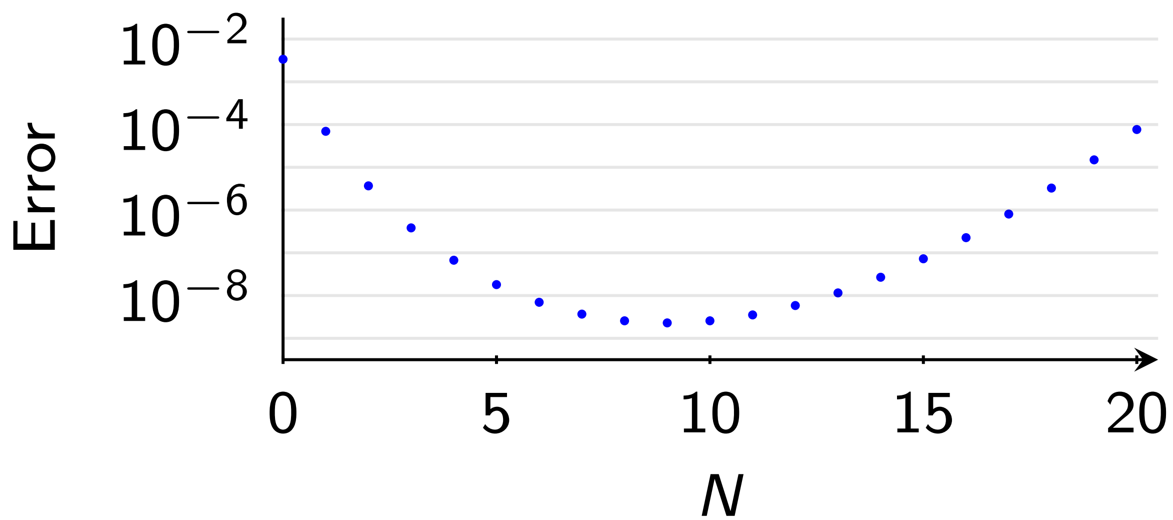
$$\sum_{n=0}^{10} \frac{n!}{100^n} = 1.0102062527748311680\ldots$$

$$\sum_{n=0}^{100} \frac{n!}{100^n} = 1.0102062527748357112\ldots$$

$$\sum_{n=0}^{281} \frac{n!}{100^n} = 732496.06921461904157\ldots$$

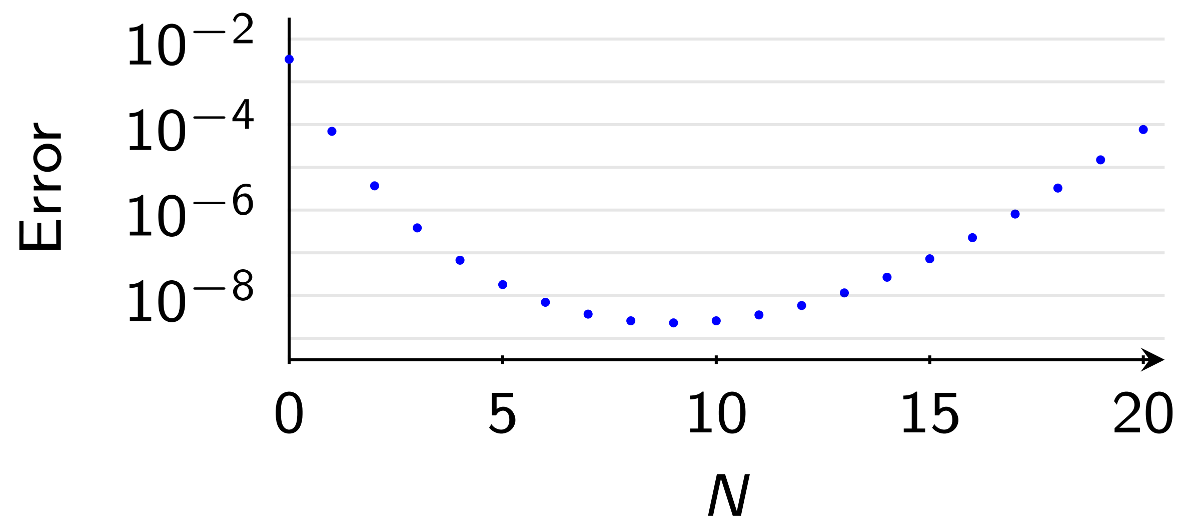
Truncation at least term



- Truncate at the least term

$$F(x) = \sum_{n=0}^{N-1} \frac{n!}{x^n} + R_N$$

Truncation at least term



- Truncate at the least term

$$F(x) = \sum_{n=0}^{N-1} \frac{n!}{x^n} + R_N$$

- *Exponentially small*
- *N* depends on *x*

The identification problem

- How do we distinguish

$$F(x) \sim \sum_{n=0}^{\infty} \frac{n!}{x^n}$$

$$\widetilde{F}(x) \sim \sum_{n=0}^{\infty} \frac{n!}{x^n} + \frac{e^{-x}C}{x}$$

in the limit $x \to +\infty$?

- E.g., if C=1

$$\frac{e^{-100}}{100} = 3.7200759760208359630 \times 10^{-46}$$

The identification problem

- How do we distinguish

- *Discreetly* hidden beyond all orders

$$F(x) \sim \sum_{n=0}^{\infty} \frac{n!}{x^n}$$

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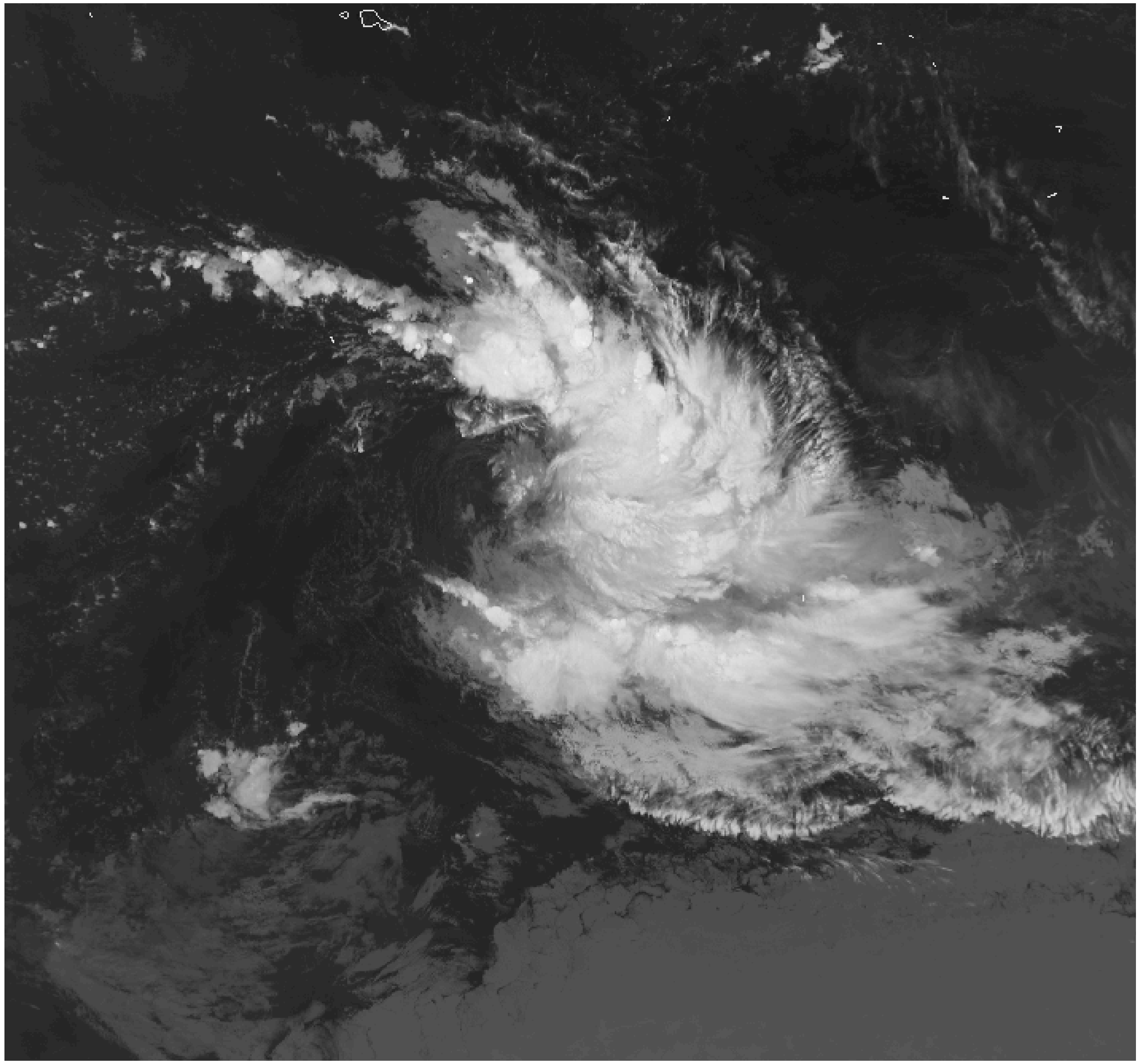
$$\frac{e^{-100}}{100} = 3.7200759760208359630 \times 10^{-46}$$

Why should we care?

Chaos

- "Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?"
- To forecast future behaviour, we need to known initial states with infinite precision.
- This has become synonymous with *chaos*, but is also present in ordered, non-chaotic systems.



TC Gavin 1997 SW Pacific

Order

- The Painlevé equations, which arise as reductions of soliton equations

$$w_\tau + 6\,w\,w_\xi + w_{\xi\xi\xi} = 0$$

$$\left\{ \begin{array}{l} w = -2\,y(x) - 2\,\tau \\ x = \xi + 6\,\tau^2 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{ll} w_\tau & = -24\,\tau\,y_x - 2 \\ w_\xi & = -2\,y_x \\ w_{\xi\xi\xi} & = -2\,y_{xxx} \end{array} \right.$$

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$$\rightarrow \qquad y'' = 6\,y^2 - x$$

Applications

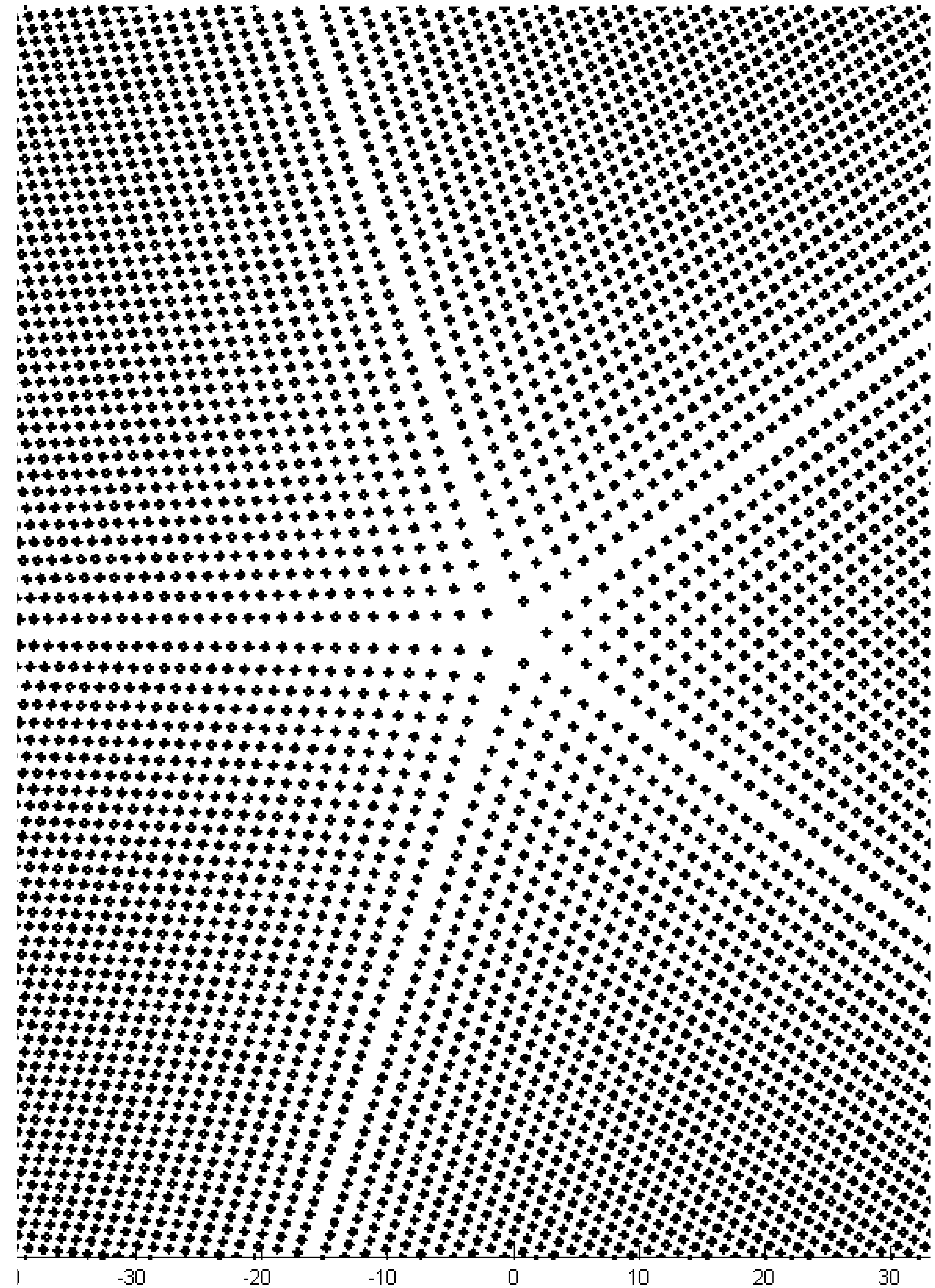
- Electrical structures of interfaces in steady electrolysis *L. Bass, Trans Faraday Soc 60 (1964)1656–1663*
- Spin-spin correlation functions for the 2D Ising model *TT Wu, BM McCoy, CA Tracy, E Barouch Phys Rev B13 (1976) 316–374*
- Spherical electric probe in a continuum gas *PCT de Boer, GSS Ludford, Plasma Phys 17 (1975) 29–41*
- Cylindrical Waves in General Relativity *S Chandrashekar, Proc. R. Soc. Lond. A 408 (1986) 209–232*
- Non-perturbative 2D quantum gravity *Gross & Migdal PRL 64(1990) 127-130*
- Orthogonal polynomials with non-classical weight function *AP Magnus J. Comput Appl. Anal. 57 (1995) 215–237*
- Level spacing distributions and the Airy kernel *CA Tracy, H Widom CMP 159 (1994) 151–174*
- Spatially dependent ecological models: *Joshi& Morrison Anal Appl 6 (2008) 371-381*
- Gradient catastrophe in fluids: *Dubrovin, Grava & Klein J. Nonlin. Sci 19 (2009) 57-94*

What do we know about the solutions of these equations?

