THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Tutorial for Week 2

MATH1921/1931: Calculus of One Variable (Advanced) — Semester 1, 2018

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1921/
Lecturer: Daniel Daners

Material covered

☐ Set notation, and number systems $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$, interval notation.
☐ Polynomial equations; solving quadratic equations over $\mathbb{C}$.
☐ Plotting regions in the complex plane.
☐ Polar and Cartesian forms of a complex number, complex exponential.
☐ Modulus, argument, and principal argument of a complex number.

Outcomes

After completing this tutorial you should

☐ understand set notation and apply it in context of the number system;
☐ solve simple examples of polynomial equations over the complex numbers;
☐ construct proofs related to properties of the number system;
☐ be able to plot regions in the complex plane;
☐ understand the geometric interpretation of complex numbers;
☐ efficiently convert between polar and Cartesian forms;
☐ perform arithmetic in polar form.

Summary of essential material

Rational and irrational numbers: A real number $r \in \mathbb{R}$ is called *rational* if there are integers $p, q \in \mathbb{Z}$ with $q \neq 0$ such that $r = p/q$. If it is not rational, it is called *irrational*. Interval notation if $a \leq b$:

$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}, \quad (a, b) := \{x \in \mathbb{R} \mid a < x < b\}, \quad (a, \infty) := \{x \in \mathbb{R} \mid a < x\}, \text{ etc.}\dots$$

Intersections and unions: If A, B are subsets of a larger set X we define

- the *union* of A and B: $A \cup B := \{x \in X \mid x \in A \text{ or } x \in B\}$;
- the *intersection* of A and B: $A \cap B := \{x \in X \mid x \in A \text{ and } x \in B\}$;
- the *complement* of A: $A^c := \{x \in X \mid x \notin A\}$;
- the *complement of B in A*: $A \setminus B := A \cap B^c = \{x \in A \mid x \notin B\}$.

Cartesian and Modulus–argument form (polar form) of complex numbers: Every complex number $z = x + iy$ represents a point on the plane with coordinates (x, y). With that identification we obtain the *complex plane* or *Argand diagram*. We call $x + iy$ the *Cartesian form* of z. We can represent each point (x, y) in polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ where r is the distance from the origin and θ is the angle from the positive x-axis measured anti-clockwise, usually in radians. The *modulus–argument form* (or *polar form*) of z is

$$z = r(\cos\theta + i\sin\theta).$$

We call $r = |z| = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}$ the *modulus* and θ an *argument* of z, written $\arg z$. The argument is determined up to a multiple of 2π. The unique argument in the interval $(-\pi, \pi]$ is called the *principal argument* and denoted by $\operatorname{Arg} z$.

The complex exponential function: For any complex number $z = x + iy$, $x, y \in \mathbb{R}$ we let

$$e^z := e^x(\cos y + i \sin y)$$

We also sometimes write $\exp(z)$. Just as with the real exponential funciton the usual index laws apply, that is, $e^{z+w} = e^z e^w$ and $e^{-z} = 1/e^z$ for all $z, w \in \mathbb{C}$. It coincides with the real exponential function on $\mathbb{R}$ and is 2π-periodic on $i\mathbb{R}$, that is, $e^{i(2\pi+\theta)} = e^{i\theta}$ for all $\theta \in \mathbb{R}$. Moreover, $|e^{i\theta}| = 1$ for all $\theta \in \mathbb{R}$.

De Moivre's Theorem: Let $n \in \mathbb{Z}$. If $r > 0$ and $\theta \in \mathbb{R}$ then

$$\big(r(\cos\theta + i\sin\theta)\big)^n = r^n\big(\cos(n\theta) + i\sin(n\theta)\big) \qquad \text{or} \qquad \big(re^{i\theta}\big)^n = r^n e^{in\theta}.$$

The Fundamental Theorem of Algebra: The complex polynomial

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n \quad \text{with } a_k \in \mathbb{C} \text{ and } a_n \neq 0$$

factorises into precisely n linear factors (possibly with repetition) over the complex numbers. Moreover, if $a_0, \ldots, a_n \in \mathbb{R}$, then for every root $\alpha \in \mathbb{C}$ of $p(z)$, the complex conjugate $\overline{\alpha}$ is also a root.

Questions to complete during the tutorial

Questions marked with * are more difficult questions.

1. Express the following subsets of $\mathbb{R}$ as a union of intervals.

(a) $\{x \in \mathbb{R} \mid -1 \leq x < 5\}$

Solution: $[-1, 5)$

(b) $(-\infty, 3] \setminus (-6, 10]$.

