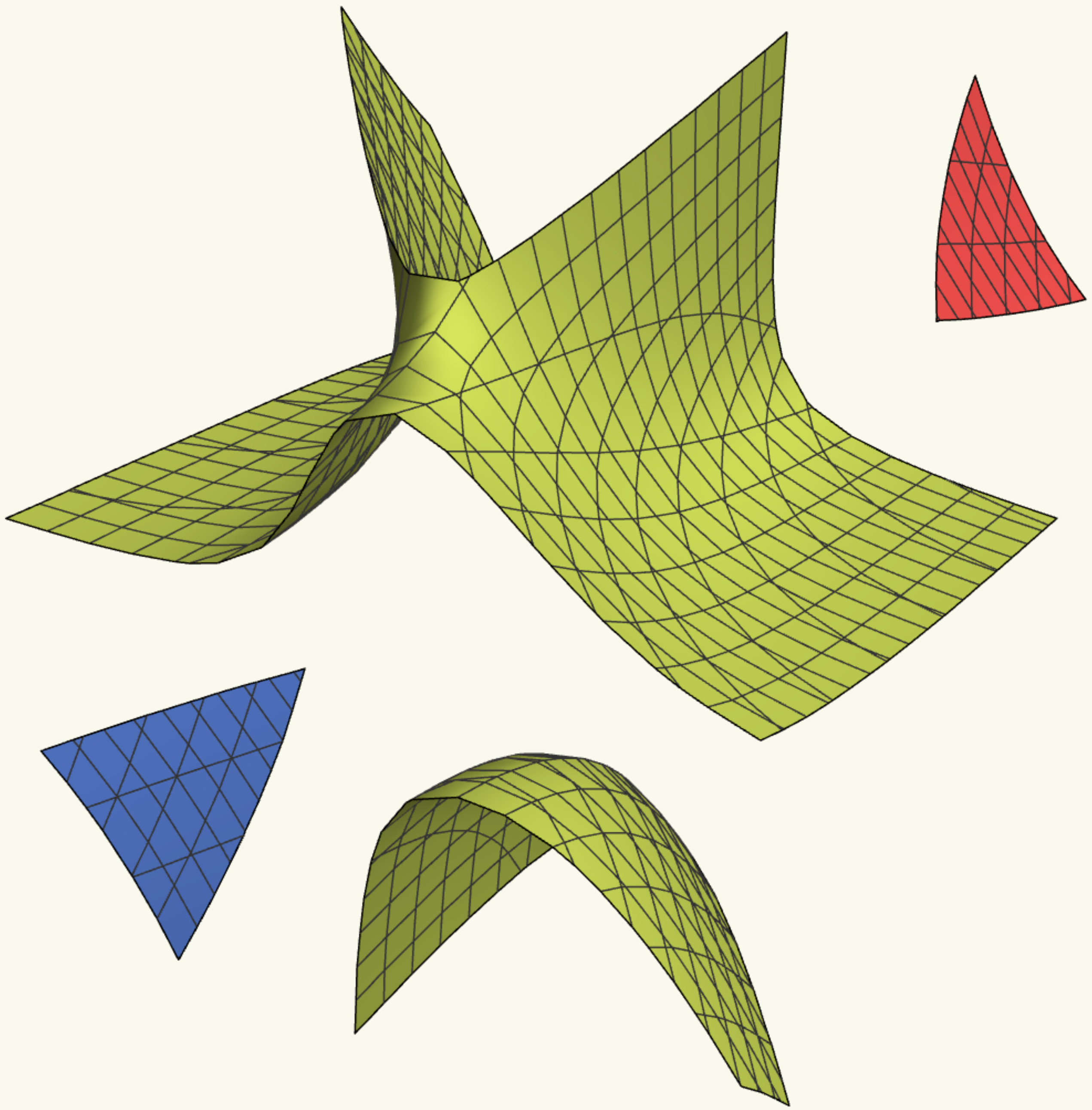
Segre surfaces and the Painlevé equations



Nalini Joshi

with Marta Mazzocco & Pieter Roffelsen arXiv:2405.10541

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🎉 Happy Birthday Roderick! 🎂🥂

生日快樂

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Hardy and Ramanujan

“I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. ‘No,’ he replied, ‘it is a very interesting number…’”

1729

Hardy, 1921

Hardy and Ramanujan

"I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.'"

$$1729$$

$$12^3 + 1^3 = 10^3 + 9^3$$

Hardy, 1921

Hardy and Ramanujan

"I remember once going to see him when he was ill at Putney. I had ridden in taxi cab number 1729 and remarked that the number seemed to me rather a dull one, and that I hoped it was not an unfavourable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways.'"

Hardy, 1921

$$x^3 + y^3 = z^3 + w^3$$

Fermat surface

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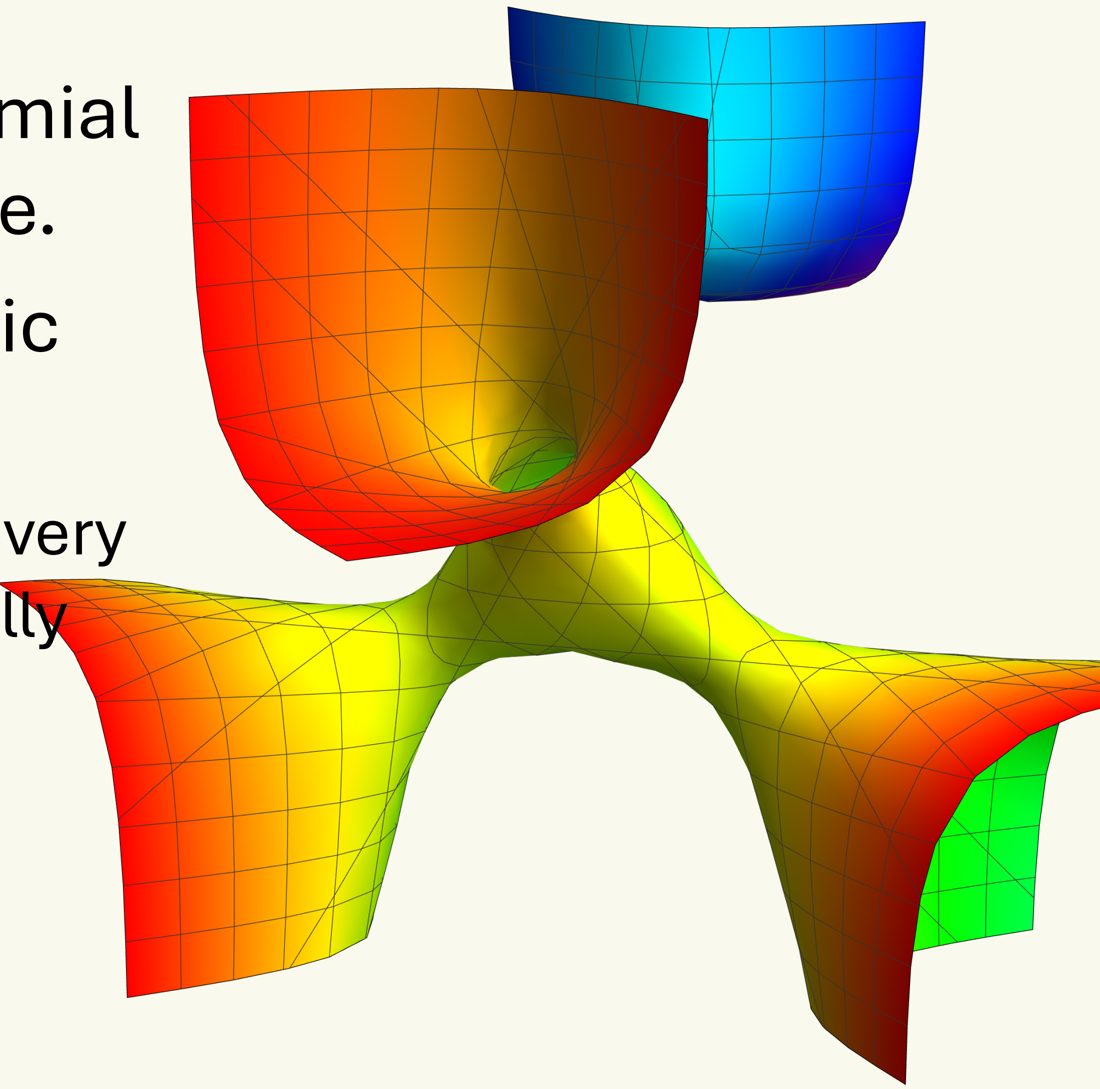
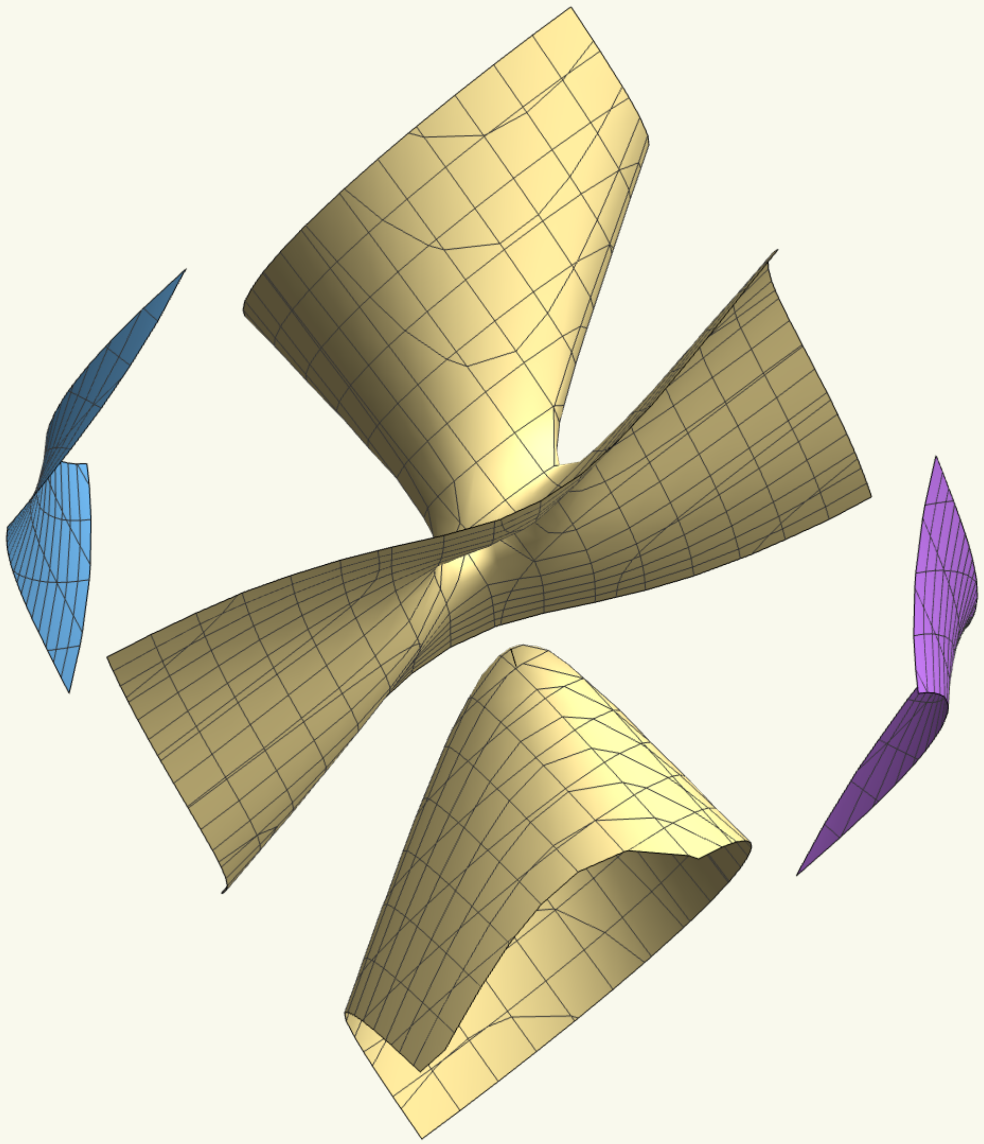
Cubic surfaces