General Solutions

- Movable poles
- Transcendentality of general solutions
- Special solutions
- Asymptotic behaviours

$u(0) = 0, \quad u'(0) = 0$



Fornberg & Weideman 2009

$\mathrm{P_I}$

Tronquée Solutions

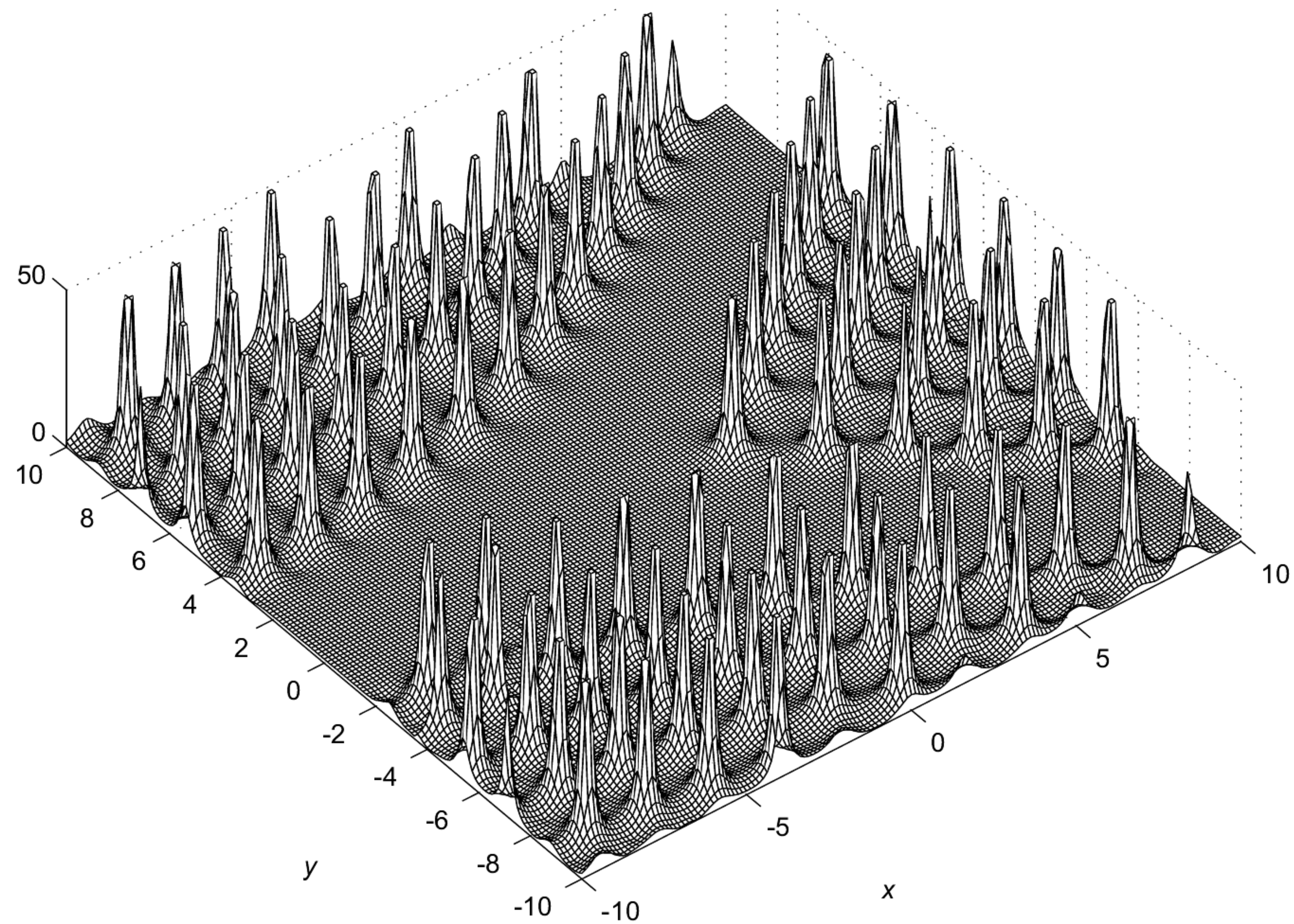
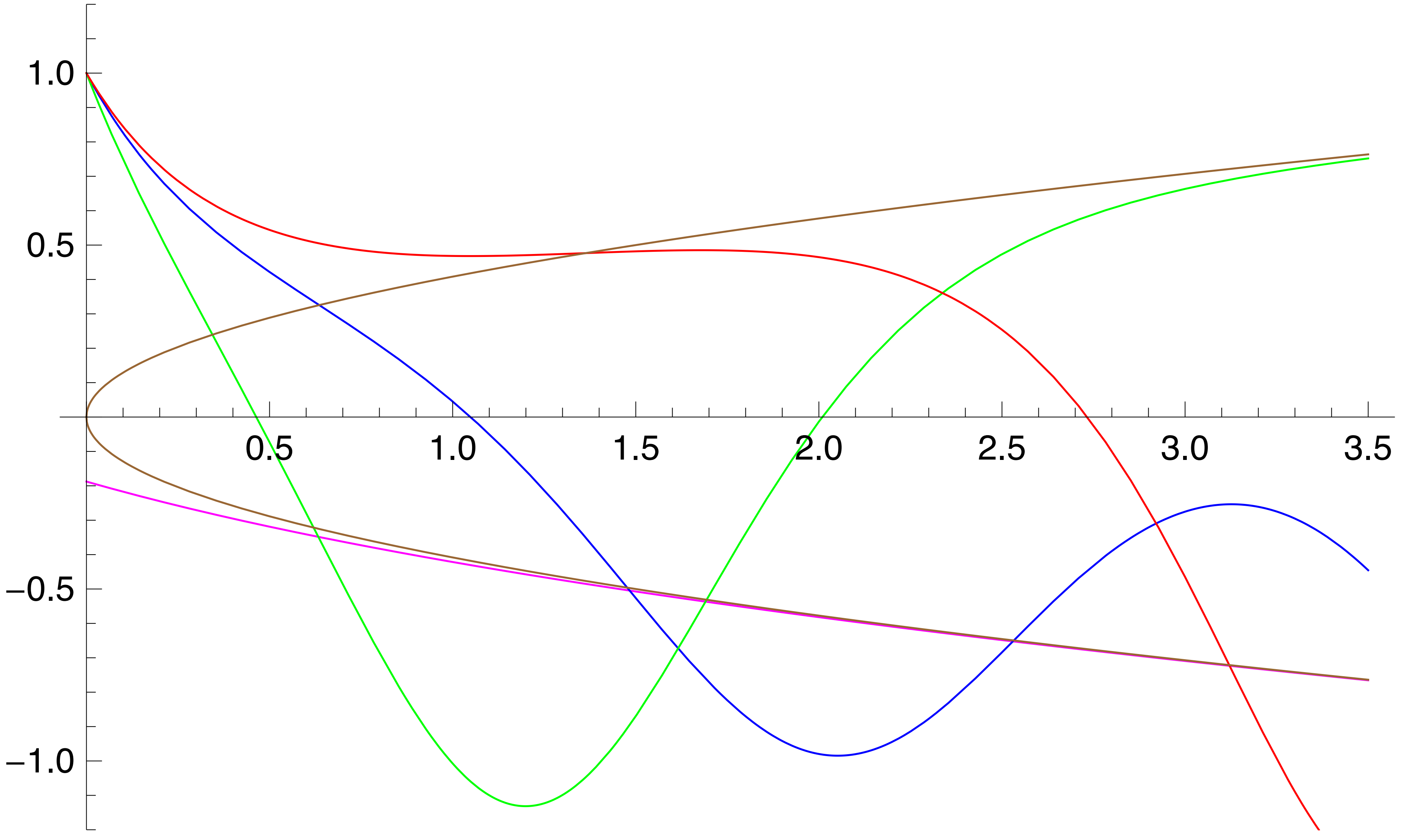


FIG. 3.1. *Magnitude of the solution* $u(z)$ *to the* P_I *equation in case of ICs* $u(0) = -0.1875$, $u'(0) = 0.3049$, *displayed over the domain* $z = x + iy$, $-10 \leq x \leq 10$, $-10 \leq y \leq 10$.

Real Solutions

Consider $\mathrm{P_I}$ $w_{tt} = 6w^2 - t$ for $w(t), t \in \mathbb{R}$



Hidden Solutions of P_I

- Solutions asymptotic to

$$\Pi_\pm = \left\{(x,y) \mid x>0, y=\pm\sqrt{x/6}\right\}$$

have formal expansions

$$y_f \quad = \quad \frac{x^{1/2}}{\sqrt{6}} \sum_{k=0}^{\infty} \frac{a_k}{(x^{1/2})^{5k}}$$

$$a_k \underset{k\to\infty}{=} -2\,c((k-1)!)^2 \left(25/(8\sqrt{6})\right)^k$$

The coeffts a_k are important in 2D quantum gravity (Di Francesco, Ginsparg, Zinn-Justin 1994).

The Real Tritronquée

- Theorem: ∃ unique solution Y(x) of PI which has asymptotic expansion

$$y_f = -\sqrt{\frac{x}{6}}\sum_{k=0}^{\infty}\frac{a_k}{(x^{1/2})^{5k}}, \text{ in } |\arg(x)| \leq 4\pi/5$$

  and

  - *Y(x)* is real for real *x*
  - Its interval of existence *I* contains $\mathbb{R}$
  - *Y(x)* lies below Π_-
  - It is monotonically decaying in *I*.

From Joshi & Kitaev Studies in Appl Math (2001)

What about global dynamics?

Perturbed Form

- In Boutroux's coordinates:

$$w_1 = t^{1/2}\, u_1(z),\ w_2 = t^{3/4} u_2(z),\ z = \frac{4}{5} t^{5/4}$$

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ 6u_1^2 - 1 \end{pmatrix} - \frac{1}{5z} \begin{pmatrix} 2u_1 \\ 3u_2 \end{pmatrix}$$

- a perturbation of an elliptic curve as $|z| \to \infty$

$$E = \frac{u_2^2}{2} - 2u_1^3 + u_1 \ \Rightarrow\ \frac{dE}{dz} = \frac{1}{5z}\,(6E + 4u_1)$$

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Projective Space

- What if *x*, *y* become unbounded?
- Use projective geometry: $x = \frac{u}{w}, y = \frac{v}{w}$

$$[x, y, 1] = [u, v, w] \ \in \mathbb{CP}^2$$

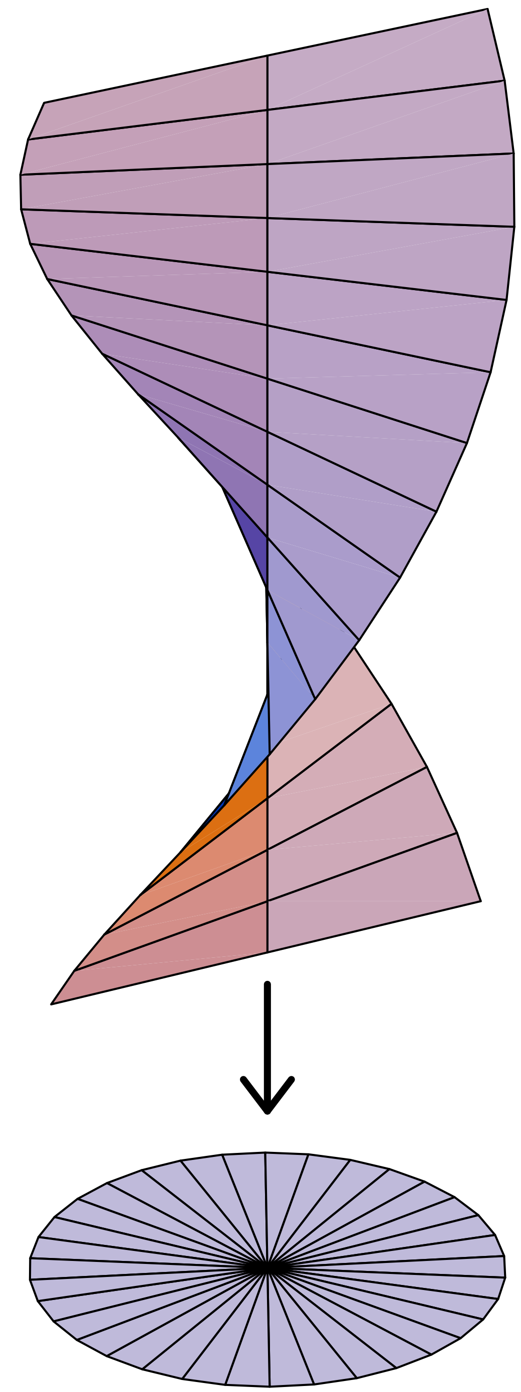
- The level curves of $\mathrm{P_I}$ are now

$$F_{\mathrm{I}} = wv^2 - 4u^3 + g_2 u w^2 + g_3 w^3$$

all intersecting at the base point [0, 1, 0].