Solution: $(-\infty, 6]$ (Draw a picture to see this).

(c) $\{x \in \mathbb{R} \mid x^2 + x > 2\}$

Solution: $x^2 + x - 2 = (x+2)(x-1)$ is a concave up parabola cutting the x-axis at $x = -2$ and $x = 1$. Thus $\{x \in \mathbb{R} \mid x^2 + x \geq 2\} = (-\infty, -2) \cup (1, \infty)$.

2. Sets are written between curly brackets in the form $\{\text{typical member} \mid \text{defining properties}\}$. Use this notation to write down the following sets.

(a) The set of odd integers.

Solution: Every odd number is of the form $2k + 1$, where k is an integer. Hence the set is

$$\{2k + 1 \mid k \in \mathbb{Z}\}.$$

(b) The complex numbers in the upper half line, excluding the real axis.

Solution: These are the numbers with positive imaginary part:

$$\{x + iy \in \mathbb{C} \mid x, y \in \mathbb{R} \text{ and } y > 0\} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}.$$

(c) The set of complex numbers in the sector with arguments strictly between $\pi/6$ and $3\pi/2$.

Solution: We can use the modulus-argument form to write this set:

$$\{re^{it} \mid r > 0, \pi/6 < \theta < \pi/2\} = \{z \in \mathbb{C} \mid \pi/6 < \arg(z) < \pi/2\}$$

*(d) The set of points on the complex plane on the ellipse with focal points 0 and $1+i$ and major semi-axis of length 2.

Solution: Recall that the sum of the lenghts of the focal chords in an ellipse is constant. The sum is the length of the major axis, that is 4 in our case. Hence the set is

$$\{z \in C \mid |z| + |z - 1 - i| = 4\}.$$

3. Prove that is $\log_2 3$ irrational. (*Hint*: Assume that $\log_2 3 = p/q$ is rational use the definition of $\log_2$ and derive a contradiction.)

Solution: If $\log_2 3$ is rational then $\log_2 3 = \frac{p}{q}$ for some integers p and q. Then

$$\log_2 3 = \frac{p}{q} \Longrightarrow 3 = 2^{p/q} \Longrightarrow 3^q = 2^p.$$

This is impossible, since the left hand side is odd and the right hand side is even. alternatively, the left and right hand side do not share any prime number factors.

4. What are the complex numbers obtained from z by the following geometric transformations?

(a) $180°$ rotation about 0.

Solution: $180°$ rotation about 0 takes the point with co-ordinates (x, y) to the point with co-ordinates $(-x, -y)$. So it takes z to $-z$.

(b) Reflection at the imaginary axis.

Solution: Reflection at the imaginary axis takes $z = x + iy$ to $-\overline{z} = -x + iy$ (the real part changes sign, and the imaginary part stays the same).

(c) $45°$ clockwise rotation about 0.

Solution: $45°$ clockwise rotation about 0 sends $z = re^{i\theta}$ to $re^{i\theta - i\frac{\pi}{4}}$. By the rule for multiplying complex numbers in polar form, this is the same as multiplying by $e^{-i\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$. So the answer is $ze^{-i\frac{\pi}{4}}$.

*(d) Reflection at the line $y = x$.

Solution: Reflection in the line $y = x$ takes the point with co-ordinates (x, y) to the point with co-ordinates (y, x). So it takes $z = x + iy$ to $i\overline{z} = y + ix$. Another way to see this is to carry out this reflection by a sequence of other rotations and reflections. For instance, the line we want to reflect in is such that if we rotate it $45°$ clockwise, it becomes the real axis. So we can carry out the reflection in three steps: first a $45°$ clockwise rotation taking z to $ze^{i-\frac{\pi}{4}}$; then a reflection in the real axis, taking this to $\overline{ze^{-i\frac{\pi}{4}}} = \overline{z}e^{i\frac{\pi}{4}}$; then a $45°$ anticlockwise rotation, taking this to $\overline{z}e^{i\frac{\pi}{4}+\frac{\pi}{4}} = i\overline{z}$.