- A cubic surface is defined by a polynomial of degree 3 in three-dimensional space.
- They are celebrated objects in algebraic geometry:
  - Cayley and Salmon (1849) showed that every smooth cubic surface over an algebraically closed field contains 27 lines.
  - Clebsch (1866) showed that every such surface is the blow-up of 6 points in $\mathbb{P}^2$.
  - The moduli space of (projective) cubic surfaces is 4 dimensional.



Segre surface



- A Segre surface: intersection of 2 quadrics (quadratic polynomials) in 4-dim space.
- Every smooth Segre surface over an algebraically closed field contains 16 lines.
- Every such surface is the blow-up of 5 points in $\mathbb{P}^2$.

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An example

Consider

$$
\begin{aligned}
&u + v + w + x + y + z = 0 \\
&a_1 v + a_2 w + x + a_3 y + a_4 z = 1 \\
&xw - b_1 u v = 0 \\
&yz - b_2 u v = 0
\end{aligned}
$$

- Elimination of two variables $\rightarrow$ two quadric equations in 4D.
- Smooth for generic parameters.
- Contains lines such as $v = w = z = 0$, $x + y + u = 0$, $x + a_3 y = 0$
- Contains 2 generic quadrics at infinity.

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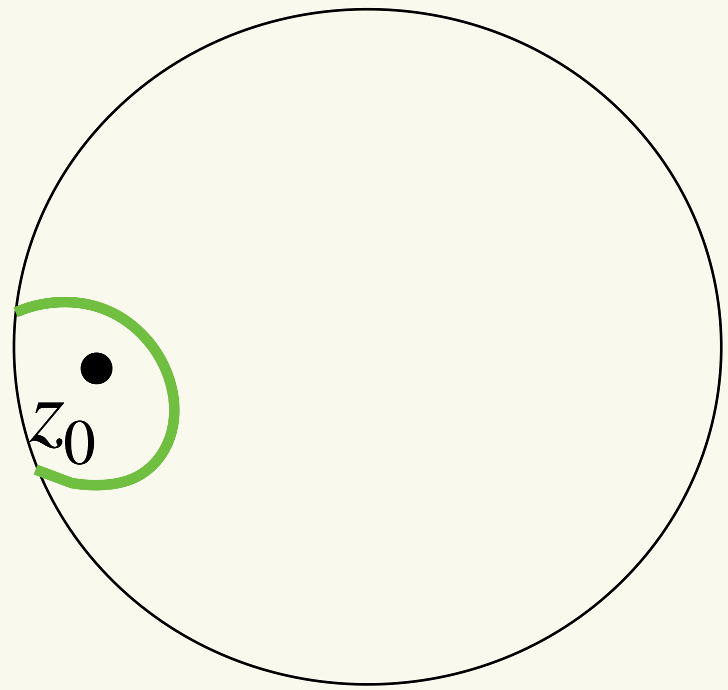
How related to analysis?

Differential equations $\leftrightarrow$ monodromy groups

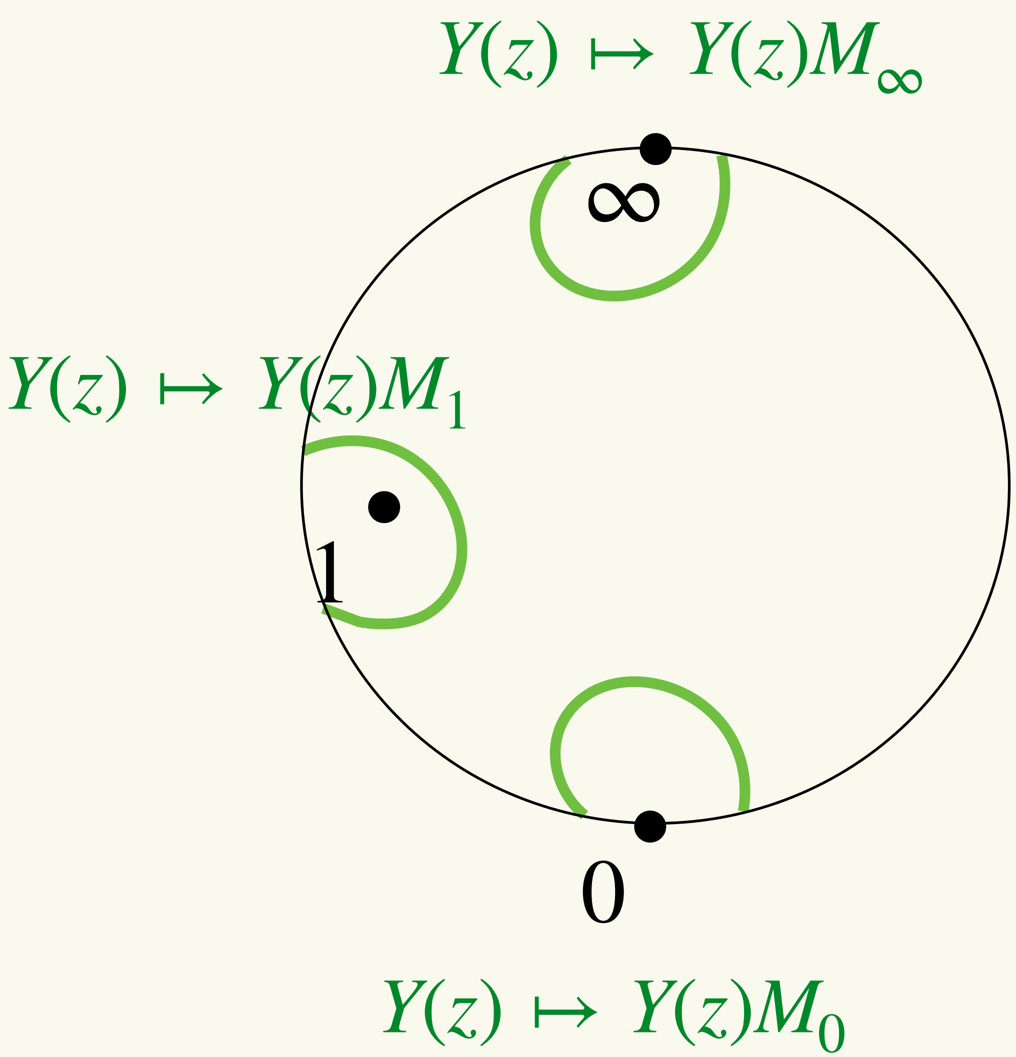
$$\frac{dY}{dz} = A(z)Y \qquad \mapsto$$



$$Y(z) \mapsto Y(z)M_0$$

Relations between solutions leads to equations satisfied by *monodromy data* $\rightarrow$ surfaces called *monodromy manifolds*.