$\Rightarrow$ To describe solutions, resolve the flow through this point

Resolving a base pt



From JJ Duistermaat, QRT Maps and Elliptic Surfaces, Springer Verlag, 2010

Resolution

- "Blow up" the singularity or base point:

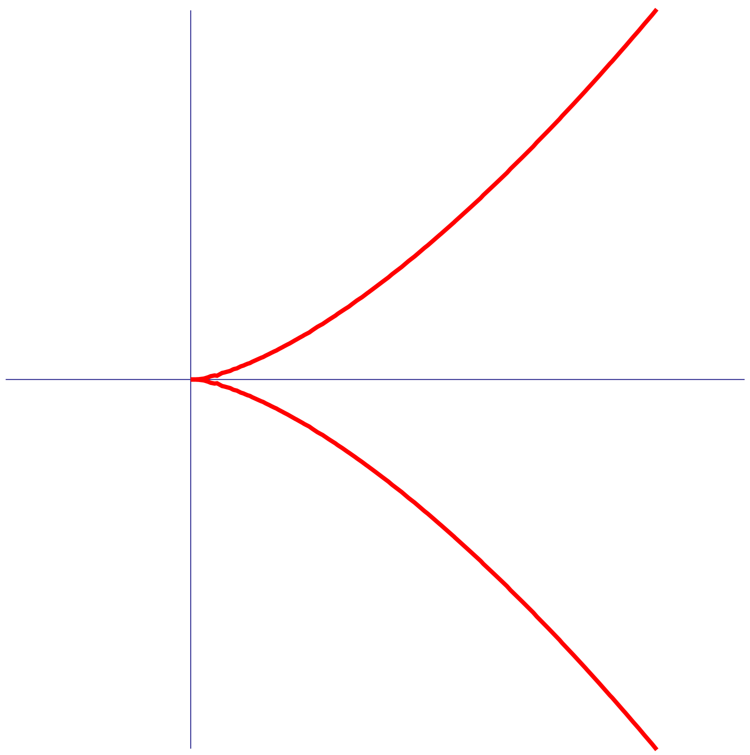
$$f(x, y) = y^2 - x^3$$
$$(x, y) = (x_1, x_1\, y_1)$$
$$\Rightarrow x_1^2\, y_1^2 - x_1^3 = 0$$
$$\Leftrightarrow x_1^2\, (y_1^2 - x_1) = 0$$

- Note that

$$x_1 = x, y_1 = y/x$$

Method

$$y^2 = x^3$$



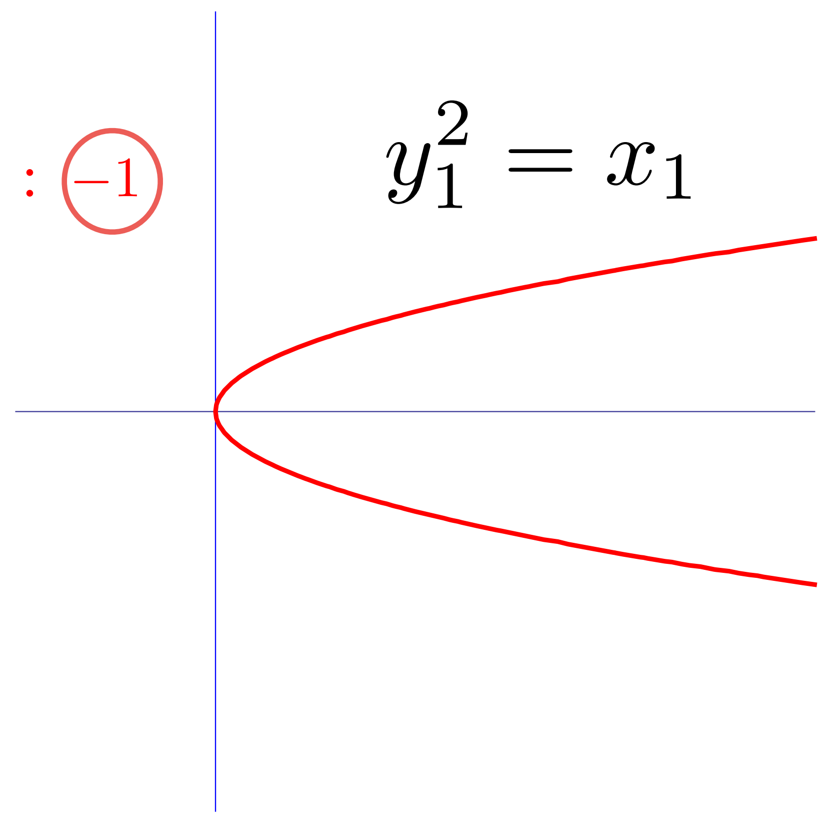
$$f(x, y) = y^2 - x^3$$

$$(x, y) = (x_1, x_1\, y_1)$$

$$f(x_1, x_1\, y_1) = x_1^2(y_1^2 - x_1)$$

$L_1 : -1$

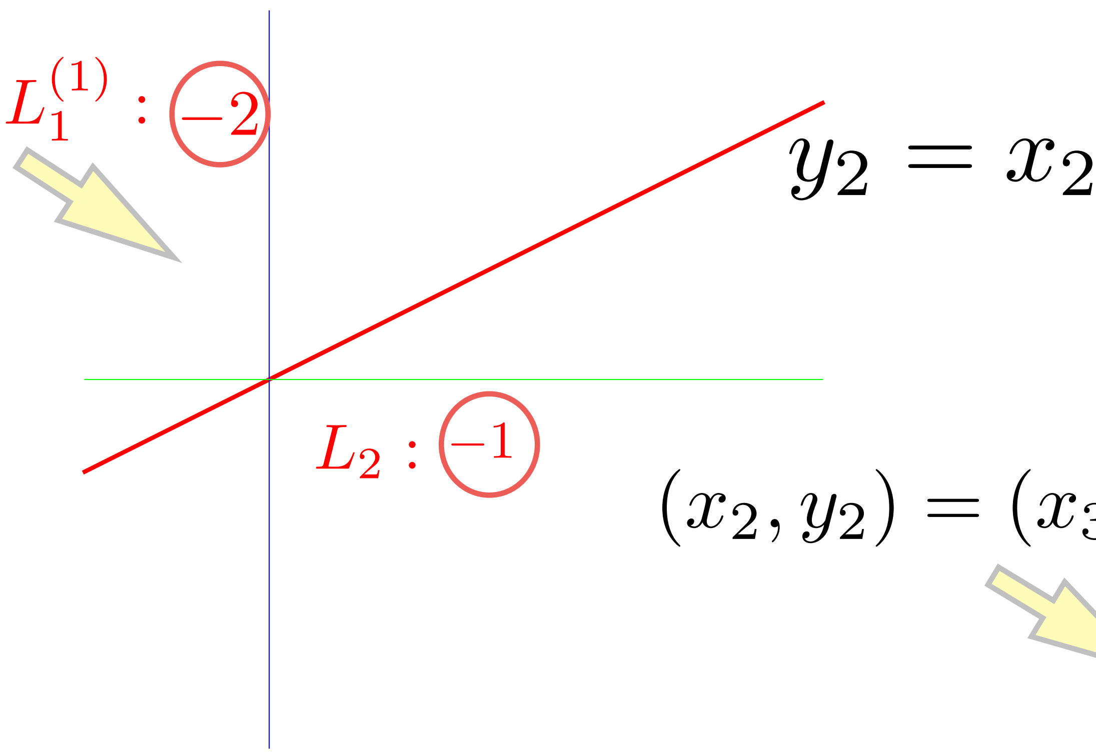
$$y_1^2 = x_1$$



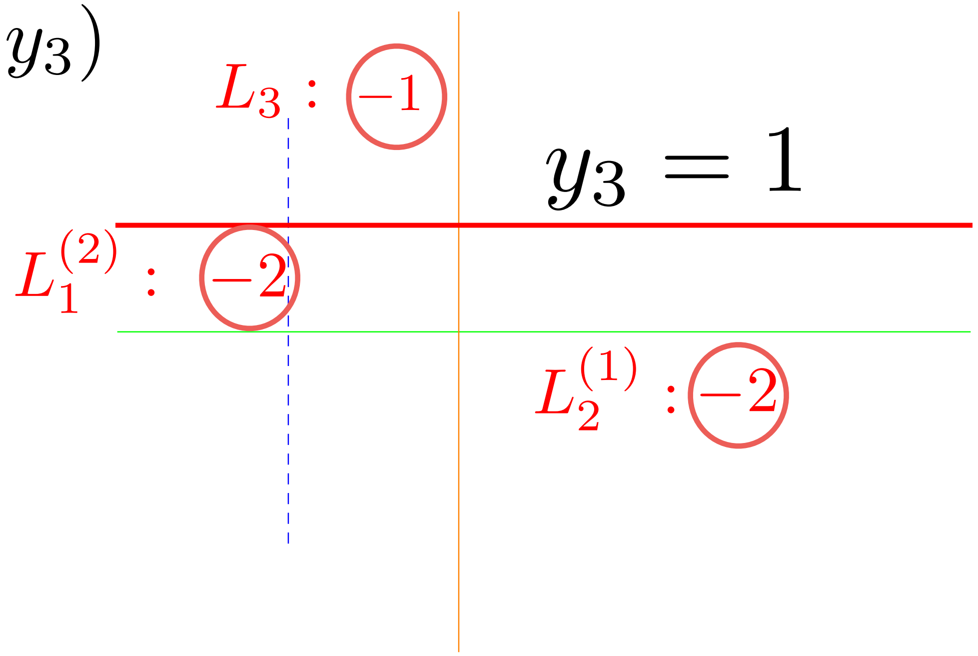
$$f_1(x_1, y_1) = y_1^2 - x_1$$

$$f_1(x_2\,y_2, y_2) = y_2(y_2 - x_2)$$

$$(x_1, y_1) = (x_2\,y_2, y_2)$$



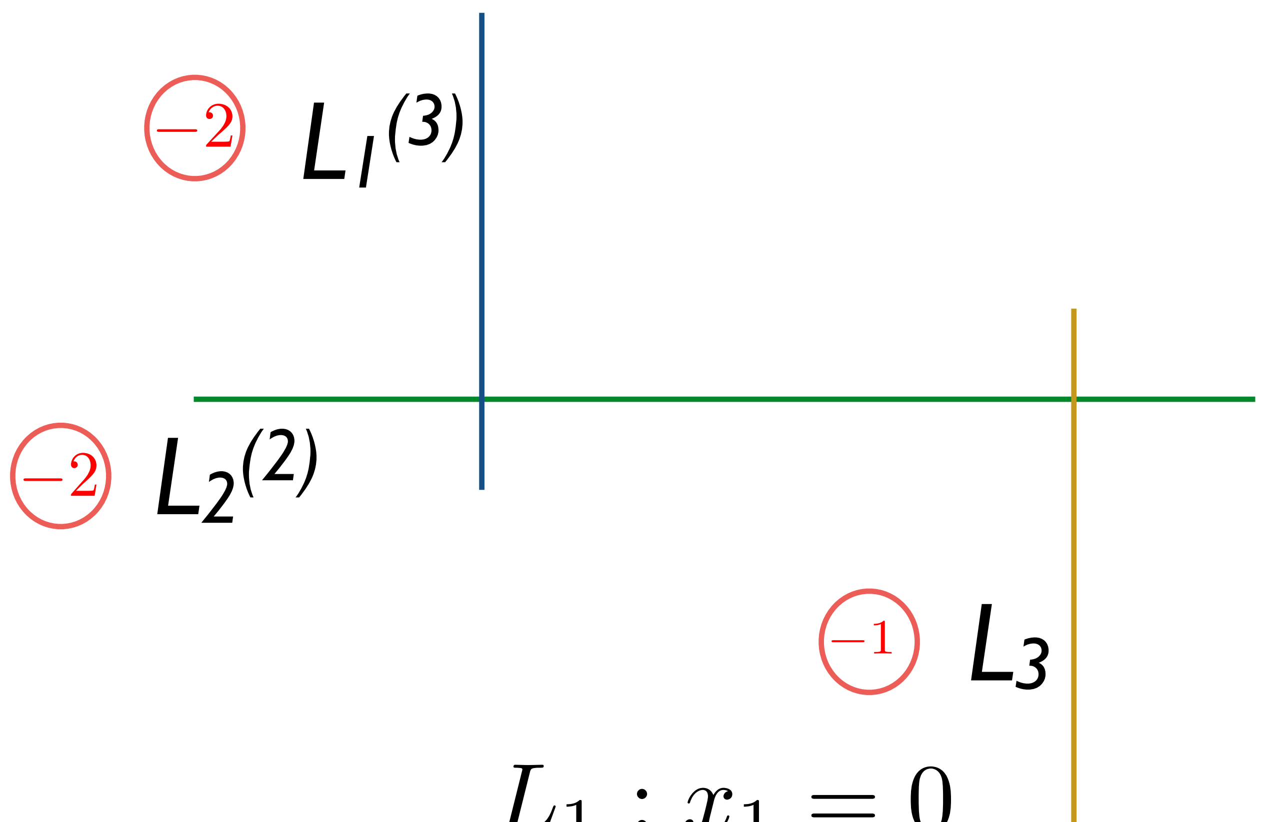
$$(x_2, y_2) = (x_3, x_3\,y_3)$$



$$f_2(x_2, y_2) = y_2 - x_2$$

$$f_2(x_3, x_3\,y_3) = x_3(y_3 - 1)$$

Initial-Value Space



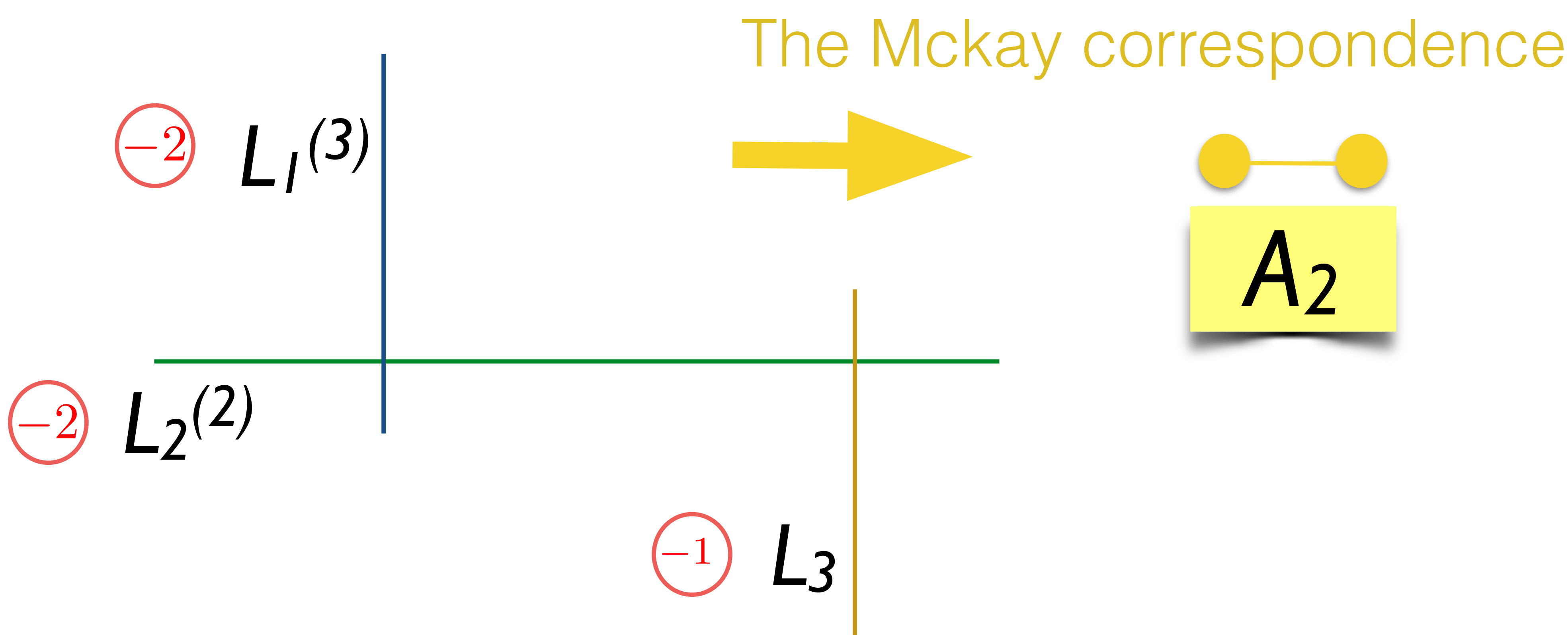
$$L_1 : x_1 = 0$$

$$L_2 : y_2 = 0$$

$$L_3 : x_3 = 0$$

Now the space is compactified and regularised.