5. Write the following complex numbers in Cartesian form:

(a) $e^{i\frac{\pi}{4}}e^{i\frac{2\pi}{5}}e^{i\frac{\pi}{3}}e^{i\frac{\pi}{2}}e^{i\frac{11\pi}{60}}$

Solution: When multiplying numbers in polar form we need to multiply the modulii (all one in this case) and add the arguments. Thus the number is

$$\begin{aligned}\exp\left(i\frac{\pi}{4}+i\frac{2\pi}{5}+i\frac{\pi}{3}+i\frac{\pi}{2}+i\frac{11\pi}{60}\right) &= \exp\left(i\frac{15+24+20+30+11}{60}\pi\right)\\ &= \exp\left(i\frac{100}{60}\pi\right)\\ &= \exp\left(i\frac{5}{3}\pi\right)\\ &= \frac{1}{2}-\frac{\sqrt{3}}{2}i.\end{aligned}$$

(b) $(1+\sqrt{3}\,i)^{107}$

Solution: We have $1+\sqrt{3}\,i = 2e^{i\pi/3}$. Thus by de Moivre's Theorem $\left(1+i\sqrt{3}\right)^{107} = 2^{107}e^{107i\pi/3}$. Since $107/3 = 35+\frac{2}{3}$ we have

$$(1+\sqrt{3}\,i)^{107} = 2^{107}\exp\left(35i\pi+\frac{2i\pi}{3}\right) = 2^{106} - 2^{107}i\sqrt{3}.$$

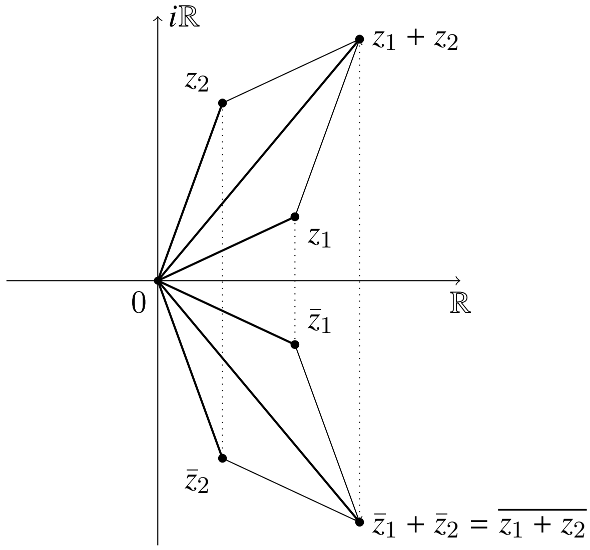
(c) $(1-i)^{-76}$

Solution: $(1-i)^{-76} = \left(\sqrt{2}e^{-i\pi/4}\right)^{-76} = 2^{-38}e^{+76i\pi/4} = 2^{-38}e^{19i\pi} = -2^{-38}$.

6. Given $z_1, z_2 \in \mathbb{C}$, discuss the geometric significance of the following operations.

(a) $\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$

Solution: The complex conjugates are obtained by reflecting the corresponding point on the complex plane on the real axis. Hence

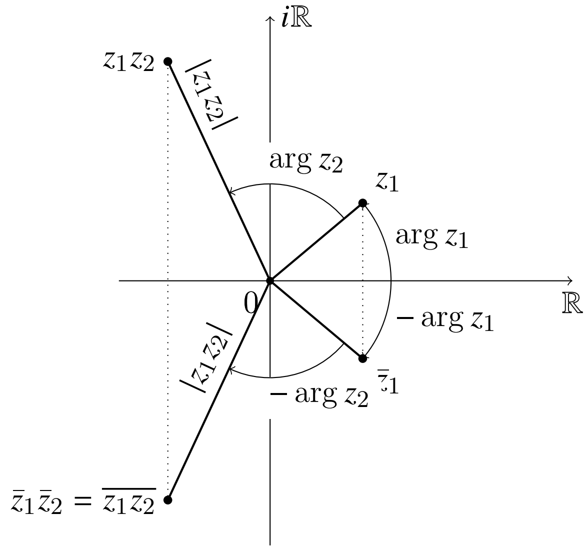


To verify computationally, let $z_1 = x_1+iy_1$ and $z_2 = x_2+iy_2$, where x_1, x_2, y_1, y_2 are real numbers. Then,

$$\begin{aligned}\overline{z_1+z_2} &= \overline{(x_1+x_2)+i(y_1+y_2)}\\ &= (x_1+x_2)-i(y_1+y_2)\\ &= (x_1-iy_1)+(x_2-iy_2)\\ &= \overline{z_1}+\overline{z_2}.\end{aligned}$$

(b) $\overline{z_1 z_2} = \bar{z}_1 \, \bar{z}_2$

Solution: We can see this geometrically since multiplication of two complex numbers means adding their arguments and multiplying the modulus. Taking the complex conjugates, the modulus stays the same but the arguments are negative. Here is an illustration:



Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, where x_1, x_2, y_1, y_2 are real numbers. Then

$$\begin{aligned}
\overline{z_1 z_2} &= \overline{(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)} \\
&= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1) \\
&= (x_1 - iy_1)(x_2 - iy_2) \\
&= \bar{z}_1 \, \bar{z}_2.
\end{aligned}$$

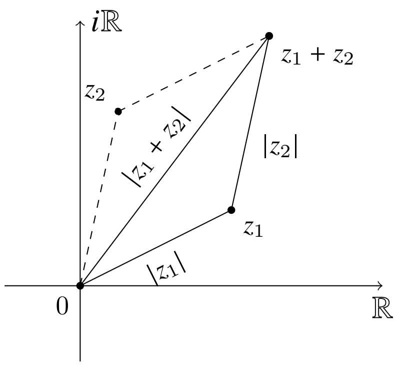
(c) $|z_1 + z_2| \leq |z_1| + |z_2|$

Solution: There are quite a few approaches to this important result.

(1) Here is a geometric proof, which gives the real 'intuition' behind the triangle inequality. Let T be the triangle in complex plane with vertices $0, z_1, z_1 + z_2$. Then the lengths of the three sides are $|z_1 - 0| = |z_1|$, $|(z_1 + z_2) - z_1| = |z_2|$, and $|(z_1 + z_2) - 0| = |z_1 + z_2|$. Geometrically we know that the length of one side of a triangle is lest than or equal to the sum of the other two side lengths (this is because the shortest distance between two points is via a straight line), and so

$$|z_1 + z_2| \leq |z_1| + |z_2|.$$

The picture accompanying this proof is as follows:



(2) Here is a brute-force algebraic proof. Let $z_1 = a + ib$ and $z_2 = c + id$, where a, b, c, d are real numbers. The inequality we need to prove is

$$\sqrt{(a+c)^2 + (b+d)^2} \le \sqrt{a^2+b^2} + \sqrt{c^2+d^2}.$$

Because both sides are nonnegative, it is equivalent to prove that the square of the left-hand side is less than or equal to the square of the right-hand side:

$$(a+c)^2 + (b+d)^2 \le a^2 + b^2 + c^2 + d^2 + 2\sqrt{a^2+b^2}\sqrt{c^2+d^2}.$$

After tidying up, this becomes:

$$ac + bd \le \sqrt{a^2+b^2}\sqrt{c^2+d^2}.$$

Again, since the right-hand side is nonnegative, it suffices to prove that the square of the left-hand side less than or equal to the square of the right-hand side:

$$a^2c^2 + b^2d^2 + 2abcd \le (a^2+b^2)(c^2+d^2).$$

After tidying up, this becomes:

$$0 \le a^2d^2 + b^2c^2 - 2abcd,$$

which is true because the right-hand side is $(ad - bc)^2$. Hence the result.

(3) Here is a more elegant algebraic proof. First, recall the following facts: If z is a complex number, then $|z|^2 = z\overline{z}$, and $z + \overline{z} = 2\,\mathrm{Re}(z)$, and $\mathrm{Re}(z) \le |z|$. To see this last statement, write $z = x + iy$, and notice that $\mathrm{Re}(z) = x \le |x| = \sqrt{x^2} \le \sqrt{x^2+y^2} = |z|$. Then,

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\
&= (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) \\
&= z_1\overline{z}_1 + z_2\overline{z}_2 + z_1\overline{z}_2 + \overline{z}_1 z_2 \\
&= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\overline{z}_2) && (\text{since } \overline{z_1\overline{z}_2} = \overline{z}_1 z_2) \\
&\le |z_1|^2 + |z_2|^2 + 2|z_1\overline{z}_2| && (\text{using } \mathrm{Re}(z) \le |z|) \\
&= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| && (\text{using } |zw| = |z||w| \text{ and } |\overline{z}| = |z|) \\
&= (|z_1| + |z_2|)^2.
\end{aligned}$$

Therefore $|z_1 + z_2| \le |z_1| + |z_2|$.

7. For $t \in \mathbb{R}$ we defined $e^{it} := \cos t + i \sin t$ arguing it behaves like the exponential function in the sense that it satisfies the familiar index laws such as $e^{it}e^{is} = e^{i(t+s)}$. This is true for every base, not just base e. The question motivates the fact that the base should be the Euler number e: If $a \in \mathbb{R}$, then

$$\frac{d}{dt}e^{at} = ae^t.$$

This is only true for base e, not for any other base. Assuming you differentiate a complex valued function by differentiating its real and imaginary part, show that

$$\frac{d}{dt}e^{it} = ie^{it},$$

similar to the real exponential function.