Model problem



The hypergeometric differential equation

$$
\begin{aligned}
z(1-z)w_{zz}& \\
&+\big(c-(a+b+1)z\big)w_z \\
&-a\,b\,w=0
\end{aligned}
$$

has Fuchsian singularities at $0,1,\infty$.

Consider 2×2 solution matrix $Y(z)$.

It changes as z moves on a closed path around each singularity.

M_0, M_1, M_∞ are monodromy matrices.

Only two are independent.

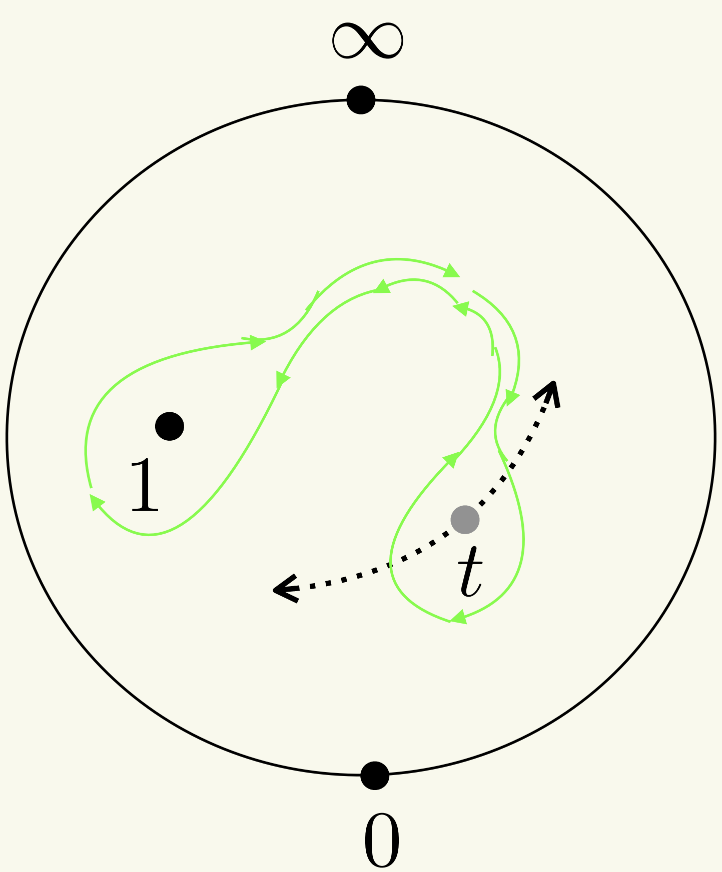
Their trace and determinant are monodromy data.

Richard Fuchs' problem

Add one more singularity at $z = t$

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} \right) Y$$

Find the condition under which *monodromy* data of this system stays invariant under deformation of t .



R. Fuchs *1905*

Isomonodromy condition

The monodromy data stays invariant as *t* varies only under certain conditions on entries of *A* $\Rightarrow$

$$
\begin{aligned}
w'' = {} & \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-t}\right){w'}^2 \\
& - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{w-t}\right)w' \\
& + \frac{w(w-1)(w-t)}{t^2(t-1)^2}\left(\alpha+\frac{\beta t}{w^2}+\frac{\gamma(t-1)}{(w-1)^2}+\frac{\delta t(t-1)}{(w-t)^2}\right)
\end{aligned}
$$

This is the sixth Painlevé equation P_{VI}.

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But should really be called Richard Fuchs' equation.

Monodromy surface

The monodromy group is generated by $M_i \in \mathrm{SL}_2(\mathbb{C})$ with monodromy data

$$x = \mathrm{Tr}(M_2M_3), y = \mathrm{Tr}(M_1M_3), z = \mathrm{Tr}(M_1M_2)$$

satisfying

$$xyz + x^2 + y^2 + z^2 + b_1x + b_2y + b_3z + c = 0$$

where

$$\begin{aligned} m_1 &= \mathrm{Tr}(M_1), \\ m_2 &= \mathrm{Tr}(M_2), \\ m_3 &= \mathrm{Tr}(M_3), \\ m_4 &= \mathrm{Tr}(M_1M_2M_3), \end{aligned}$$

$$\begin{aligned} b_1 &= -(m_1m_4 + m_2m_3), \\ b_2 &= -(m_2m_4 + m_1m_3), \\ b_3 &= -(m_3m_4 + m_1m_4), \\ c &= m_1m_2m_3m_4 - 4 + m_1^2 + m_2^2 + m_3^2 + m_4^2 \end{aligned}$$

known as Fricke's relation.

Jimbo-Fricke surfaces

$$xyz + x^2 + y^2 + z^2 + b_1x + b_2y + b_3z + c = 0$$

Fricke and Klein (1889)
Jimbo (1982)

- Symmetric: $b_1 = b_2 = b_3$
- Markov cubic surface: $b_i = c = 0$
- Cayley's nodal cubic surface: $b_i = 0, c = -4$

$$\vdots$$

A coalescence limit

- Take

$$t \mapsto 1 + \epsilon\, t$$
$$\delta \mapsto \frac{\delta}{\epsilon^2}$$
$$\gamma \mapsto \frac{\gamma}{\epsilon} - \frac{\delta}{\epsilon^2}$$

$\epsilon \to 0$ →

$$\mathrm{P_V}:\ w_{tt} = \left(\frac{1}{2w} + \frac{1}{2-1}\right)(w_t)^2 - \frac{w_t}{t} + \frac{(w-1)^2}{t^2}\left(\alpha\, w + \frac{\beta}{w}\right) + \frac{\gamma\, w}{t} + \frac{\delta\, w\,(w+1)}{w-1}$$

- The monodromy surface of $\mathrm{P_{VI}}$ becomes

$$xyz - x^2 - y^2 + \omega_1 x + \omega_2 y + \omega_3 z + \omega_4 = 0$$

where ω_i are parameters related to $\alpha, \beta, \gamma, \delta$.

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Coalescence limits lead to cubic monodromy surfaces for all the Painlevé equations.

The Painlevé equations

$$\mathrm{P_I}: \quad w'' = 6w^2 + t$$

$$\mathrm{P_{II}}: \quad w'' = 2w^3 + tw + \alpha$$

$$\mathrm{P_{III}}: \quad w'' = \frac{w'^2}{w} - \frac{w}{t} + \frac{1}{t}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}$$

$$\mathrm{P_{IV}}: \quad w'' = \frac{w'^2}{2w} + \frac{3w^3}{2} + 4tw^2 + 2(t^2 - \alpha)w + \frac{\beta}{w}$$

$$\mathrm{P_V}: \quad w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)w'^2 - \frac{w'}{t} + \frac{(w-1)^2}{t^2 w}(\alpha w^2 + \beta) + \frac{\gamma w}{t} + \frac{\delta w(w+1)}{w-1}$$

$$\mathrm{P_{VI}}: \quad w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t}\right)w'^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t}\right)w' + \frac{w(w-1)(w-t)}{t^2(t-1)^2}\left(\alpha + \frac{\beta t}{w^2} + \frac{\gamma(t-1)}{(w-1)^2} + \frac{\delta t(t-1)}{(w-t)^2}\right)$$

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Cubic surfaces

$$
\begin{aligned}
\mathrm{P_{VI}}: &\quad xyz - x^2 - y^2 - z^2 + \omega_1 x + \omega_2 y + \omega_3 z + \omega_4 = 0 \\
\mathrm{P_{V}}: &\quad xyz - x^2 - y^2 + \omega_1 x + \omega_2 y + \omega_3 z + \omega_4 = 0 \\
\mathrm{P_{IV}}: &\quad xyz - x^2 + \omega_1 x + \omega_2 y + \omega_3 z + \omega_4 = 0 \\
\mathrm{P_{III,D_6}}: &\quad xyz - x^2 - y^2 + \omega_1 x + \omega_2 y + \omega_4 = 0 \\
\mathrm{P_{II,FN}}: &\quad xyz + x + \omega_2 y - z - 1 = 0 \\
\mathrm{P_{I}}: &\quad xyz - x - z + 1 = 0
\end{aligned}
$$

where ω_i are parameters.

Iwasaki, *2002*.

Vanderput & Saito*, 2009*.

Chekhov, Mazzocco & Rubtsov *2021*.