Initial-Value Space



$$L_1 : x_1 = 0$$

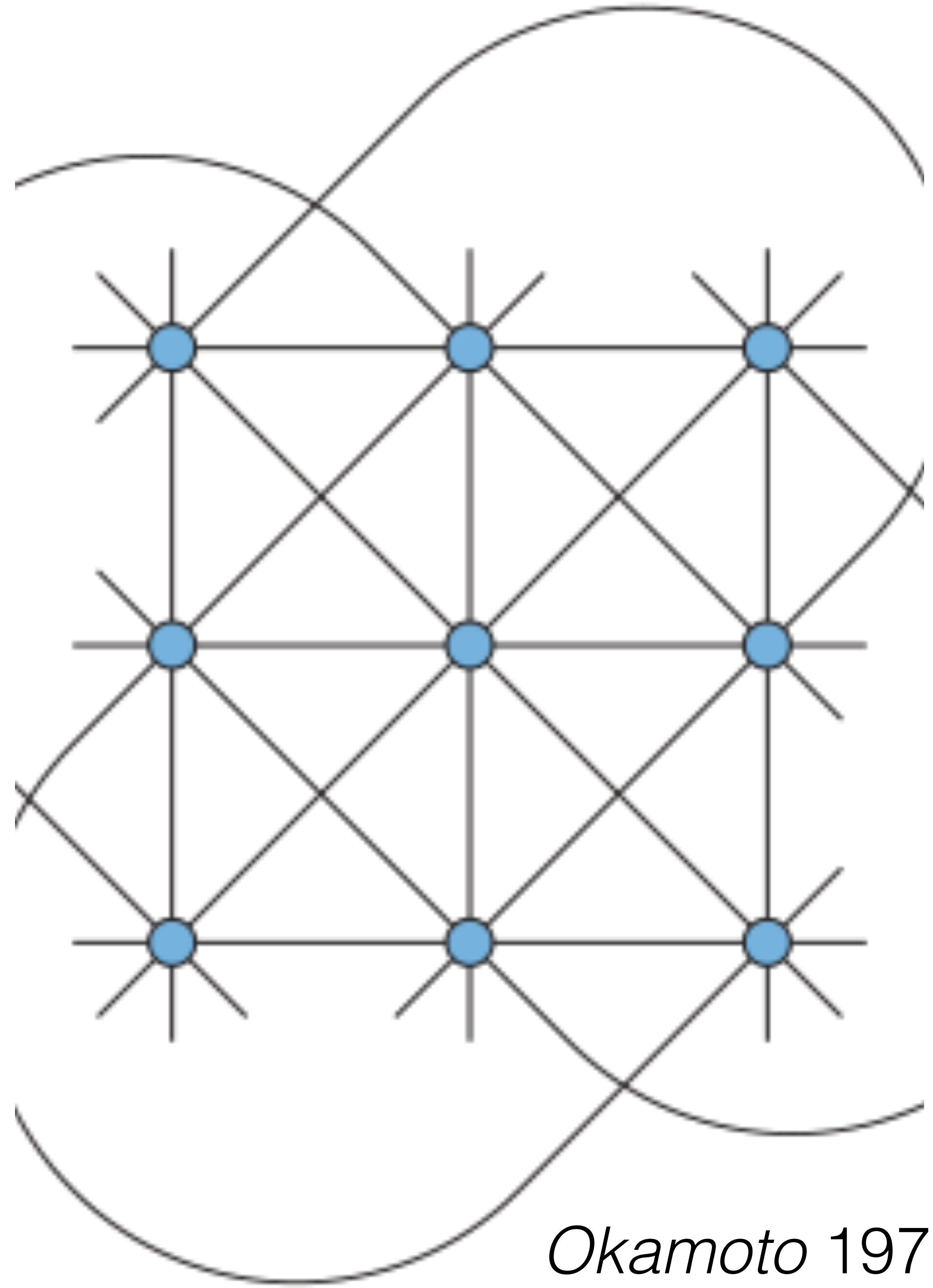
$$L_2 : y_2 = 0$$

$$L_3 : x_3 = 0$$

Now the space is compactified and regularised.

Unifying Property

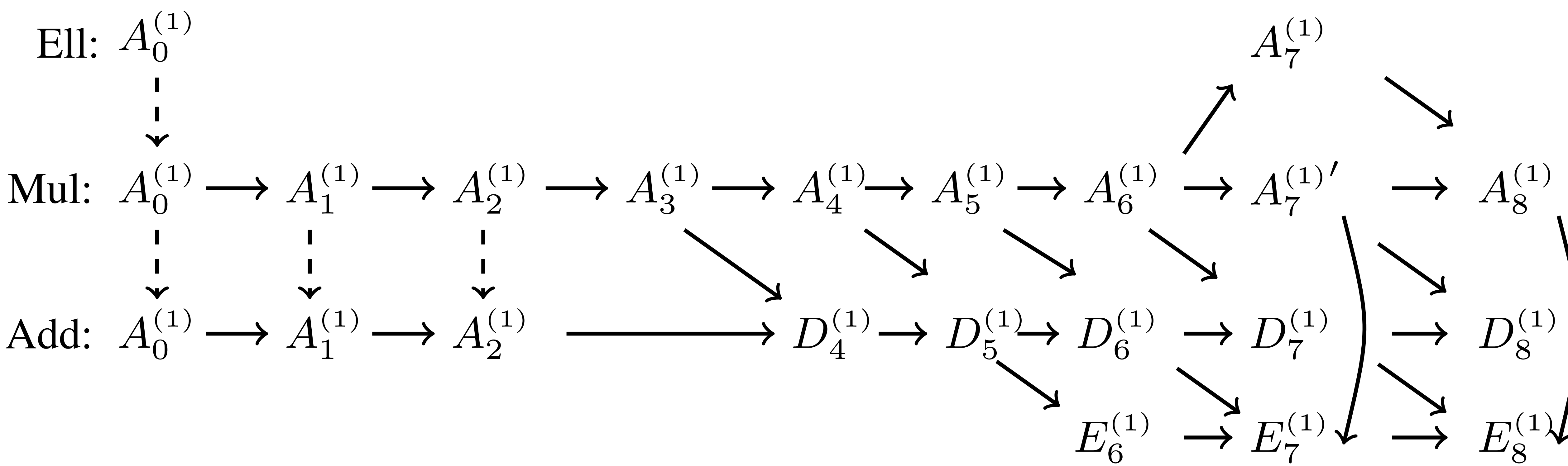
The space of initial values of a Painlevé system is resolved by “blowing up” 9 points in $\mathbb{CP}^2$
(or 8 points in $\mathbb{P}^1$x$\mathbb{P}^1$)