Solution: Differentiating real and imaginary parts separately we have

$$\frac{d}{dt}e^{it} = \frac{d}{dt}(\cos t + i\sin t) = -\sin t + i\cos t = i(\cos t + i\sin t) = ie^{it}$$

as required.

8. Solve the following equations for $z \in \mathbb{C}$:

(a) $z^2 + z + 1 = 0$

Solution: The quadratic formula is perfectly valid when solving quadratic equations in $\mathbb{C}$ (after all, it just comes from completing the square). Thus $z = \frac{-1\pm\sqrt{-3}}{2}$. The expression $\pm\sqrt{-3}$ means either of the two complex square roots of -3, namely $\pm i\sqrt{3}$. The solutions are therefore $z = \frac{-1\pm i\sqrt{3}}{2}$.

(b) $z^2 + 2\bar{z} + 1 = 0$

Solution: Note that this is not a quadratic (because of the $\bar{z}$), and so we cannot use the quadratic formula. Instead, we set $z = a + bi$ for a, b real. Then

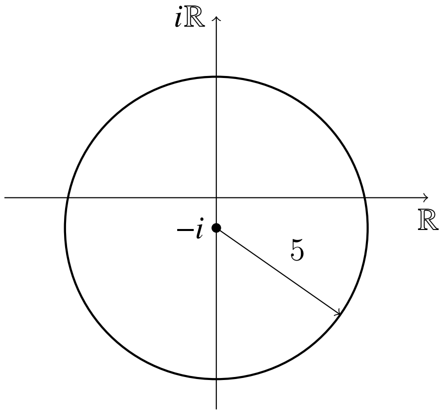
$$\begin{aligned} z^2 + 2\bar{z} + 1 &= (a^2 - b^2) + 2abi + 2(a - bi) + 1 \\ &= (a^2 - b^2 + 2a + 1) + (2ab - 2b)i. \end{aligned}$$

This is zero if and only if both the real and imaginary parts are zero, that is, $a^2 - b^2 + 2a + 1 = 0$ and $2ab - 2b = 0$. The second equation can also be written as $2(a - 1)b = 0$, which gives two cases: either $a = 1$ or $b = 0$. If $a = 1$, the first equation becomes $1 - b^2 + 2 + 1 = 0$, which has solutions $b = \pm 2$. If $b = 0$, then the first equation becomes $a^2 + 2a + 1 = 0$, which has solution $a = -1$. So the solutions to the original equation are $z = -1$ and $z = 1 \pm 2i$.

9. Sketch the following sets in the complex plane. Recall that $|z - \alpha|$ is the distance between z and α in the Argand diagram.

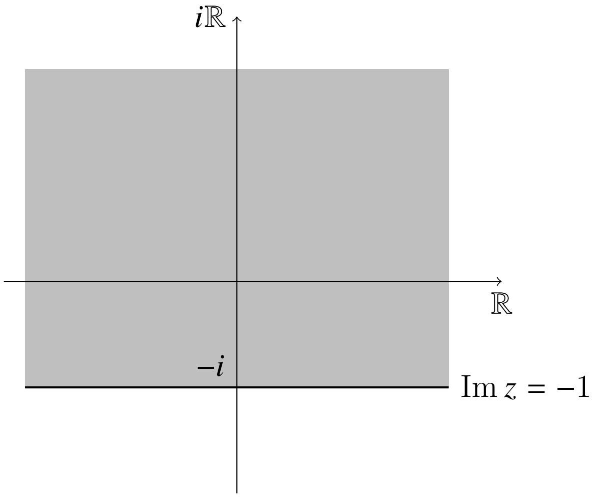
(a) $\{z \in \mathbb{C} \mid |z + i| = 5\}$

Solution: The set $\{z \in \mathbb{C} \mid |z + i| = 5\}$ is a circle of radius 5, centred at $-i$, illustrated below:



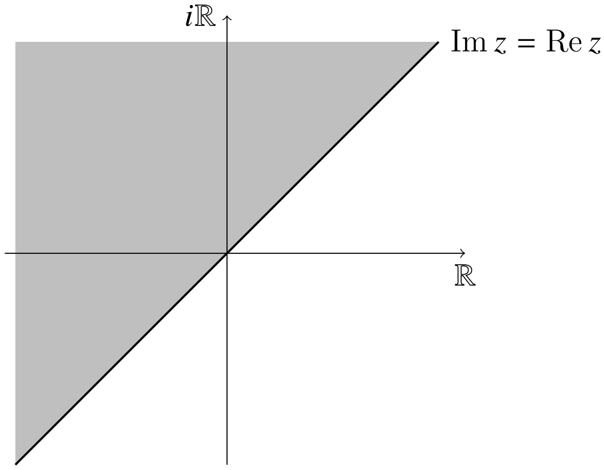
(b) $\{z \in \mathbb{C} \mid \operatorname{Im} z \geq -1\}$

Solution: The set $\{z \in \mathbb{C} \mid \operatorname{Im} z \geq -1\}$ is shaded below (it includes the horizontal line):



(c) $\{z \in \mathbb{C} \mid |z - i| \le |z - 1|\}$

Solution: The set $\{z \in \mathbb{C} \mid |z - i| \le |z - 1|\}$ is shaded below:



(d) $\left\{z \in \mathbb{C} \mid \left|\frac{z-1}{z-2}\right| \le 3\right\}$

Solution: This set is the region outside of the circle of radius $\frac{3}{8}$ centred at the point $z_0 = \frac{17}{8} + 0i$ (including the circle itself). To see this, rewrite the condition as $|z - 1|^2 \le 9|z - 2|^2$, substitute $z = x + iy$ and simplify. Along the way you will need to complete the square (in x). The working is as follows: The condition $|z - 1|^2 \le 9|z - 2|^2$ reads

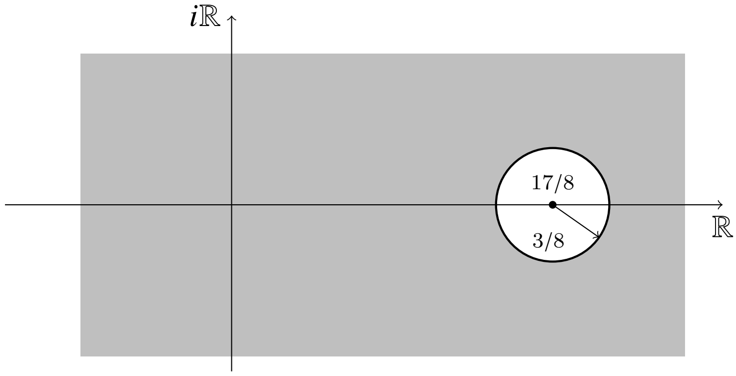
$$(x - 1)^2 + y^2 \le 9\left((x - 2)^2 + y^2\right),$$

and hence after some algebra,

$$x^2 - (17/4)x + y^2 \ge -35/8.$$

Using a completion of squares we have $x^2 - (17/4)x = x^2 - (17/4)x + (17/8)^2 - (17/8)^2 = (x - 17/8)^2 - (17/8)^2$ and thus

$$(x - 17/8)^2 + y^2 \ge (17/8)^2 - 35/8 = 9/64 = (3/8)^2.$$



Extra questions for further practice

10. You are told that $\alpha = 2 + i$ is a root of the polynomial

$$p(z) = z^6 - 4z^5 + 8z^4 - 12z^3 + 5z^2 + 40z - 50.$$

You will need to use polynomial long division and the fact that $\bar{\alpha} = 2 - i$ is a root as well.

(a) Factorise the polynomial $p(z)$ into linear factors over $\mathbb{C}$.

Solution: Since the polynomial has real coefficients we know that $\overline{2+i} = 2 - i$ is also a root. Thus

$$(z - (2 + i))(z - (2 - i)) = z^2 - 4z + 5$$

is a factor of $z^6 - 4z^5 + 8z^4 - 12z^3 + 5z^2 + 40z - 50$. We can use polynomial long division to factorise:

$$
\begin{array}{rr}
 & z^4 \qquad\quad + 3z^2 \qquad\quad - 10 \\
z^2 - 4z + 5 \,\big) & z^6 - 4z^5 + 8z^4 - 12z^3 + 5z^2 + 40z - 50 \\
 & -z^6 + 4z^5 - 5z^4 \qquad\qquad\qquad\qquad\qquad\quad \\
 & 3z^4 - 12z^3 + 5z^2 \qquad\qquad\quad \\
 & -3z^4 + 12z^3 - 15z^2 \qquad\qquad\quad \\
 & -10z^2 + 40z - 50 \\
 & 10z^2 - 40z + 50 \\
 & 0
\end{array}
$$