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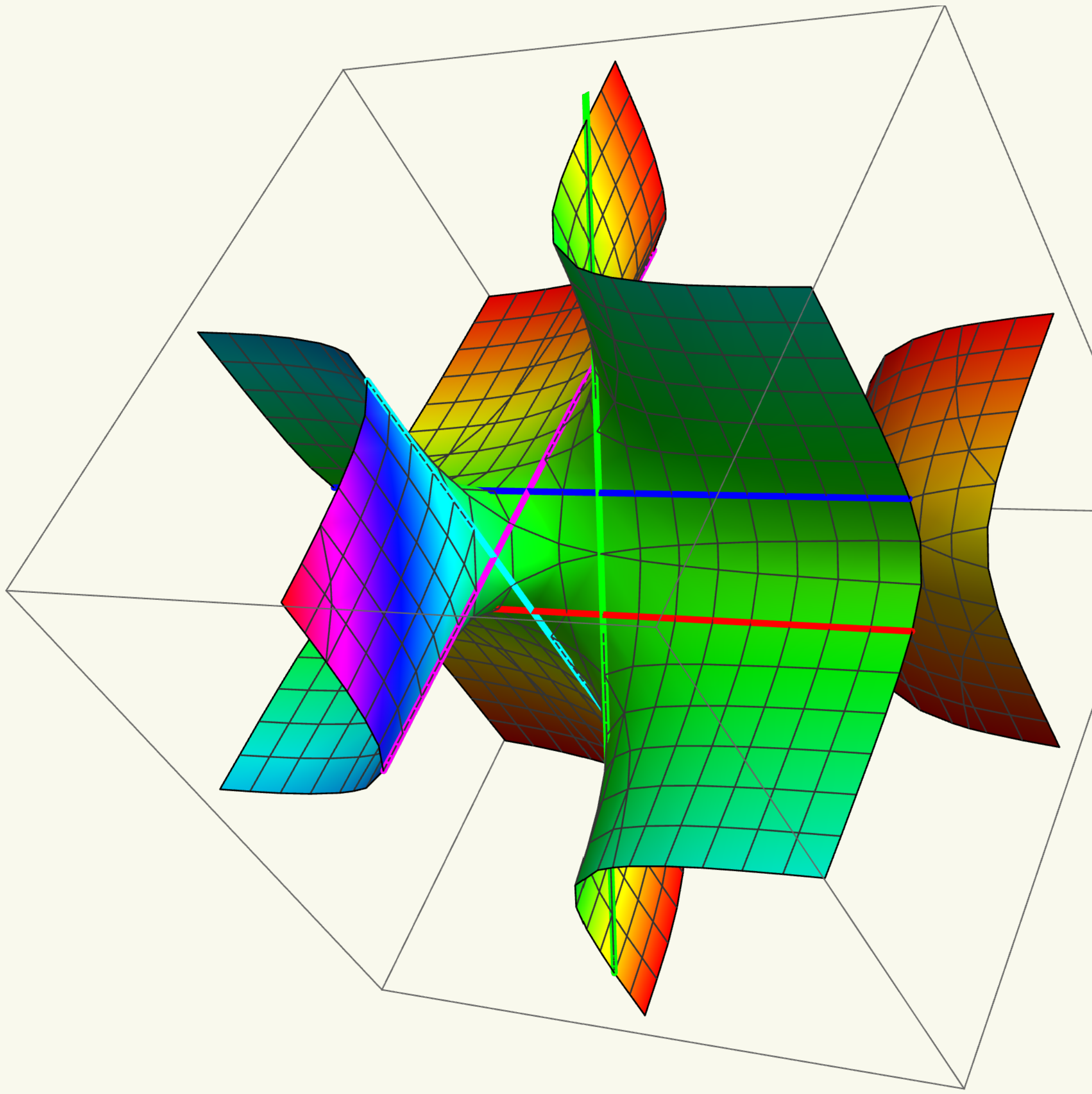
Cubic surfaces and asymptotics

The cubic surface for $\mathrm{P_I}$

$$xyz - x - z + 1 = 0$$

contains 5 (affine) lines.

$$L_1 : \ \{x = 0, z = 1\}$$
$$L_2 : \ \{x = 1, z = 0\}$$
$$L_3 : \ \{y = 0, x + z = 1\}$$
$$L_4 : \ \{y = 1, z = 1\}$$
$$L_5 : \ \{x = 1, y = 1\}$$



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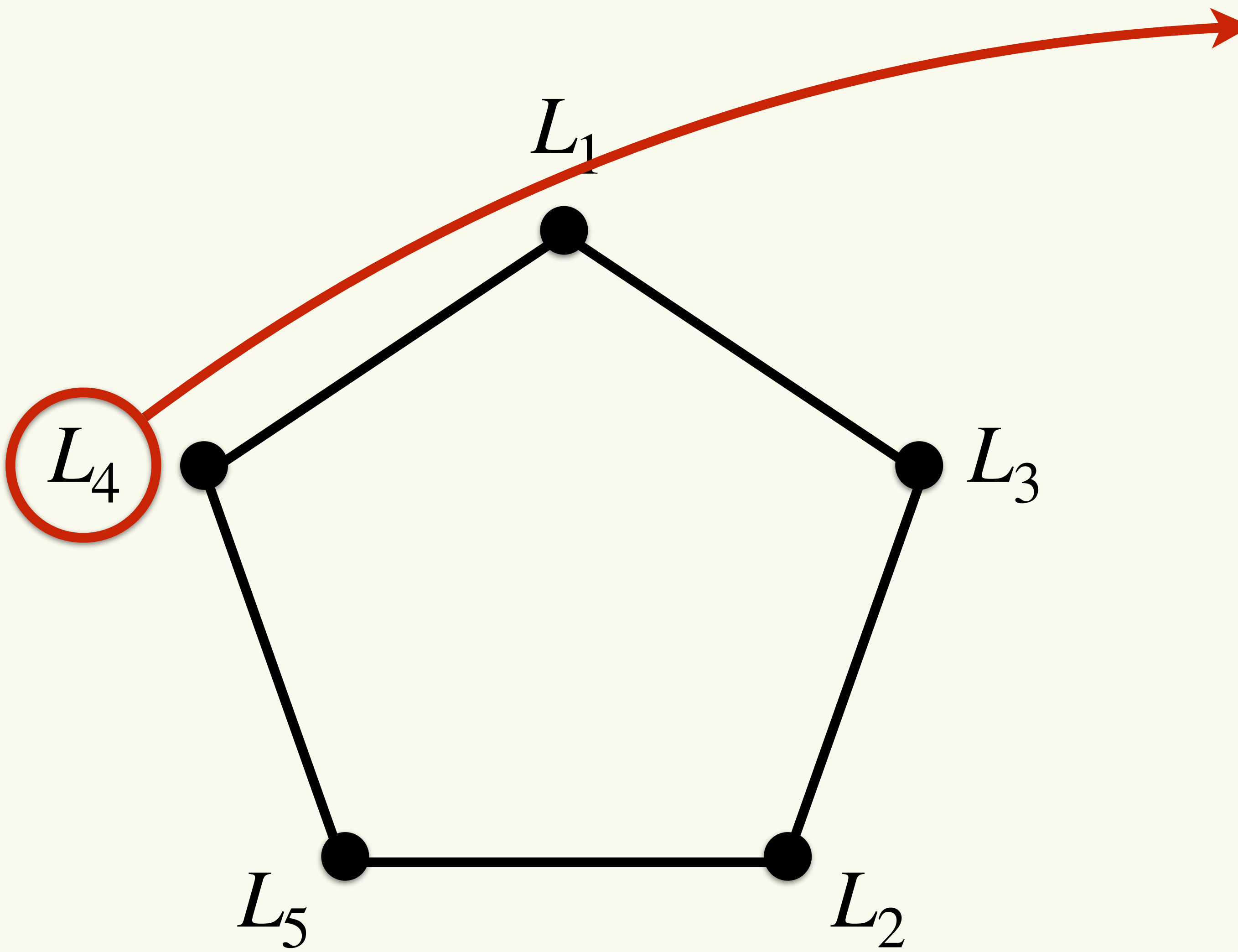
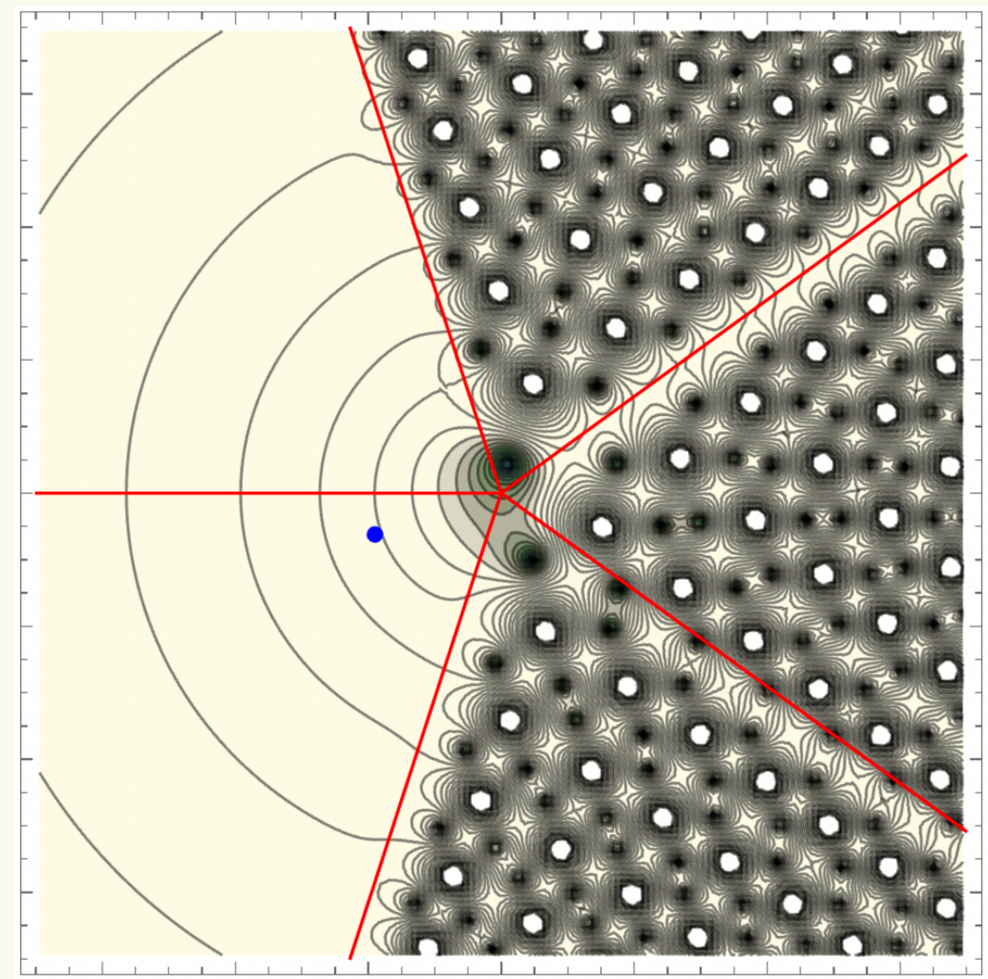
Tronquée solutions