Okamoto 1979

Sakai 2001

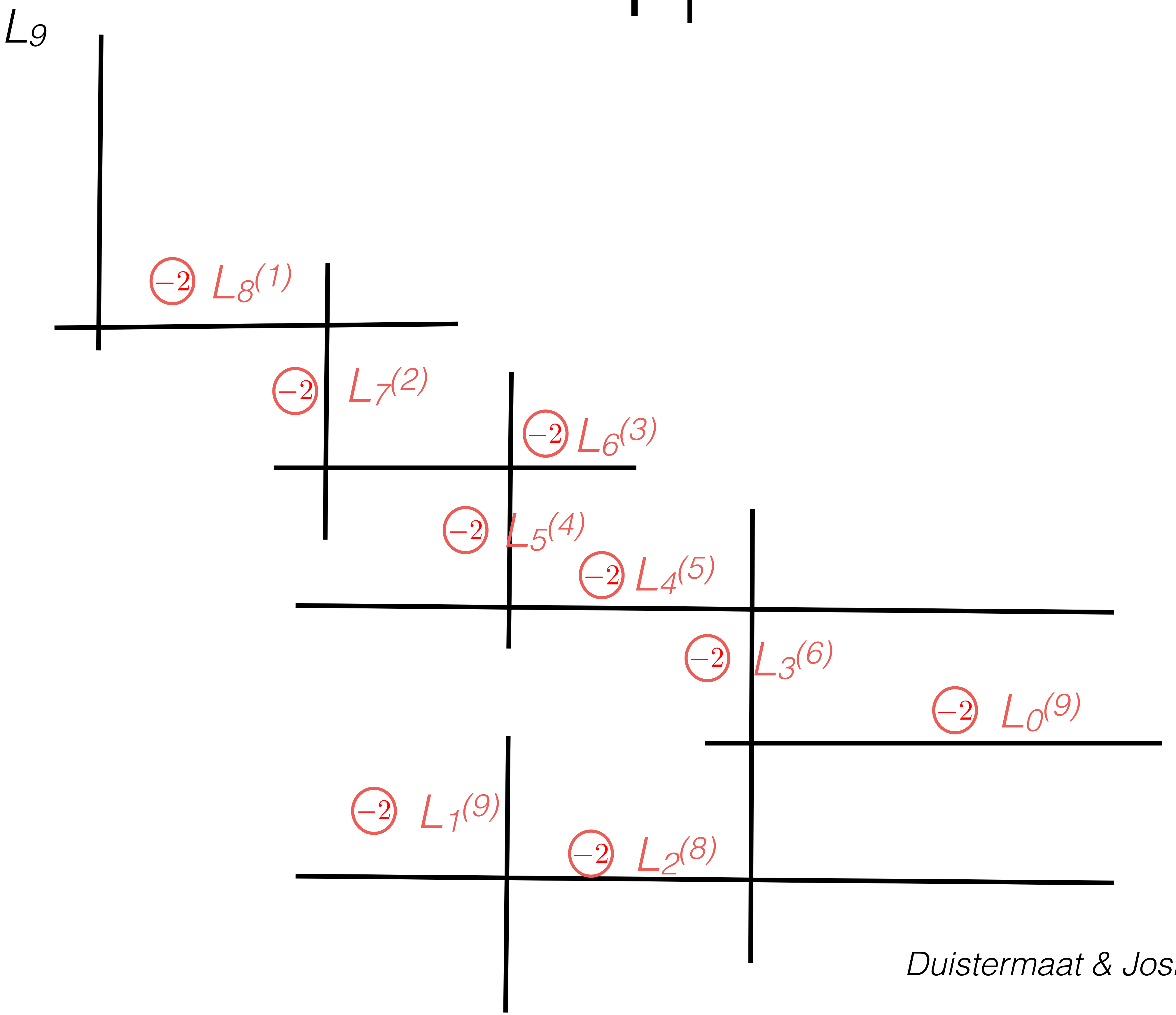
Sakai's Description I



Initial-value spaces of all continuous and discrete Painlevé equations

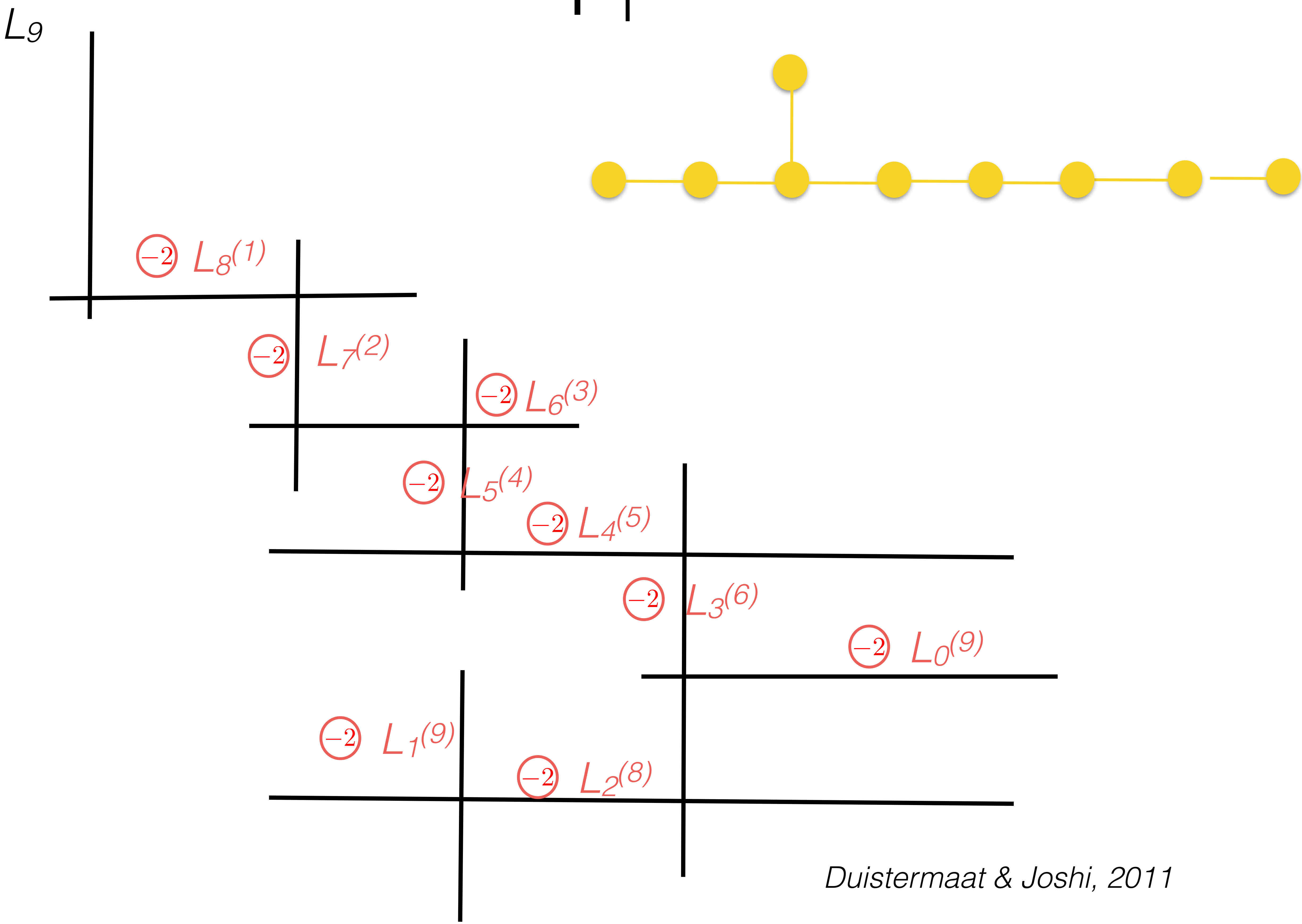
Sakai 2001

$\mathrm{P_I}$



Duistermaat & Joshi, 2011

P_I



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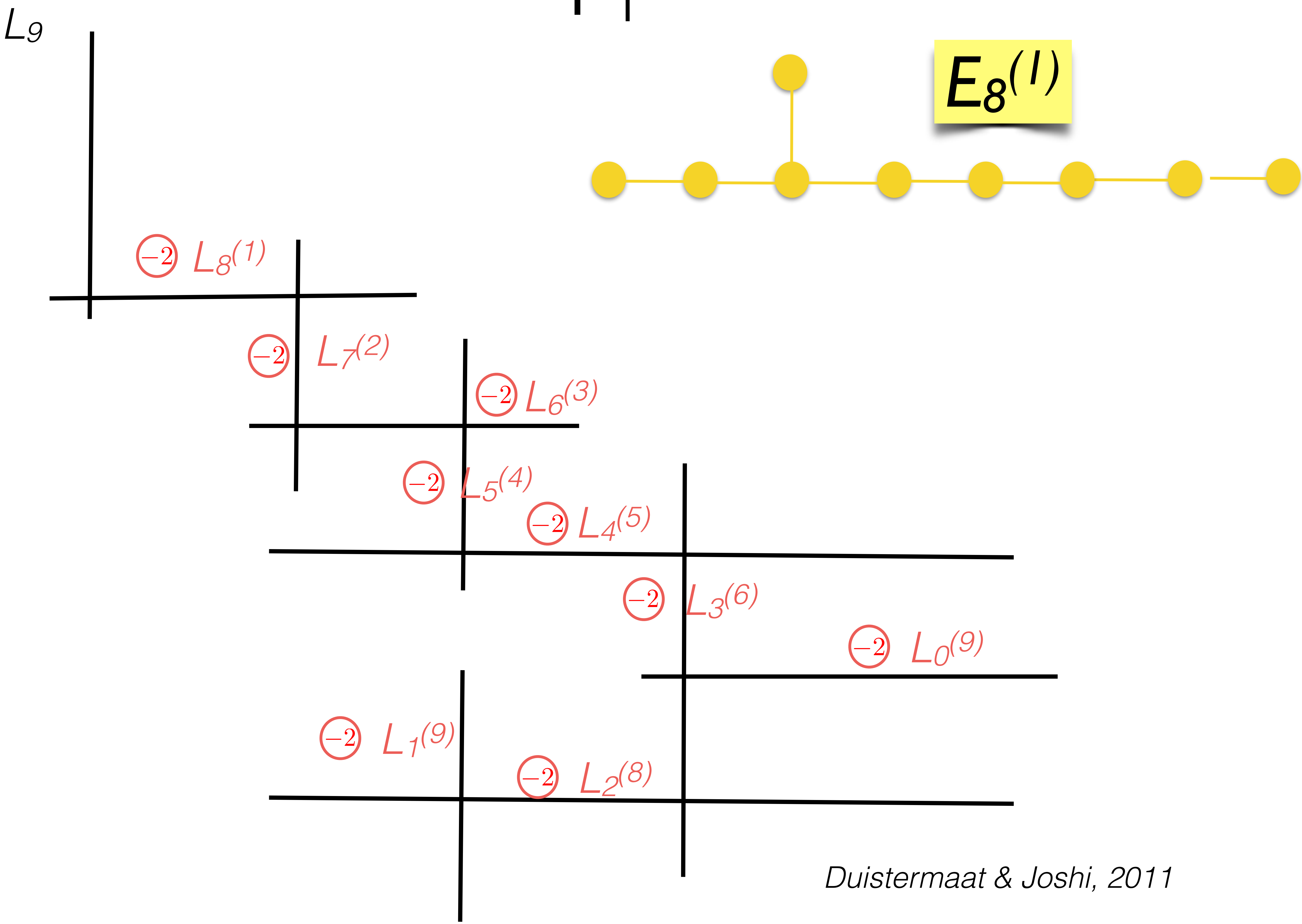

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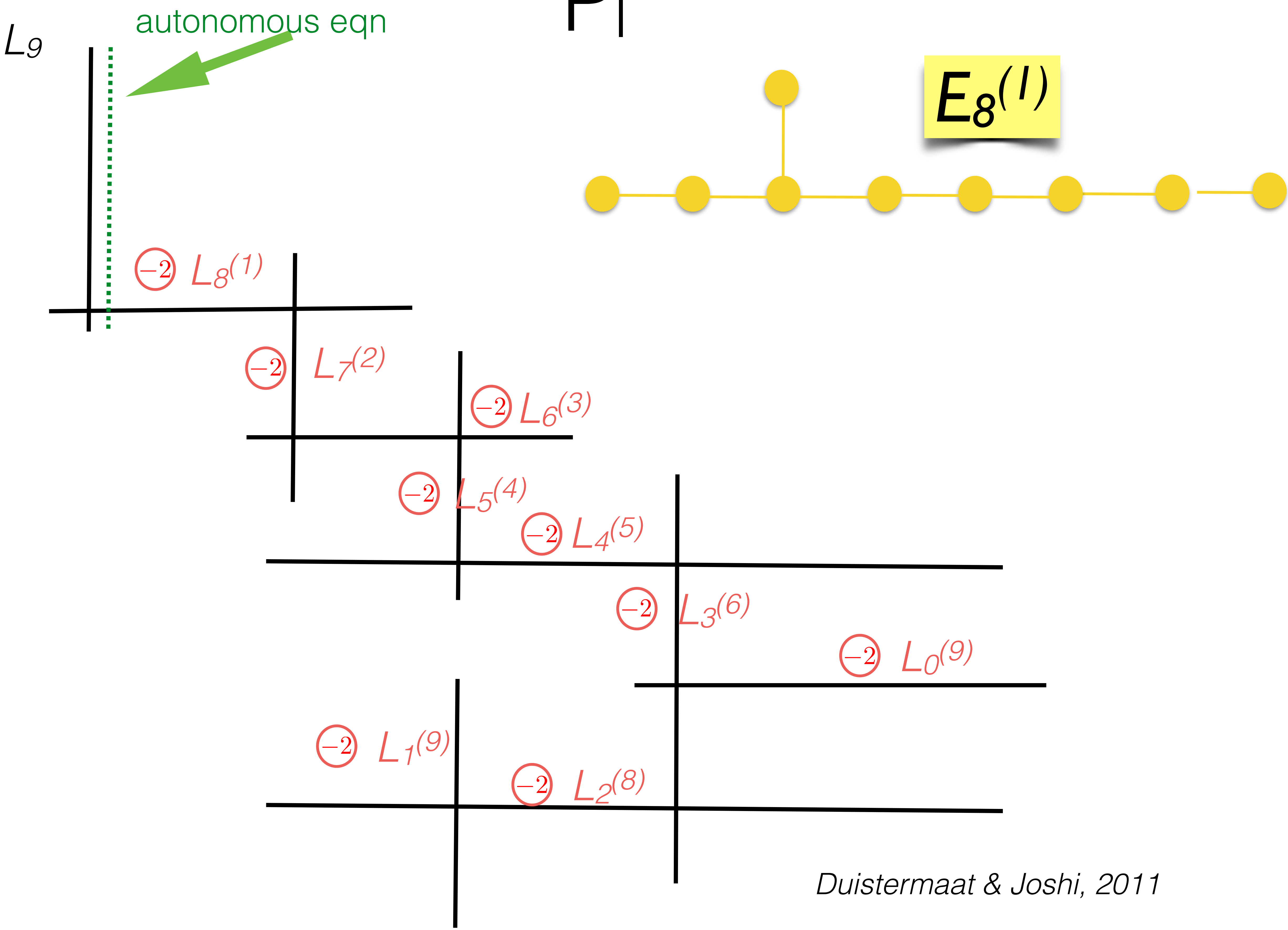
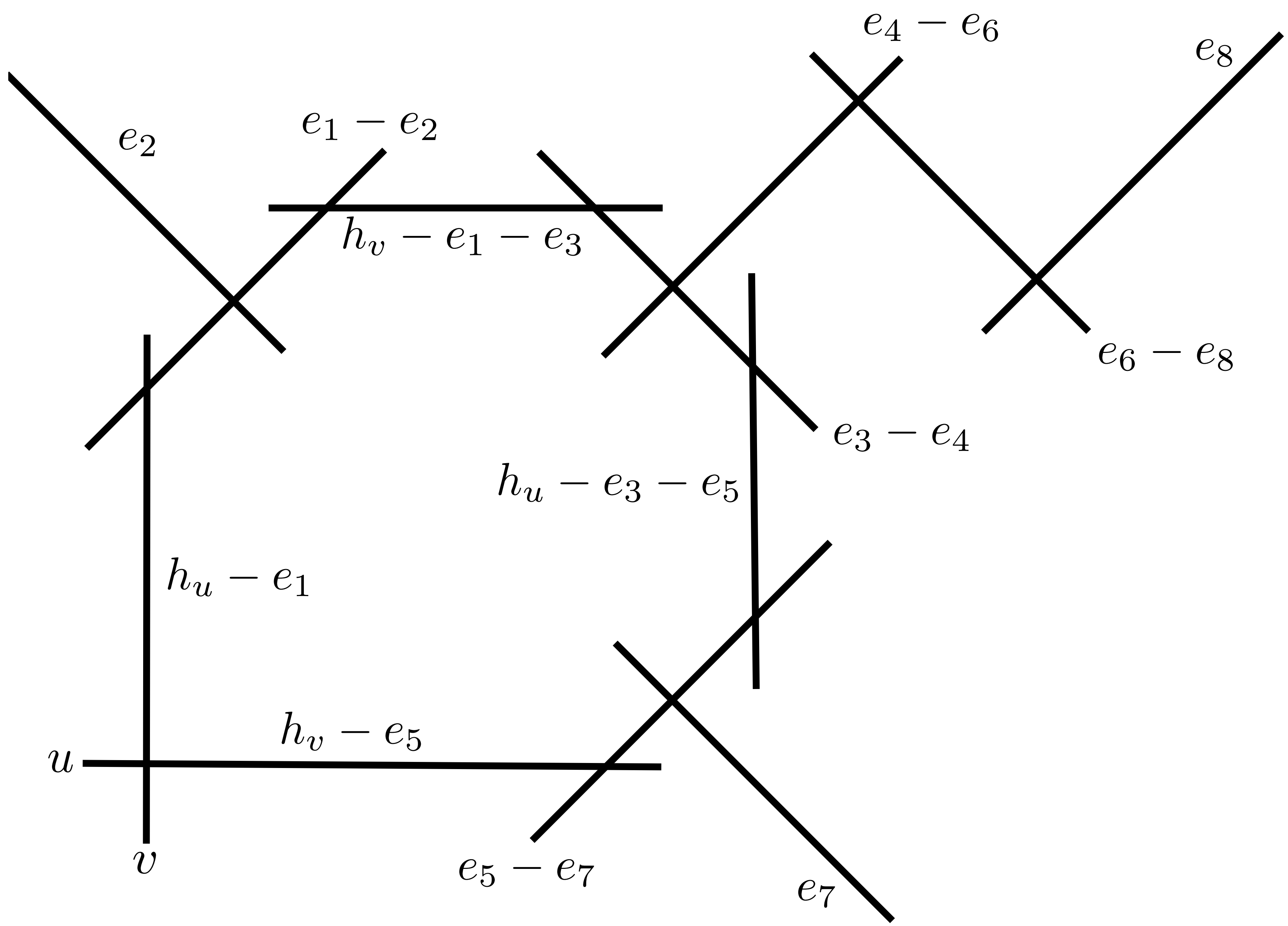
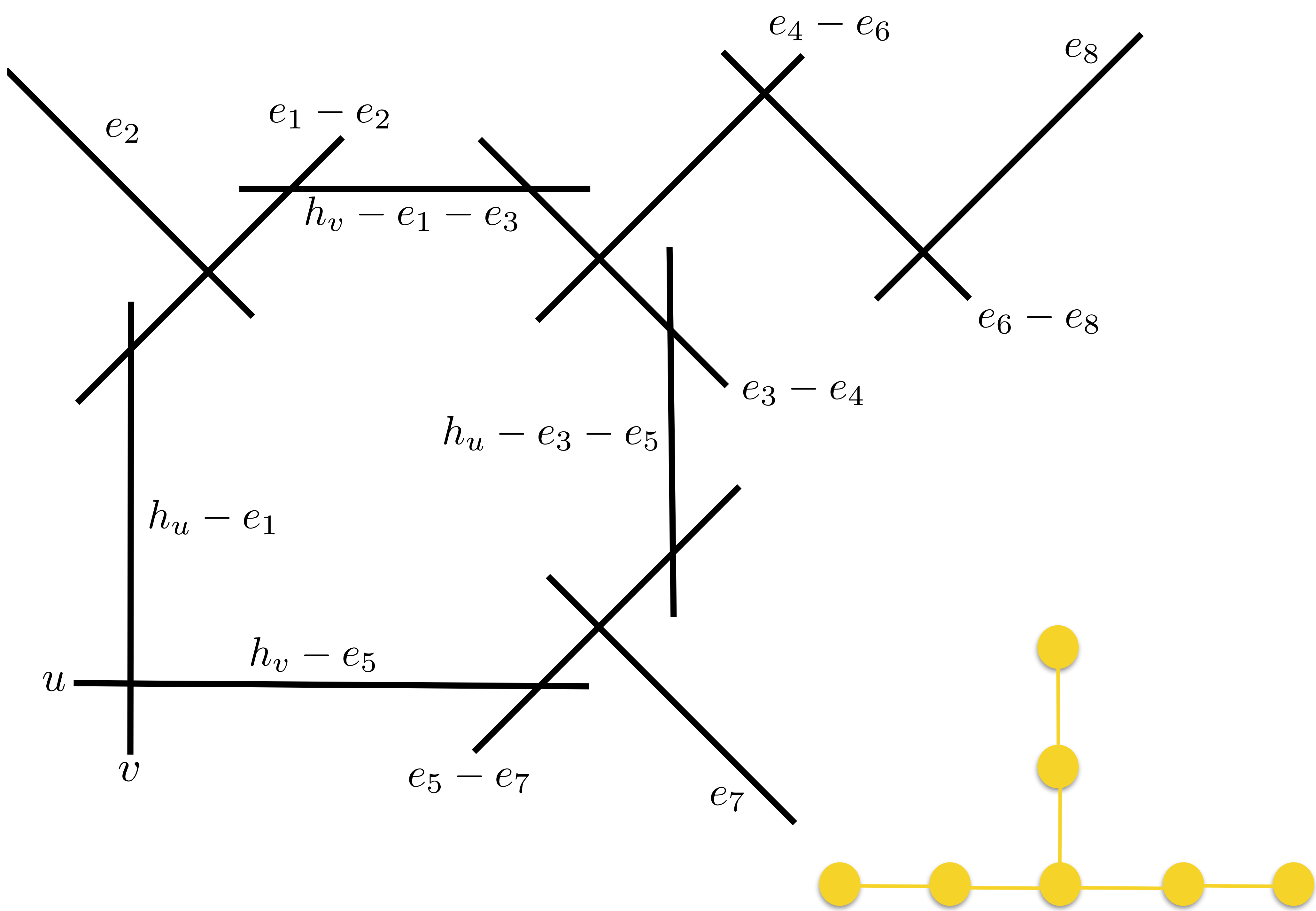