to see that $z^6 - 4z^5 + 8z^4 - 12z^3 + 5z^2 + 40z - 50 = (z^2 - 4z + 5)(z^4 + 3z^2 - 10)$. Another way to do this step is to argue as follows: Consider

$$z^6 - 4z^5 + 8z^4 - 12z^3 + 5z^2 + 40z - 50 = (z^2 - 4z + 5)(z^4 + az^3 + bz^2 + cz + d),$$

where we are trying to find a, b, c, d. By considering the constant terms on both sides we get $-50 = 5d$, and hence $d = -10$. Then by looking at the coefficients of z on both sides we get $40 = -4d + 5c$, and hence $c = 0$. Now looking at coefficients of z^2 we get $5 = d - 4c + 5b$, and hence $b = (5 + 10)/5 = 3$. Finally, looking at the coefficient of z^3 we get $-12 = c - 4b + 5a$, and hence $a = 0$.

In any case, we have arrived at

$$z^6 - 4z^5 + 8z^4 - 12z^3 + 5z^2 + 40z - 50 = (z^2 - 4z + 5)(z^4 + 3z^2 - 10).$$

We have $z^4 + 3z^2 - 10 = (z^2 - 2)(z^2 + 5)$, and hence the complete factorisation of $p(z)$ is

$$p(z) = (z - (2 + i))(z - (2 - i))(z - \sqrt{2})(z + \sqrt{2})(z - i\sqrt{5})(z + i\sqrt{5}).$$

(b) What can you say about factorisations of $p(z)$ over $\mathbb{R}$ and $\mathbb{Q}$?

Solution: First a clarification. When we say "the factorisation over $\mathbb{R}$" we mean the decomposition of the polynomial into factors of the lowest possible degrees subject to the condition that all of the coefficients of each factor are in $\mathbb{R}$. The last few lines of the solution to the previous part tells us that this factorisation is:

$$p(z) = (z^2 - 4z + 5)(z - \sqrt{2})(z + \sqrt{2})(z^2 + 5).$$

Note that neither $z^2 - 4z + 5$ nor $z^2 + 5$ can be factorised further over $\mathbb{R}$, since the roots are complex numbers.

The factorisation in the previous part is not the factorisation over $\mathbb{Q}$, because $\sqrt{2}$ is an irrational number. The factorisation over $\mathbb{Q}$ is

$$p(z) = (z^2 - 4z + 5)(z^2 - 2)(z^2 + 5).$$

None of these quadratics factorise further over $\mathbb{Q}$, since they either have complex roots, or irrational roots.

11. Solve the following equations.

(a) $z^4 - 16 = 0$

Solution: $z^4 - 16 = 0 \iff z^2 = \pm 4 \iff z = \pm 2, \pm 2i.$

(b) $z^2 + 3z + 2 = 0$

Solution: We have $z^2 + 3z + 2 = (z + 2)(z + 1)$, and so $z = -2$ or $z = -1$.

(c) $z^2 + z + 1 + i = 0$.

Solution: Using the quadratic formula we find

$$z = \frac{-1 \pm \sqrt{1 - 4(1 + i)}}{2} = \frac{-1 \pm \sqrt{-3 - 4i}}{2}.$$

The expression $\pm\sqrt{-3 - 4i}$ represents the two numbers whose square is $-3-4i$. In other words, we need to find the numbers $a + ib$ such that $(a + ib)^2 = -3 - 4i$. That is,

$$(a^2 - b^2) + 2abi = -3 - 4i.$$

This yields the two equations:

$$\begin{aligned} a^2 - b^2 &= -1 \\ 2ab &= -4. \end{aligned}$$

From the second we have $b = -2/a$ and substituting this into the first yields

$$a^2 - \frac{4}{a^2} = -3.$$

Rearranging gives $a^4 + 3a^2 - 4 = 0$, which is a quadratic in a^2. Thus by the quadratic formula

$$a^2 = \frac{-3 \pm \sqrt{3^2 - 4(-4)}}{2} = \frac{-3 \pm 5}{2}.$$

Since a is real, a^2 must be non-negative and hence $a^2 = 1$. Thus $a = \pm 1$, $b = \mp 2$ and there are two solutions, $1 - 2i$ and $-1 + 2i$.

So substituting these values in for $\pm\sqrt{-3 - 4i}$ we see that the required solutions are $z = -i$ and $z = -1 + i$.