$$w(t) \sim \left(\frac{-t}{6}\right)^{1/2} \sum_{j=0}^{\infty} \frac{a_j}{t^{5j/2}}$$

$$|t| \to \infty, \ \frac{3\pi}{5} < \arg(t) < \frac{7\pi}{5}$$



Poles of a tronqué solution of PI in t-plane from arXiv:2204.09062 Figure 1(b) by Alexander van Spaendonck and Marcel Vonk. (Figure is reflected.)

$$w_{tt} = 6w^2 + t$$

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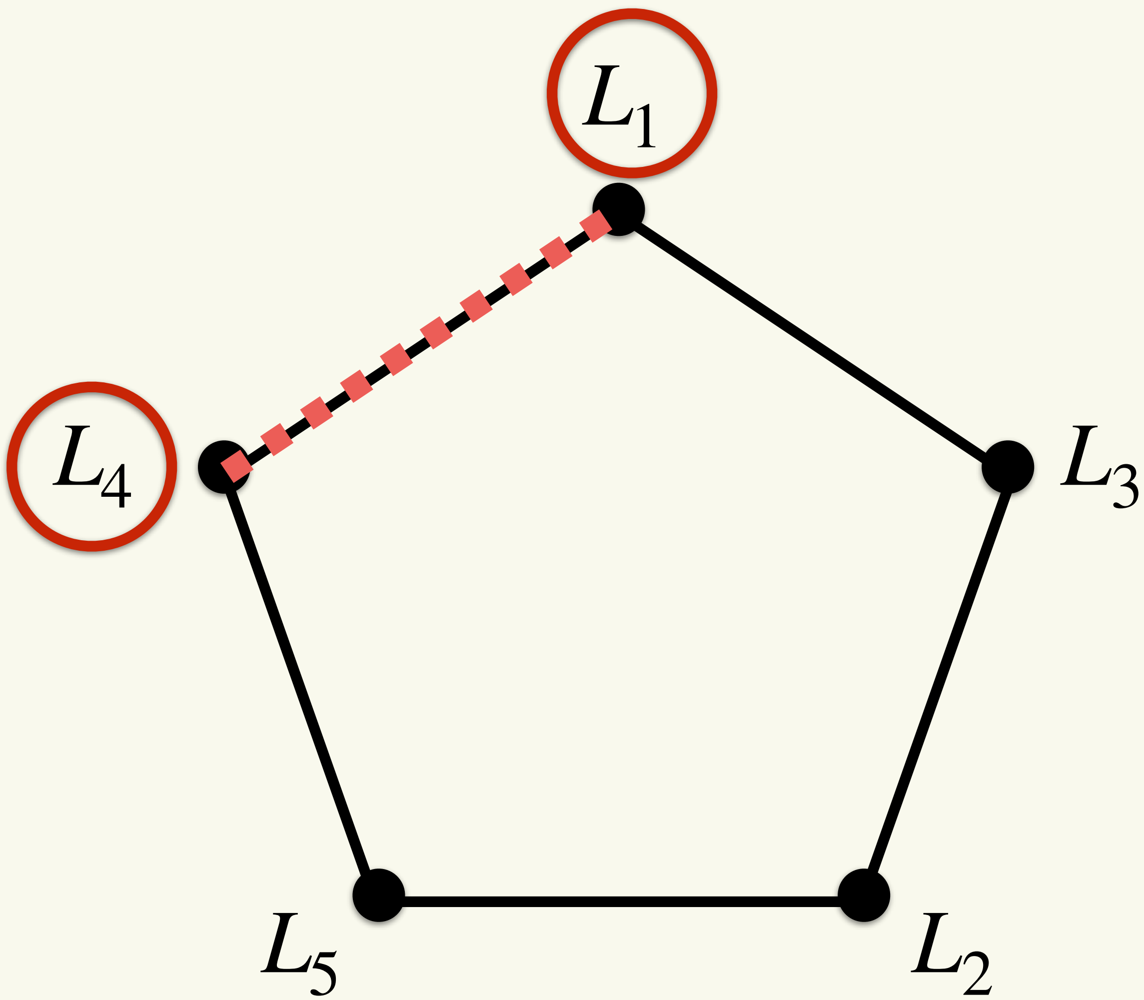
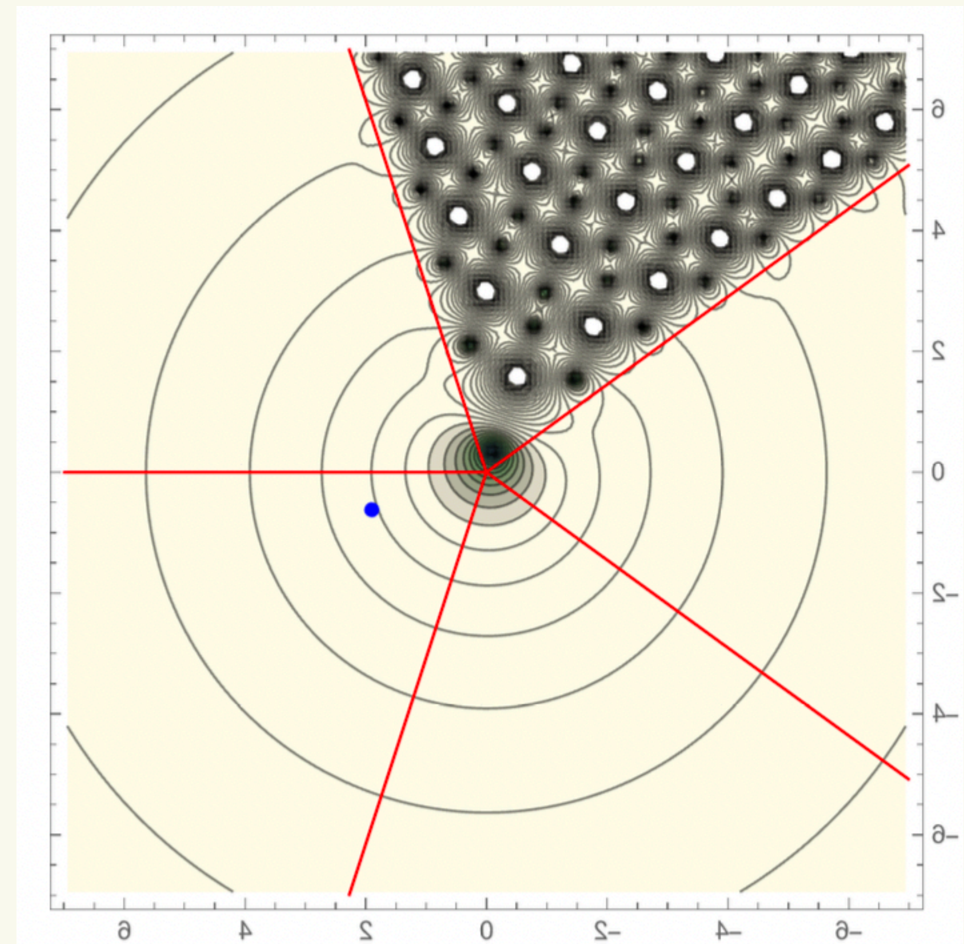
Tritronquée solutions



$$L_1 \cap L_4 : \ \{x = 0, y = 1, z = 1\}$$



Poles of a tritronqué solution of PI in t-plane from arXiv:2204.09062 Figure 1(a) by Alexander van Spaendonck and Marcel Vonk. (Figure is reflected.)