Duistermaat & Joshi, 2011

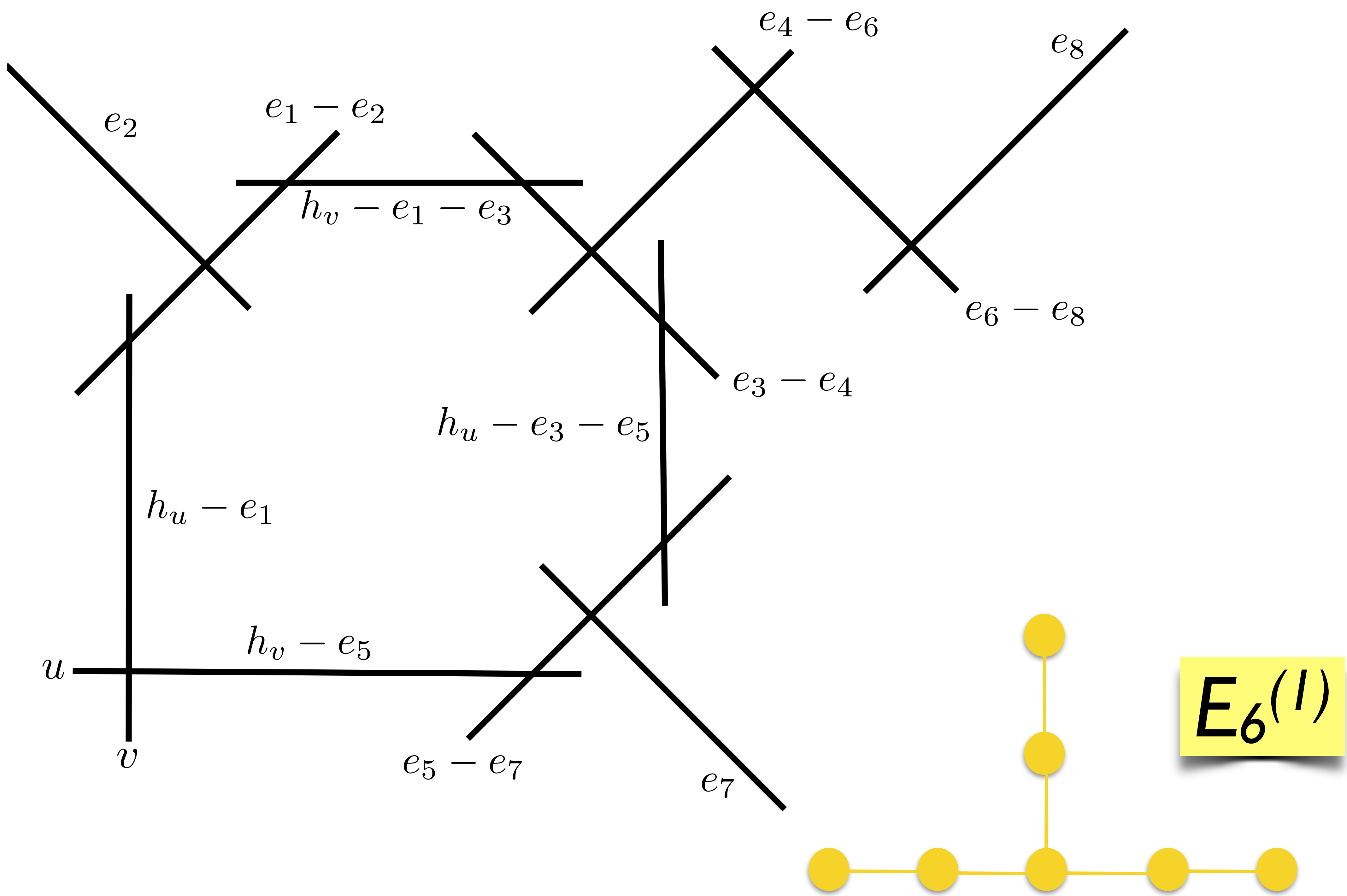
$\mathrm{dP_I}$



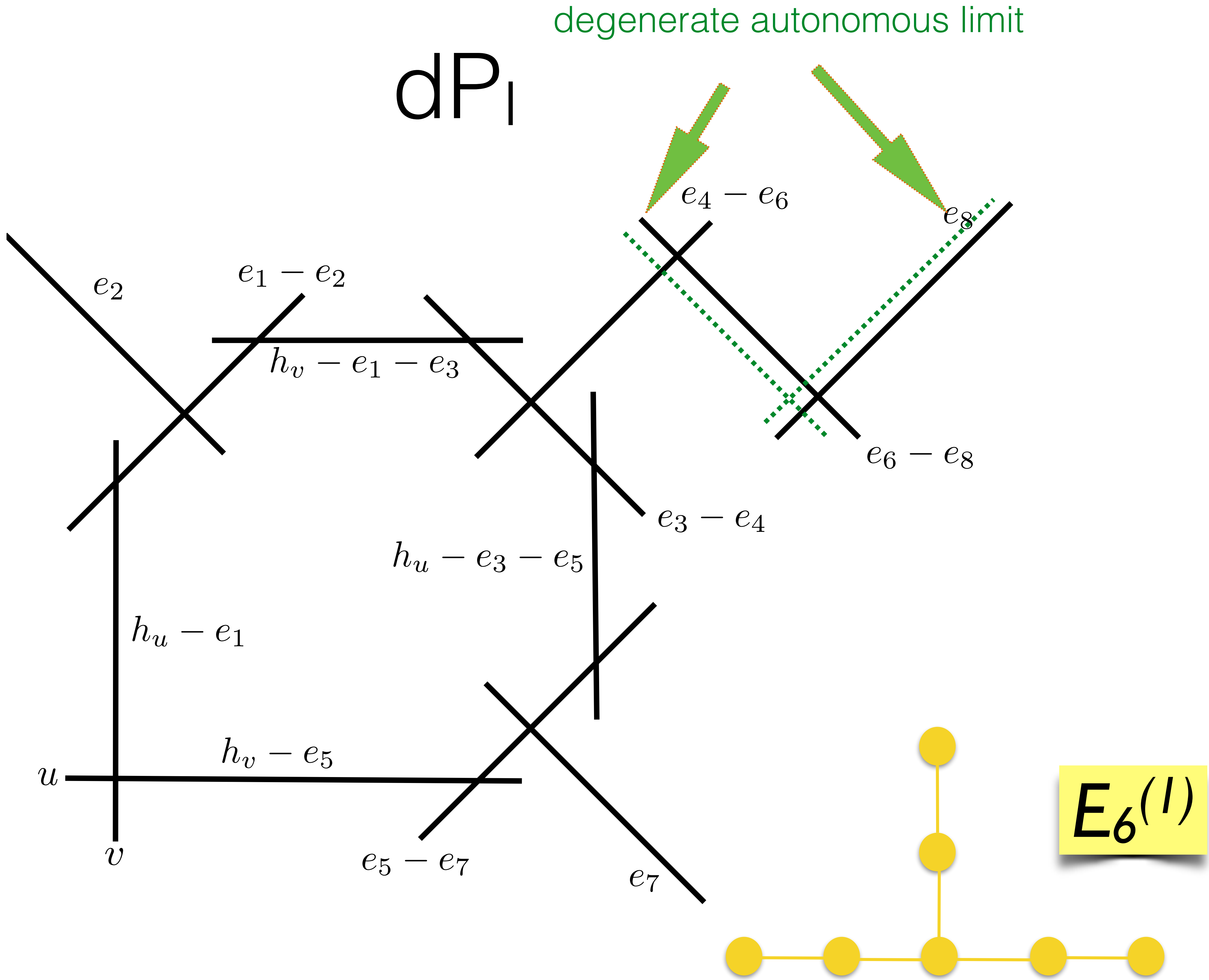
dP$_I$



$\mathrm{dP_I}$

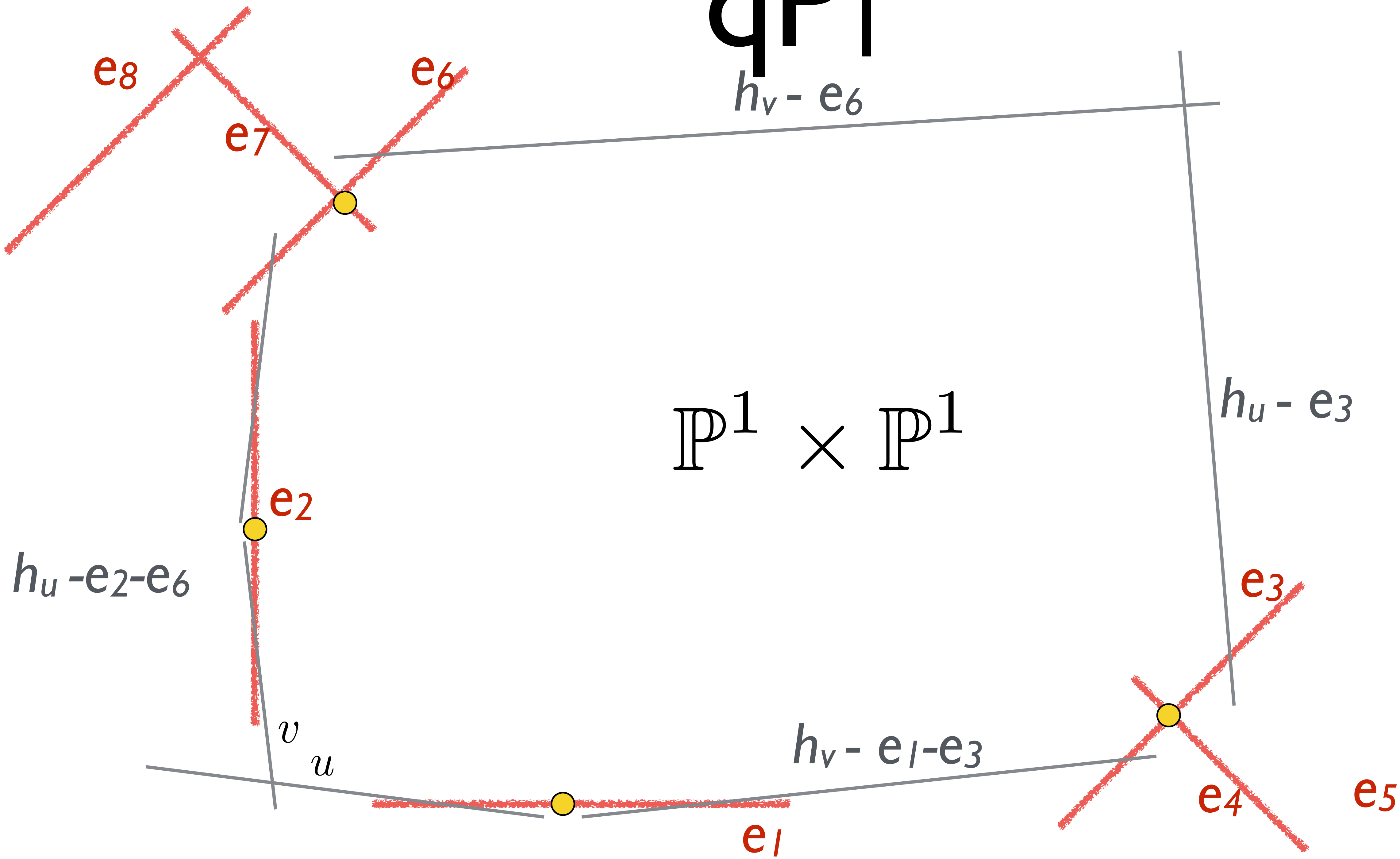

$e_4 - e_6$
e_8
e_2
$e_1 - e_2$
$h_v - e_1 - e_3$
$e_6 - e_8$
$e_3 - e_4$
$h_u - e_3 - e_5$
$h_u - e_1$
$h_v - e_5$
u
v
$e_5 - e_7$
e_7
$E_6^{(1)}$

dP_I

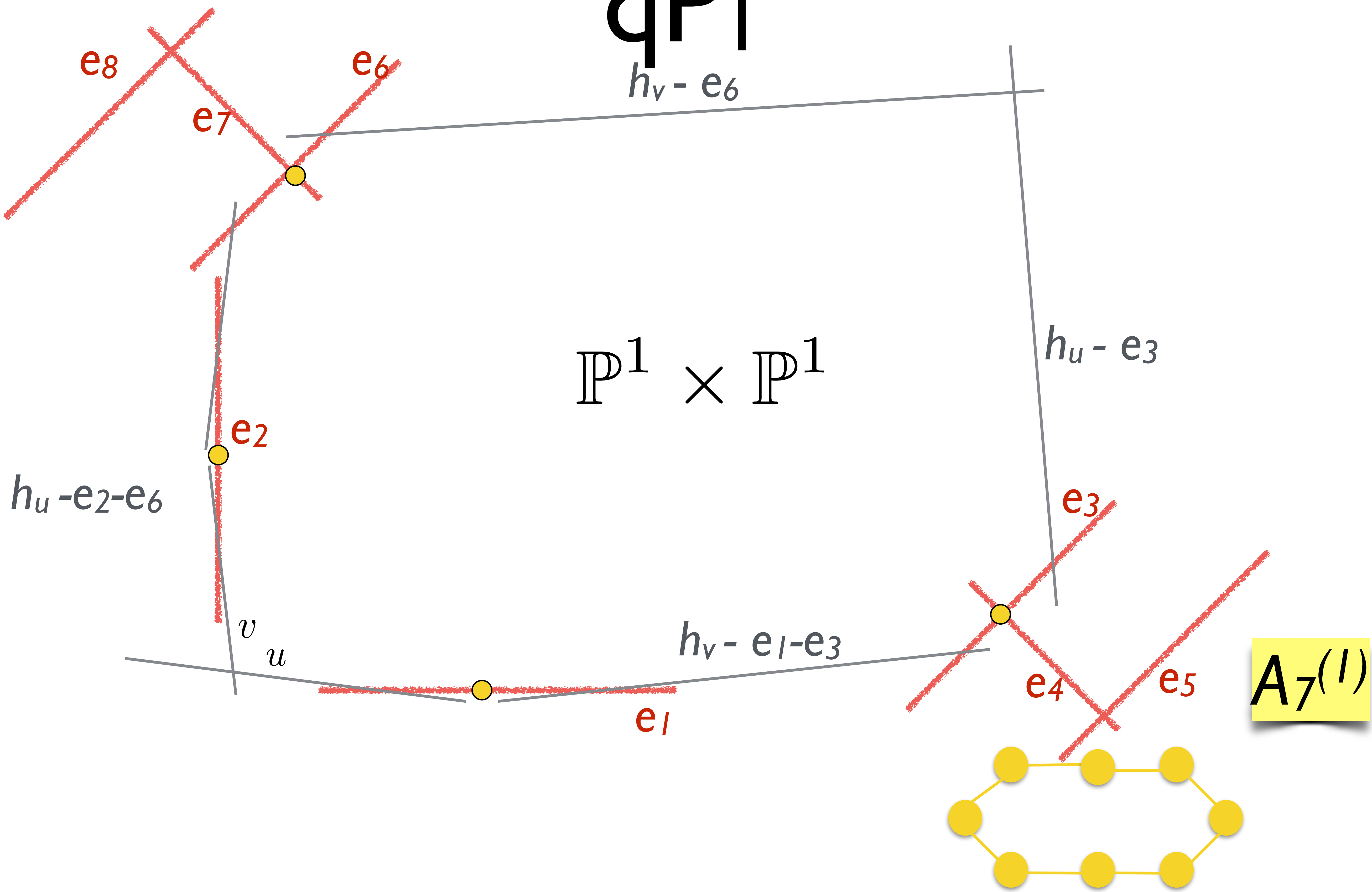


degenerate autonomous limit
$e_4 - e_6$
e_8
e_2
$e_1 - e_2$
$h_v - e_1 - e_3$
$e_6 - e_8$
$e_3 - e_4$
$h_u - e_3 - e_5$
$h_u - e_1$
$h_v - e_5$
u
v
$e_5 - e_7$
e_7
$E_6^{(1)}$

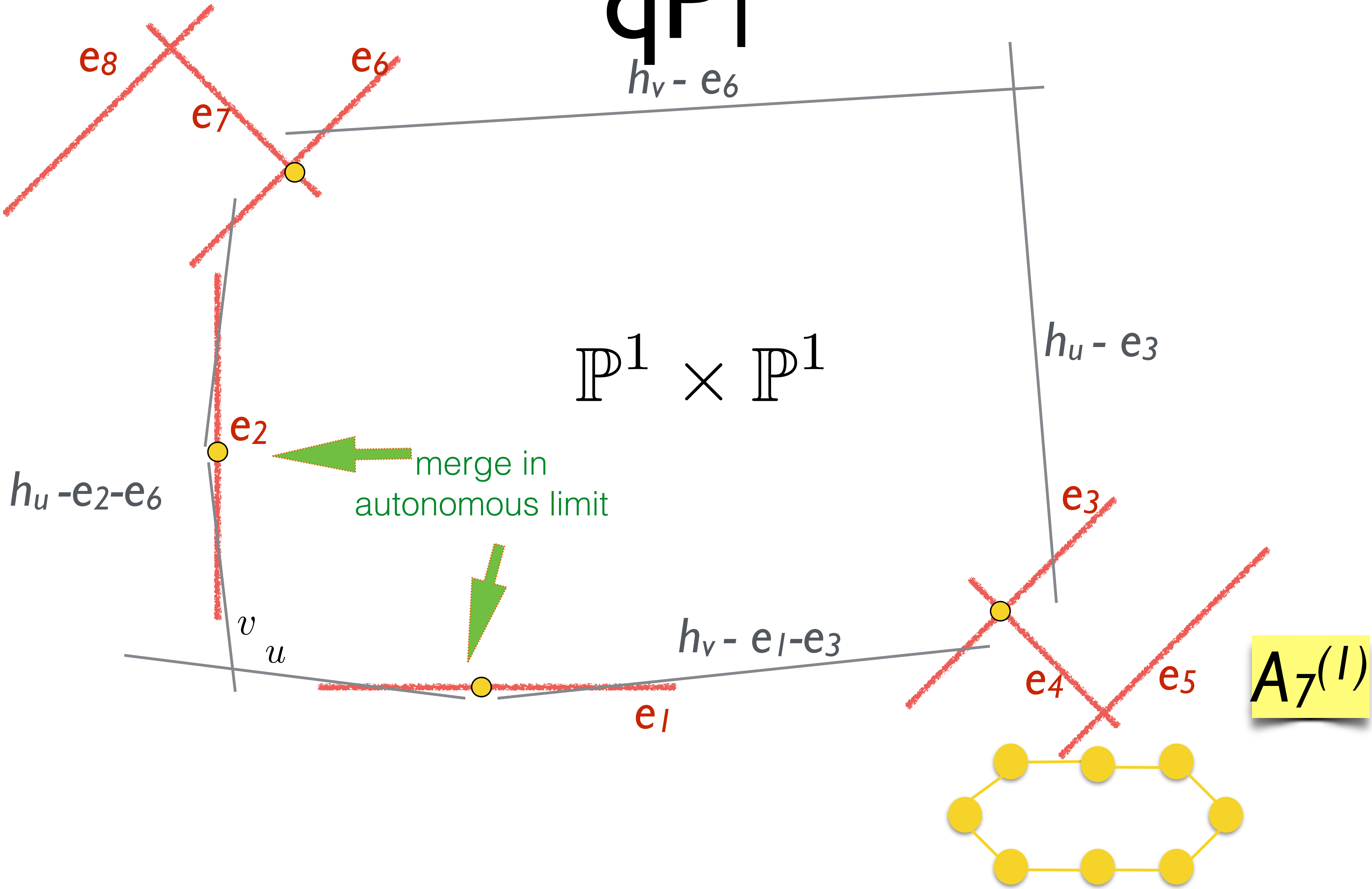
qP_1


e_8
e_6
e_7
$h_v - e_6$
$h_u - e_3$
$\mathbb{P}^1 \times \mathbb{P}^1$
e_2
$h_u - e_2 - e_6$
e_3
v
u
$h_v - e_1 - e_3$
e_4
e_5
e_1

qP_I



qP_{I}



e8
e6
e7
hv - e6
hu - e3
$\mathbb{P}^1 \times \mathbb{P}^1$
e2
merge in
autonomous limit
hu -e2-e6
e3
v
u
hv - e1-e3
e4
e5
e1
$A_7^{(1)}$

Symmetric dP1

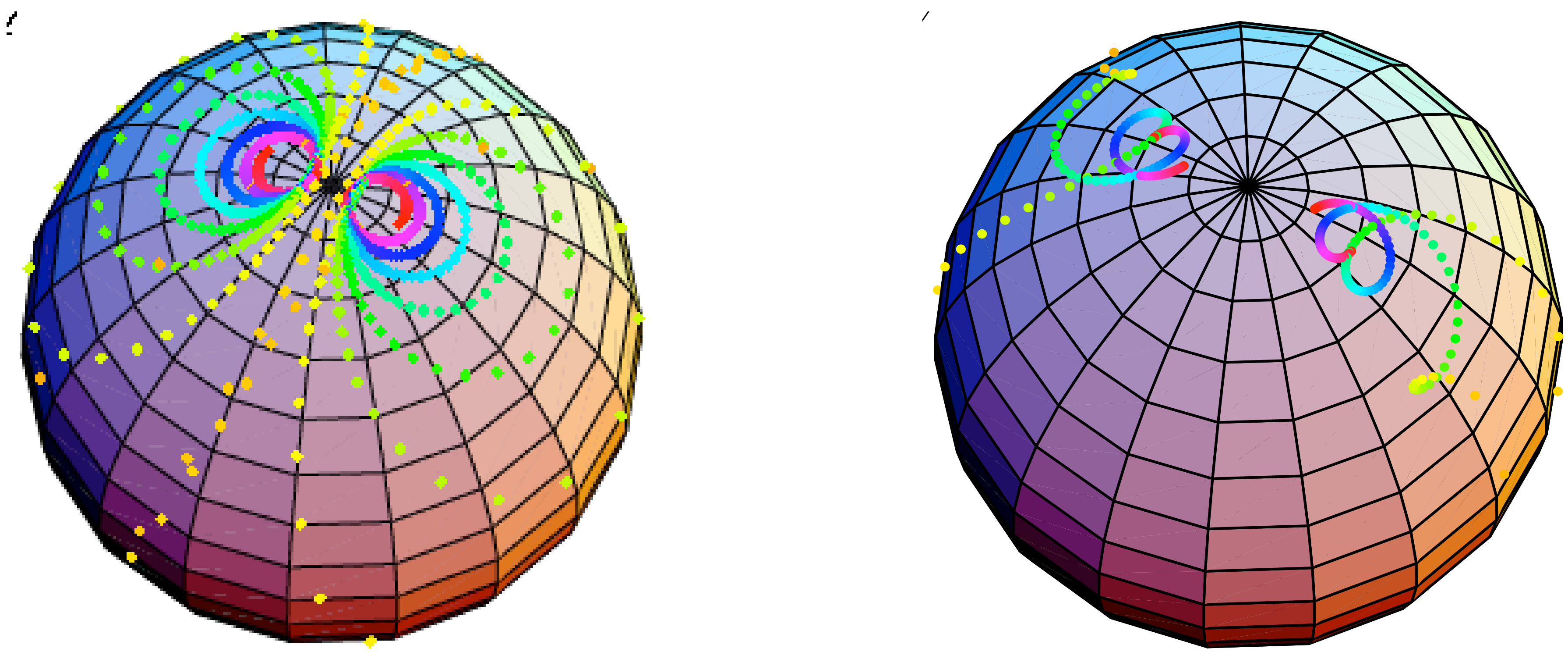
$$w_{n+1} + w_n + w_{n-1} = \frac{\alpha\, n + \beta}{w_n} + \gamma$$

- Consider $n \to \infty$
- General behaviours are close to elliptic functions
- Special solutions are given by power series

Joshi 1997

Vereschagin 1995

Solutions



Solution orbits of scalar dP1 on the Riemann sphere (where the north pole is infinity).

Scaling

$$\begin{cases} w_{2k} & = \frac{u(s)}{\epsilon^{1/2}} \\ w_{2k-1} & = \frac{v(s)}{\epsilon^{1/2}} \end{cases} \qquad s = \epsilon\, n$$

- dPI becomes

$$\big(v(s+\epsilon) + u(s) + v(s-\epsilon)\big)u(s) = \alpha\, s + \epsilon\, \beta + \epsilon^{1/2}\, \gamma\, u(s)$$

$$\big(u(s+\epsilon) + v(s) + u(s-\epsilon)\big)v(s) = \alpha\, s + \epsilon\, \beta + \epsilon^{1/2}\, \gamma\, u(s)$$

- Series expansions as $\epsilon \to 0$

$$u(s) \sim \sum_{m=0}^{\infty} \epsilon^{m/2}\, u_m(s)$$

$$v(s) \sim \sum_{m=0}^{\infty} \epsilon^{m/2}\, v_m(s)$$

Types of solutions

- Type A

$$u \sim \pm\sqrt{-\alpha s} + \frac{\gamma\epsilon^{1/2}}{2} \mp \frac{(4\beta - \gamma^2)\epsilon}{8\sqrt{-\alpha s}} + \ldots$$

$$v \sim \mp\sqrt{-\alpha s} + \frac{\gamma\epsilon^{1/2}}{2} \pm \frac{(4\beta - \gamma^2)\epsilon}{8\sqrt{-\alpha s}} + \ldots$$

- Type B

$$u = v \sim \pm\sqrt{\frac{\alpha s}{3}} + \frac{\gamma\epsilon^{1/2}}{6} \mp \pm \frac{\sqrt{3}(12\beta + \gamma^2)\epsilon}{72\sqrt{\alpha s}} + \ldots$$