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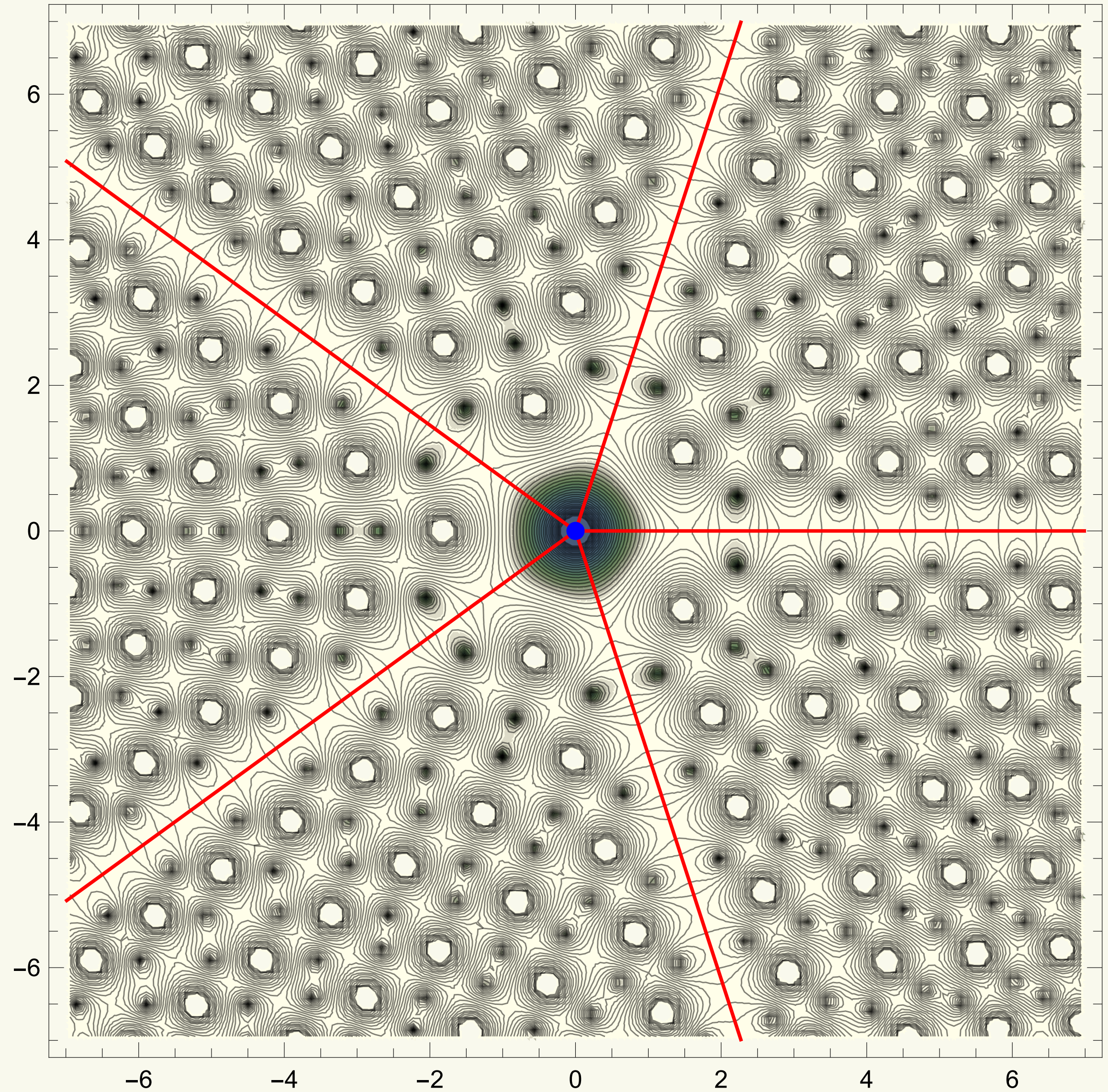
Symmetric solutions



Poles of a symmetric solution of PI with double zero at t=0 using code supplied by Marcel Vonk.

The monodromy surface

$$xyz - x - z + 1 = 0$$

contains points (p, p, p), where

$$p^3 - 2p + 1 = 0$$

$$\Leftrightarrow \quad (p-1)\left(p^2 + p + 1\right) = 0$$

Two of these points corresponds to symmetric solutions of P_I

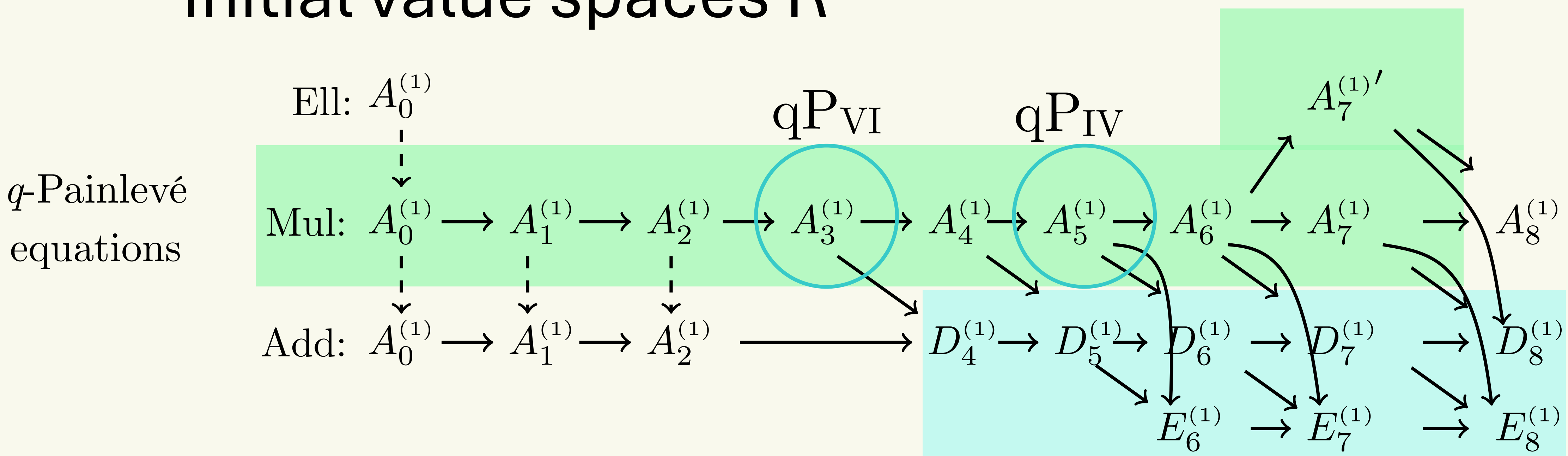
($p = 1$ is *tritronquée*).

Kitaev, 1995

What are the behaviours of solutions of discrete Painlevé equations?

Sakai’s scheme

Initial value spaces R



The Painlevé equations   Okamoto *1979*

Sakai *2001*
Rains *2016*

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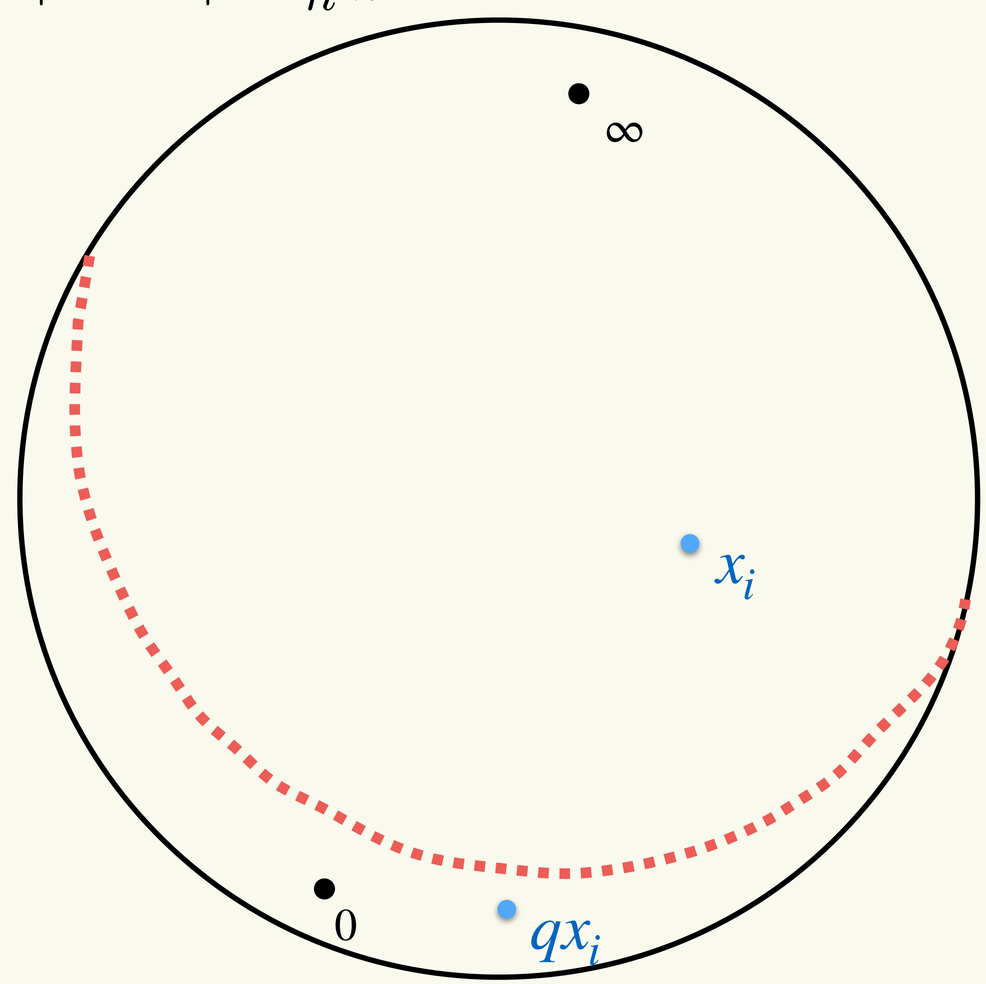
q-Riemann problem

$$Y(qz) = A(z)Y(z)$$
$$A(z) = A_0 + A_1 z + \ldots + A_n z^n$$

Existence: Analytic solutions Y_0, Y_∞ exist in domains $\mathscr{D}_0 \ni 0$ and $\mathscr{D}_\infty \ni \infty$ respectively.

under certain conditions



Connection: $Y_\infty(z) = Y_0(z)\, C(z)$.

Invertibility: We can reconstruct $A(z)$ from $C(z)$.

Singularities: $\det(A(x_i)) = 0$ move with t.

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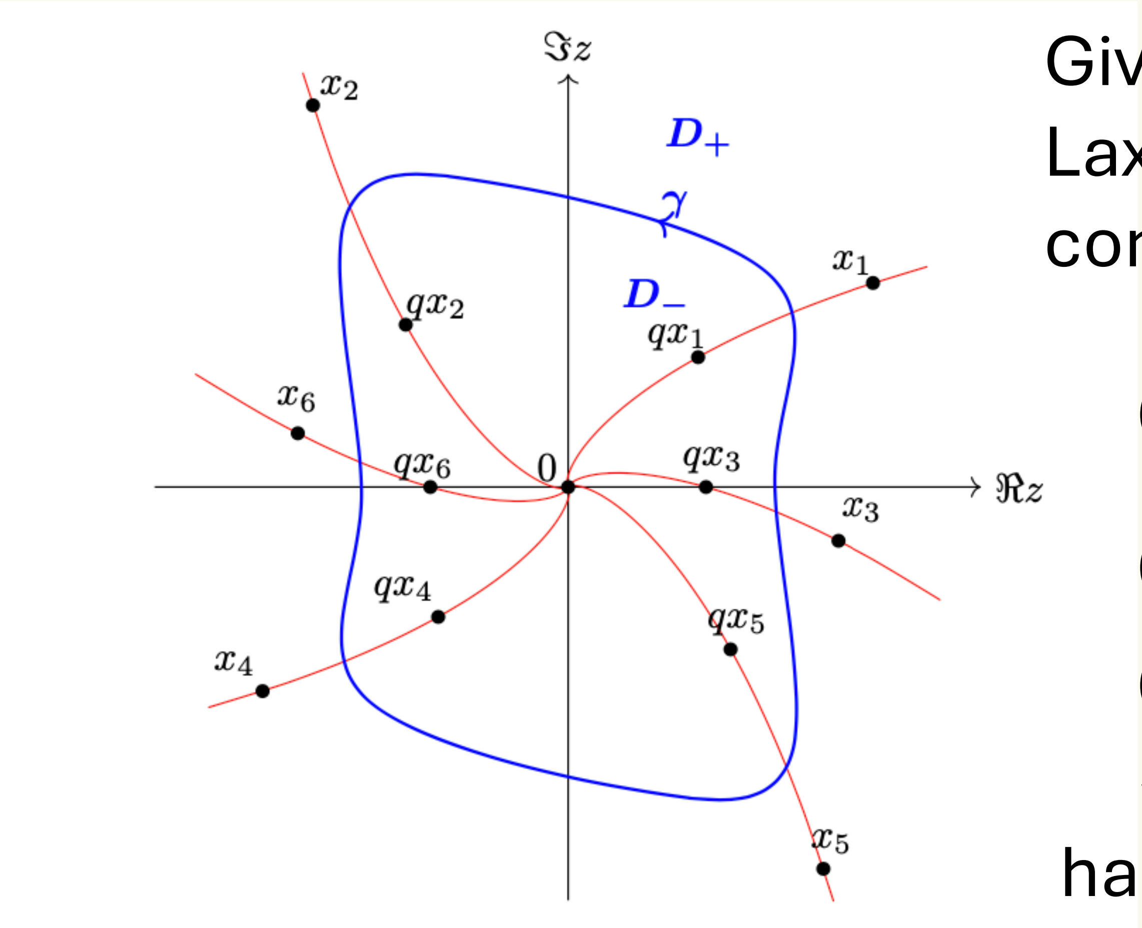
Birkhoff (1913) and Carmichael (1912)

For q-Painlevé equations



$$|A(x_k)| = 0$$

Given "q-Fuchsian" data related to the Lax pairs of qP_{IV} and qP_{VI} under certain conditions, the qRHP:

(i) $Y^{(m)}(z)$ analytic on $\mathbb{C}\backslash\gamma$

(ii) $Y_+^{(m)}(z) = Y_-^{(m)}(z)C(z),\ z \in \gamma$

(ii) $Y^{(m)}(z) = (I + \mathcal{O}(z^{-1}))\, z^{m\sigma_3}, |z| \to \infty$

has a unique solution and singularities of $C(z)$ give rise to a "monodromy" manifold explicitly.

N. Joshi and P. Roffelsen, *Commun. Math. Phys* (2021)

N. Joshi & Roffelsen, *Commun. Math. Phys* (2023)

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q-difference fourth Painlevé equation

$$q\mathrm{P}_{\mathrm{IV}}: \begin{cases} \dfrac{\overline{f}_0}{a_0a_1f_1} = \dfrac{1+a_2f_2(1+a_0f_0)}{1+a_0f_0(1+a_1f_1)}, \\ \dfrac{\overline{f}_1}{a_1a_2f_2} = \dfrac{1+a_0f_0(1+a_1f_1)}{1+a_1f_1(1+a_2f_2)}, \\ \dfrac{\overline{f}_2}{a_2a_0f_0} = \dfrac{1+a_1f_1(1+a_2f_2)}{1+a_2f_2(1+a_0f_0)}, \end{cases}$$

$$\overline{f}_j = f_j(qt)$$

$$f_0\,f_1\,f_2 = t^2, \qquad a_0\,a_1\,a_2 = q$$

$$0 < |q| < 1$$

Kajiwara, Noumi, Yamada *2001*

N. Joshi and N. Nakazono, Lax pairs of discrete Painlevé equations: $(A_2+A_1)^{(1)}$ case, Proc. R. Soc A. 472 (2016) 20160696.

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q-Monodromy surface

We found a monodromy surface:

$$
\begin{aligned}
&\theta_q(+a_0,+a_1,+a_2)\left(\theta_q(t_0)p_1p_2p_3-\theta_q(-t_0)\right)\\
&\quad-\theta_q(-a_0,+a_1,-a_2)\left(\theta_q(t_0)p_1-\theta_q(-t_0)p_2p_3\right)\\
&\quad+\theta_q(+a_0,-a_1,-a_2)\left(\theta_q(t_0)p_2-\theta_q(-t_0)p_1p_3\right)\\
&\quad-\theta_q(-a_0,-a_1,+a_2)\left(\theta_q(t_0)p_3-\theta_q(-t_0)p_1p_2\right)=0
\end{aligned}
$$

$$(\xi;q)_\infty=\prod_{k=0}^{\infty}(1-q^k\xi)$$

$$\theta_q(\xi)=(\xi;q)_\infty(q/\xi;q)_\infty$$

$$\theta_q(\xi_1,\ldots,\xi_n)=\theta_q(\xi_1)\ldots\theta_q(\xi_n)$$

N. Joshi and P. Roffelsen, On the Riemann-Hilbert Problem for a q-difference Painlevé equation, Commun. Math. Phys. 384 (2021) 549–585

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Symmetric Solutions of $q\mathrm{P}_{\mathrm{IV}}$

Symmetry:
$$f_k(i\,q^m) = \frac{1}{f_k(i\,q^{-m})},$$
$$m \in \mathbb{Z},\ \ k = 0, 1, 2$$

Initial values:
$$\big(f_0(i), f_1(i), f_2(i)\big) \in \Big\{(-1,-1,-1), (-1,1,1), (1,-1,1), (1,1,-1)\Big\}$$

The corresponding *q*-RHP is explicitly solvable in terms of Jackson's *q*-Bessel functions of the second kind

$$J_\nu^{(2)}(x;p) = \frac{(p^{\nu+1};p)_\infty}{(p;p)_\infty}\left(\frac{x}{2}\right)^\nu \ {}_0\phi_1\left[\begin{matrix} - \\ p^{\nu+1} \end{matrix}; p, -\frac{x^2 p^{\nu+1}}{4}\right],$$