Late-order terms: Type A

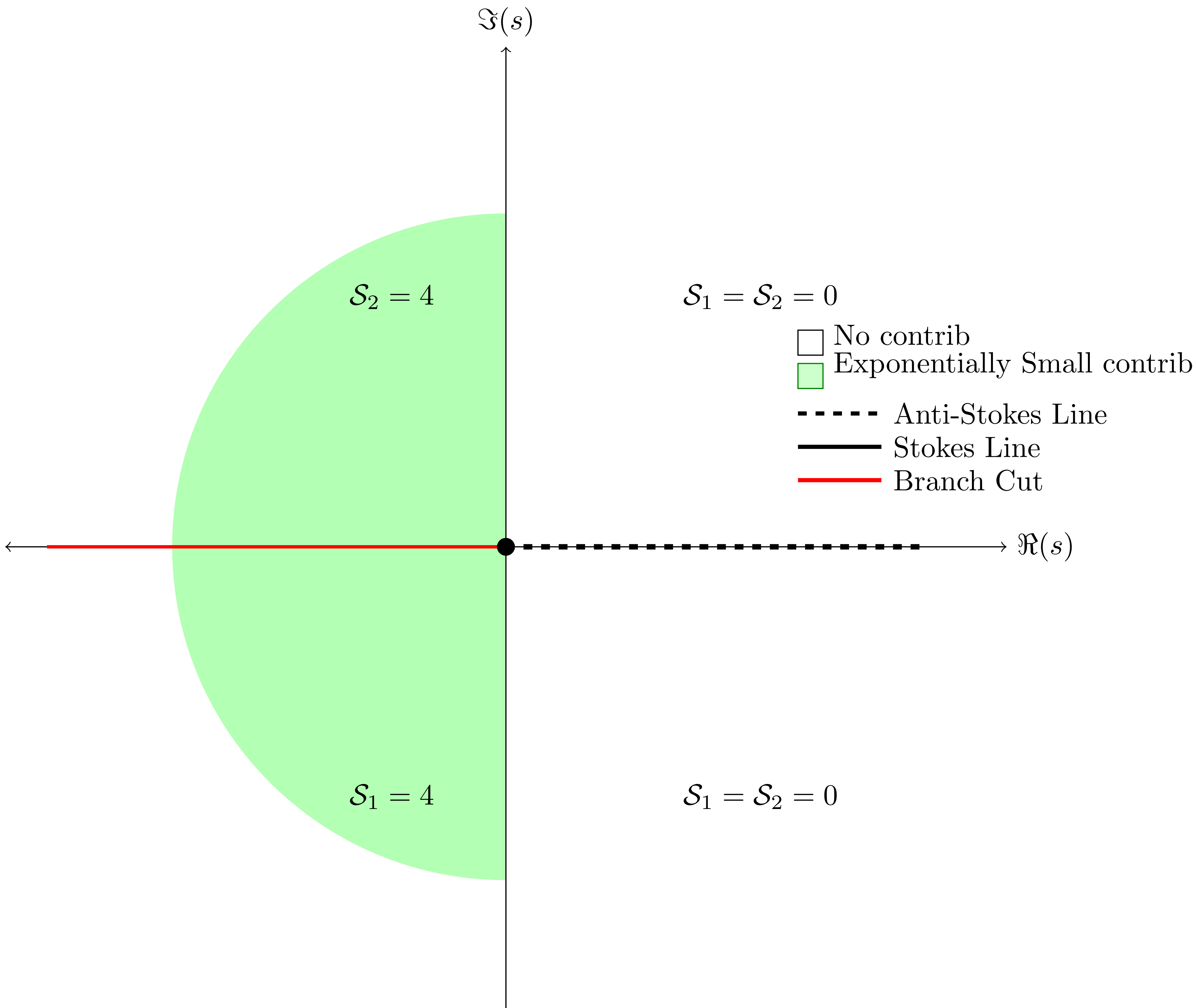
$$u_m \sim \frac{\Lambda_1\,\Gamma(\frac{m-1}{2})}{(i\pi\, s/2)^{\frac{m-1}{2}}} + \frac{\Lambda_2\,\Gamma(\frac{m-1}{2})}{(-i\pi\, s/2)^{\frac{m-1}{2}}}$$

$$v_m \sim \frac{\Lambda_3\,\Gamma(\frac{m-1}{2})}{(i\pi\, s/2)^{\frac{m-1}{2}}} + \frac{\Lambda_4\,\Gamma(\frac{m-1}{2})}{(-i\pi\, s/2)^{\frac{m-1}{2}}}$$

- Optimal truncation

$$u(s) \sim \sum_{m=0}^{N_o} \epsilon^{m/2} u_m(s) + S_1\,\Lambda_1 (-i)^{s/\epsilon} + S_2\,\Lambda_2\, i^{s/\epsilon}$$

Stokes Sectors: Type A



What about q-discrete Painlevé equations?

qP1

$$\Rightarrow \quad \overline{w}\,\underline{w} = \frac{1}{w} - \frac{1}{\xi\, w^2} \qquad (\mathrm{qP_I})$$

$$\overline{w} = w(q\,\xi), w = w(\xi), \underline{w} = w(\xi/q)$$

- A limiting form of qP3, rescaled

$$\overline{g}\,\underline{g} = \frac{\alpha\, x}{g} + \frac{\beta}{g^2}$$

Ramani & Grammaticos (1996)

$$\overline{g} = g(\tilde{q}\, x), \underline{g} = g(x/\tilde{q})$$

$\mapsto$ **PI:** $y'' = 6\, y^2 - t$ in continuum limit.

Singular Dynamics

- Near e_1 where $v_{11} << 1$

$$\begin{cases} \overline{u}_{11} & \sim & \xi(q\xi^2 u_{11} - 1), \\ \overline{v}_{11} & \sim & \dfrac{1}{\xi}, \end{cases}$$

- The flow is tangential & fast

$$\begin{cases} u_{11}(\xi) & \sim & C_1 \left(q\xi_k^3\right)^{n-1} . \\ v_{11}(\xi) & \sim & \dfrac{1}{\xi_k}, \end{cases}$$

- Result: union of e_j is a repeller.

Behaviours near fixed points

$$\overline{w} \sim w, \quad \underline{w} \sim w, \quad |\xi| \to \infty$$

$$\Rightarrow \quad w^4 = w + \mathcal{O}(1/\xi)$$

$$\Rightarrow \quad w = \begin{cases} \omega + \mathcal{O}(1/\xi) \\ \mathcal{O}(1/\xi) \end{cases} \qquad \omega^3 = 1$$

- qP_I is invariant under rotation by argument 2π/3, so ω can be replaced by unity.
- The second case lies in neighbourhood of a merger of two base points: *(1/ξ ,0), (q/ξ , 0)*.

Near unity

- Near $w = 1,\ \underline{w} = 1,\ \exists$ a formal series solution

$$w = \sum_{n=0}^{\infty} a_n\, t^n, \quad a_0 = 1$$

where

$$\begin{aligned} a_n \big(q^n + 1 + q^{-n}\big) &\\ = -\sum_{l=1}^{n-1} a_l\, a_{n-l}\, (q^{(2l-n)} + 1)&\\ - \sum_{m=1}^{n-1} \sum_{j=0}^{n-m} \sum_{l=0}^{m} a_j\, a_{n-m-j}\, a_l\, a_{m-l}\, q^{(n-m-2\,j)}& \end{aligned}$$

Near zero

- Near $w = 1/\xi,\ \underline{w} = q/\xi,\ \exists$ a formal series solution

$$w(\xi) = \sum_{n=1}^{\infty} \frac{b_n}{\xi^n}$$

where

$$b_1 = 1,\ b_2 = 0,\ b_3 = 0$$

$$b_n = \sum_{r=2}^{n-2} \sum_{k=1}^{r-1} \sum_{m=1}^{n-r-1} b_k\, b_{r-k}\, b_m\, b_{n-r-m} q^{(r-2\,k)},\ n \geq 4$$

Step 1: divergence

The coefficients of the asymptotic series grow very fast:

$$b_{3\,p+1} \underset{p\to\infty}{=} \mathcal{O}\left(|q|^{3\,p\,(p-1)/2}\prod_{k=0}^{p-1}\left(1+q^{-3k}\right)^2\right), |q|>1$$

$$b_{3\,p+2}=0,\ b_{3\,p+3}=0,\ \forall p\geq 0$$

Step 2: Analytic Sum

- Use of the Borel-Ritt theorem provides an analytic function W s.t.

$$W(\xi) \sim \sum_{n=1}^{\infty} \frac{b_n}{\xi^n}$$

Step 3: Linearisation

- The linearisation around W satisfies

$$\overline{P} + \left(2\,\frac{\overline{W}}{W} - \frac{1}{W^2\,\underline{W}}\right) P + \frac{\overline{W}}{\underline{W}}\,\underline{P} = 0$$

which has solutions with behaviours

$$P^{\pm}(\xi) \sim q^{\pm 3\,n(n\,\mp\,5/3)/2}$$

$$\xi = \xi_0\,q^n$$

Step 4: True Solutions

- The perturbed q-difference equation gives

$$v_n = \beta_0 \, P_n - \; P_n \sum_{j=n}^{n_0-1} \frac{W_j \, W_{j-1}}{P_j \, P_{j-1}} \sum_{k=k_0}^{j-1} \frac{P_k \, \mathcal{R}_2(v_k, v_{k-1}, t_k)}{W_{k+1} \, W_k}$$

where
$$\|\mathcal{R}_2(v, \underline{v}, t)\| \leq C_1 \, \|\mathbf{v}\|^2 + C_2|t|$$
$$\|\nabla\mathcal{R}_2\| \leq C_3 \, \|\mathbf{v}\| + C_4 \, |t|$$

- The contraction mapping theorem provides a true solution.

Quicksilver solution

- The vanishing solution approaches two base points.
- Its series expansion is divergent.
- We prove a true solution exists with this behaviour; it does not lie on a singularity of the underlying elliptic curve. So it is different to the *tritronquée* solutions of the Painlevé equations.

  $\Rightarrow$ new name: *quicksilver* solution

- It is unstable in initial-value space.

Joshi, Stud Appl Math (2014)

Comparison

PI

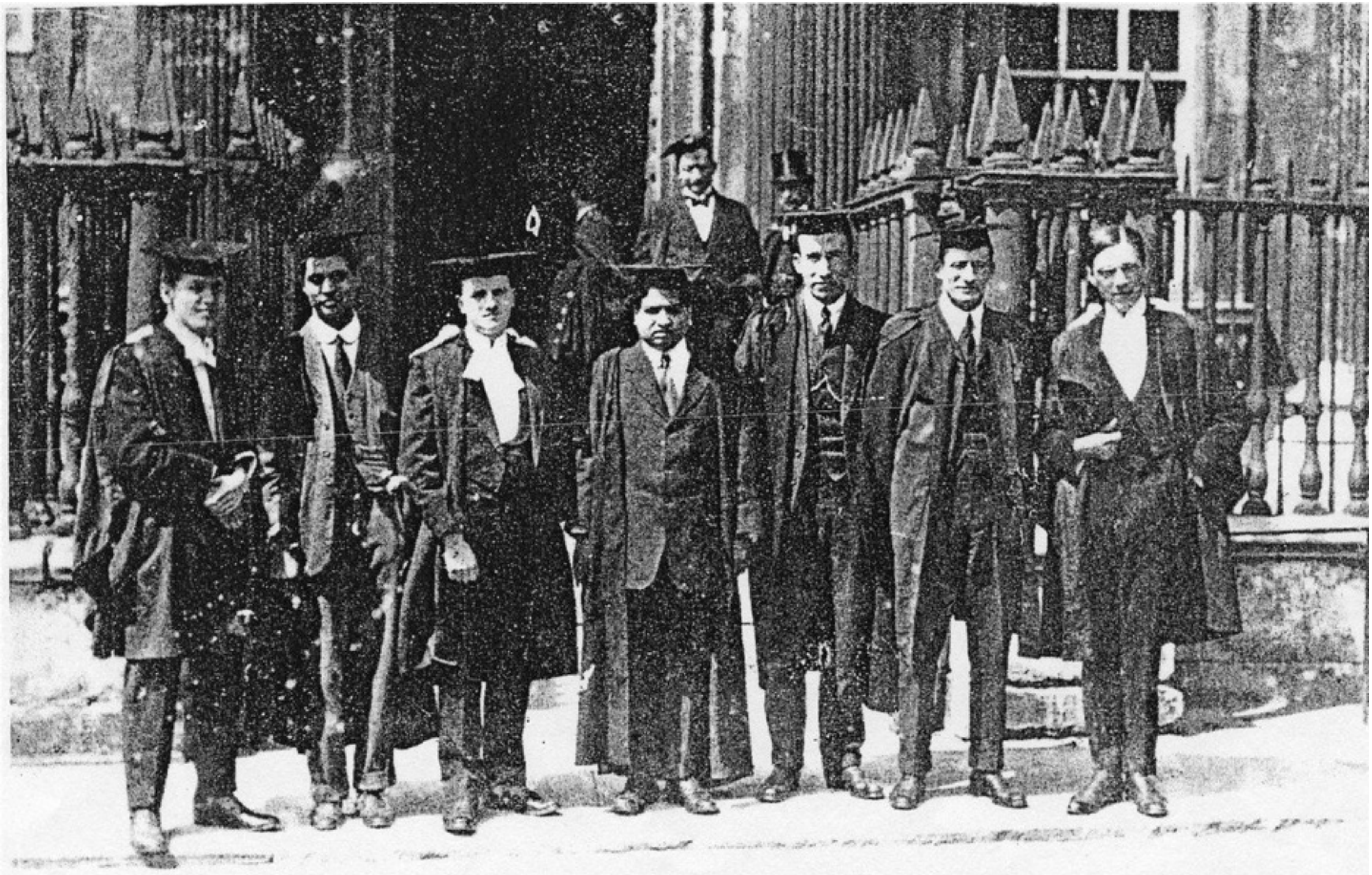
- No rational or classical solutions
- Leading-order behaviour is elliptic
- Two types of solutions described by asymptotic behaviours
- Tronquée solutions are asymptotic to a power series in a large sector

qPI

- No algebraic or solutions in terms of linear eqns *Nishioka (2010)*
- Leading-order behaviour is elliptic
- Four types of solutions described by asymptotic behaviours *J (2014)*
- Quasi-stationary solutions are asymptotic to a power series in a large region *J (2014)*

Summary

- New mathematical models of physics pose new questions for applied mathematics
- Global dynamics of solutions of non-linear equations, whether they are differential or discrete, can be found through geometry.
- Geometry provides the only analytic approach available in $\mathbb{C}$ for discrete equations.
- Tantalising questions about finite properties of solutions remain open.



The mathematician's pattern's, like those of the painter's or the poet's, must be beautiful, the ideas, like the colours or the words, must fit together in a harmonious way. *GH Hardy, A Mathematician’s Apology, 1940*