N. Joshi and P. Roffelsen, On symmetric solutions of the fourth *q*-Painlevé equation, *J. Phys. A* 56 (18) (2023) 185201

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q-difference sixth Painlevé equation

$$\begin{cases} f\bar{f} &= \dfrac{(\bar{g} - q^{\theta_0}t)(\bar{g} - q^{-\theta_0}t)}{(\bar{g} - q^{\theta_\infty - 1})(\bar{g} - q^{-\theta_\infty})}, \\ g\bar{g} &= \dfrac{(f - q^{\theta_t}t)(f - q^{-\theta_t}t)}{q(f - q^{\theta_1})(f - q^{-\theta_1})}, \end{cases}$$

Jimbo, Sakai *1996*

$$q \in \mathbb{C},\ 0 < |q| < 1,\ \bar{f} = f(qt),\ \bar{g} = g(qt)$$

$$Y(qz, t) = A(z, t)Y(z, t)$$

$$A(z, t) = A_0(t) + A_1(t)z + A_2(t)z^2$$

$$\det(A) = 0 \ \Rightarrow \ z = \kappa_J,$$

$$\kappa_0 = q^{\theta_0}, \kappa_1 = q^{\theta_1}, \kappa_t = q^{\theta_1}, \kappa_\infty = q^{-\theta_\infty}$$

$$t \notin q^{\mathbb{Z} \pm (\theta_1 + \theta_t)}, q^{\mathbb{Z} \pm (\theta_1 - \theta_t)}, q^{\mathbb{Z} \pm (\theta_0 + \theta_\infty)}, q^{\mathbb{Z} \pm (\theta_0 - \theta_\infty)}$$

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See also Y. Ohyama, J.-P. Ramis, J. Sauloy, *Ann Fac Sci Toulouse Math* (2020) 1119-1250.

q-P_{VI} Monodromy surface

The monodromy surface is a smooth Segre surface in $\mathbb{P}^6$

$$\eta_{12}+\eta_{13}+\eta_{14}+\eta_{23}+\eta_{24}+\eta_{34}=0$$

$$a_{12}\eta_{12}+a_{13}\eta_{13}+a_{14}\eta_{14}+a_{23}\eta_{23}+a_{24}\eta_{24}+a_{34}\eta_{34}+a_{\infty}=0$$

$$\eta_{13}\eta_{24}-b_1\eta_{12}\eta_{34}=0$$

$$\eta_{14}\eta_{23}-b_2\eta_{12}\eta_{34}=0$$

$$a_{12}=\prod_{\epsilon=\pm1}\frac{\theta_q(q^{+\theta_\infty}t_0)}{\theta_q(q^{\epsilon\,\theta_0+\theta_\infty}t_0)}, \qquad a_{34}=\prod_{\epsilon=\pm1}\frac{\theta_q(q^{-\theta_\infty}t_0)}{\theta_q(q^{\epsilon\,\theta_0-\theta_\infty}t_0)},$$

$$a_{13}=\prod_{\epsilon=\pm1}\frac{\vartheta_\tau(\theta_t+\theta_1+\theta_\infty)}{\vartheta_\tau(\epsilon\,\theta_0+\theta_t+\theta_1+\theta_\infty)}, \qquad a_{24}=\prod_{\epsilon=\pm1}\frac{\vartheta_\tau(-\theta_t-\theta_1+\theta_\infty)}{\vartheta_\tau(\epsilon\,\theta_0-\theta_t-\theta_1+\theta_\infty)},$$

$$a_{23}=\prod_{\epsilon=\pm1}\frac{\vartheta_\tau(-\theta_t+\theta_1+\theta_\infty)}{\vartheta_\tau(\epsilon\,\theta_0-\theta_t+\theta_1+\theta_\infty)}, \qquad a_{14}=\prod_{\epsilon=\pm1}\frac{\vartheta_\tau(\theta_t-\theta_1+\theta_\infty)}{\vartheta_\tau(\epsilon\,\theta_0+\theta_t-\theta_1+\theta_\infty)},$$

Joshi, N. and Roffelsen, P., 2023. On the Monodromy Manifold of q-Painlevé VI Its Riemann–Hilbert Problem. Commun. in Mathematical Physics, 404(1), pp.97-149.

P. Roffelsen *arXiv:2305.17912*

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This surface contains 16 lines.

$$\begin{cases} \mathcal{L}_k^0 : & p_k & = 0 \\ \widetilde{\mathcal{L}}_k^0 : & \tilde{p}_k & = 0 \\ \mathcal{L}_k^\infty : & p_k & = \infty \\ \widetilde{\mathcal{L}}_k^\infty : & \tilde{p}_k & = \infty \end{cases}$$

$$1 \leq k \leq 4$$

Each line corresponds to an asymptotic behaviour, e.g.

$$f(t) \sim F_{1,0} c^k (-t) + F_{1,1} c^k (-t)^{1+2(\theta_t - \theta_0)}$$

$$g(t) \sim G_{1,0} c^k (-t) + F_{1,1} c^k (-t)^{1+2(\theta_t - \theta_0)}$$

Corresponds to $\widetilde{\mathcal{L}}_1^0$

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Continuum limits

- The continuum limit ⇒ a Segre surface for the sixth Painlevé equation.
- Coalescence limits ⇒ Segre surfaces for all the Painlevé equations.
- Thurston's shear coordinates for Teichmuller space, given by Chekhov and Mazzocco (JPhysA, 2010) for Fricke surfaces, are useful for calculating these limits.

Painlevé eqn	$\mathcal{Z}$–Segre surface
qP_{VI}	$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 0,$ $\rho_2 z_2 + \rho_3 z_3 + z_4 + \rho_5 z_5 + \rho_6 z_6 = 1,$ $z_3 z_4 - \lambda_1 z_1 z_2 = 0, \quad z_5 z_6 - \lambda_2 z_1 z_2 = 0.$
P_{VI}	$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 0,$ $\rho_3 z_3 + z_4 + \rho_5 z_5 + \rho_6 z_6 - 1 = 0,$ $z_3 z_4 - \lambda_1 z_1 z_2 = 0, \quad z_5 z_6 - \frac{\rho_3 \lambda_1}{\rho_5 \rho_6} z_1 z_2 = 0.$
P_{V}	$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 0,$ $z_4 + \rho_5 z_5 - 1 = 0,$ $z_3 z_4 - \lambda_1 z_1 z_2 = 0, \quad z_5 z_6 - \lambda_2 z_1 z_2 = 0.$
P_{V}^{deg}	$z_1 + z_3 + z_4 + z_5 + z_6 = 0,$ $\rho_3 z_3 + z_4 + \rho_5 z_5 + \frac{\rho_3}{\rho_5} z_6 - 1 = 0,$ $z_3 z_4 - z_1 z_2 = 0, \quad z_5 z_6 - z_1 z_2 = 0.$

Painlevé eqn	$\mathcal{Z}$–Segre surface
P_{IV}	$z_1+z_2+z_3+z_4+z_5+z_6=0,$ $z_4-1=0,$ $z_3z_4-\lambda_1 z_1z_2=0, \quad z_5z_6-\lambda_2 z_1 z_2=0.$
$P_{III}^{D_6}$	$z_1+z_2+z_3+z_4+z_5=0,$ $z_4+\rho_5 z_5-1=0,$ $z_3z_4-\lambda_1 z_1z_2=0, \quad z_5z_6-z_1z_2=0.$
$P_{III}^{D_7}$	$z_1+z_2+z_3+z_4+z_5=0,$ $z_4+\rho_5 z_5-1=0,$ $z_3z_4-z_1z_2=0, \quad z_5z_6-z_1z_2=0.$
P_{II}^{JM}, P_{II}^{FN}	$z_1+z_2+z_3+z_4+z_5+z_6=0,$ $z_4-1=0,$ $z_3z_4-z_1z_2=0, \quad z_5z_6-\lambda_2 z_1z_2=0.$
P_I	$z_3+z_4+z_5+z_6=0,$ $z_4-1=0,$ $z_3z_4-z_1z_2=0, \quad z_3z_4-z_5z_6=0.$

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Main results:

Theorem 1: The monodromy manifold of each differential Painlevé equation (except $\mathrm{P}_{\mathrm{III}}^{D_8}$) is isomorphic to the corresponding $\mathscr{X}$-Segre surface as an affine variety.

Theorem 2: The blow-down of a line on the cubic monodromy manifold of each differential Painlevé equation gives an alternate $\mathscr{Y}$-Segre surface affine equivalent to the corresponding $\mathscr{X}$-Segre surface.

Theorem 3: There is a natural Poisson bracket on the $\mathscr{X}$-Segre surface. The mapping from each respective $\mathscr{X}$-Segre surface to $\mathscr{Y}$-Segre surface are Poisson maps.

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Summary

- To find the behaviours of transcendental solutions of *q*-discrete Painlevé equations, we developed a *q*-RHP method and found "monodromy" surfaces.
- Lines and points on monodromy surfaces indicate tronquée-like and symmetric solutions for *q*-Painlevé equations.
- The surface for qPVI led us to a new type of monodromy surface for each of the classical Painlevé equations.
- Open question: what are the monodromy surfaces for the remaining discrete Painlevé equations?