(d) $z^2 + (2+3i)z - 1 + 3i = 0$.

Solution: Using the quadratic formula,

$$z = -\frac{2+3i}{2} \pm \frac{\sqrt{(2+3i)^2 - 4(-1+3i)}}{2} = -1 - \frac{3}{2}i \pm \frac{1}{2}\sqrt{-1},$$

and so $z = -1 - i$ and $z = -1 - 2i$ are the solutions.

12. Solve $z^5 - 2z^4 + 2z^3 - z^2 + 2z - 2 = 0$, given that $z = 1 + i$ is a solution.

Solution: The polynomial has real coefficients, therefore since $z = 1 + i$ is a root we know that $\overline{1+i} = 1 - i$ is also a root. Thus $(z - (1+i))(z + (1+i)) = z^2 - 2z + 2$ is a factor of $z^5 - 2z^4 + 2z^3 - z^2 + 2z - 2$. By polynomial long division,

$$\begin{array}{r|l} & z^3 \qquad\qquad\quad - 1 \\ z^2 - 2z + 2 & z^5 - 2z^4 + 2z^3 - z^2 + 2z - 2 \\ & -z^5 + 2z^4 - 2z^3 \\ \hline & \qquad\qquad\quad - z^2 + 2z - 2 \\ & \qquad\qquad\quad z^2 - 2z + 2 \\ \hline & \qquad\qquad\qquad\qquad 0 \end{array}$$

and so

$$z^5 - 2z^4 + 2z^3 - z^2 + 2z - 2 = (z^2 - 2z + 2)(z^3 - 1).$$

Therefore the roots are $z = 1 + i, 1 - i$, along with the solutions to $z^3 = 1$. The latter are $z = 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

13. For all complex numbers z, prove that

(a) $|z|^2 = z\bar{z}$.

Solution: Let $z = x + iy$. By definition we have $|z| = \sqrt{x^2 + y^2}$, and so

$$|z|^2 = x^2 + y^2 = (x + iy)(x - iy) = z\bar{z}.$$

(b) $\bar{z} = z$ if and only if z is real.

Solution: Let $z = x + iy$. Then $\bar{z} = z$ if and only if $x - iy = x + iy$, which occurs if and only if $y = 0$, which occurs if and only if z is real.

(c) $\mathrm{Re}(z) \le |z|$ and $\mathrm{Im}(z) \le |z|$.

Solution: Let $z = x + iy$, so that $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$. Then

$$\mathrm{Re}(z) = x \le |x| = \sqrt{x^2} \le \sqrt{x^2 + y^2} = |z|.$$

Similarly, $\mathrm{Im}(z) = y \le \sqrt{y^2} \le |z|$.

(d) $\overline{1/z} = 1/\bar{z}$ for $z \neq 0$.

Solution: We could either argue directly, or use the fact that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$ (proved above) to note that

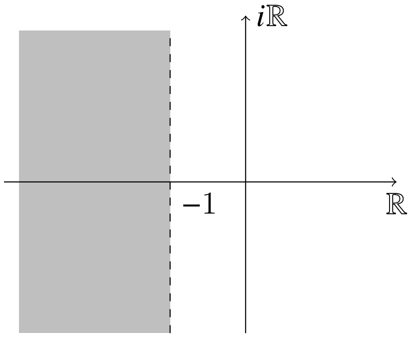
$$\bar{z} \times \overline{1/z} = \overline{z \times 1/z} = \bar{1} = 1,$$

and thus $\overline{1/z} = 1/\bar{z}$.

14. Sketch the following sets in the complex plane.

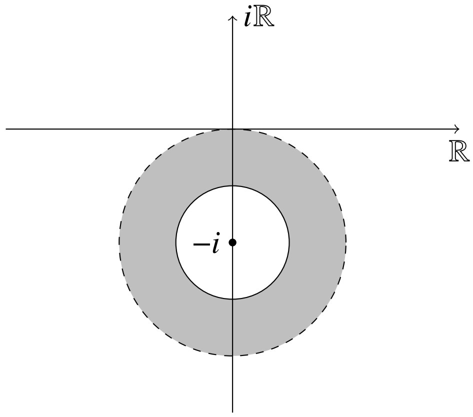
(a) $\{z \in \mathbb{C} \mid \operatorname{Re} z < -1\}$

Solution: The set $\{z \in \mathbb{C} \mid \operatorname{Re} z < -1\}$ is shaded below (it does not include the vertical line):



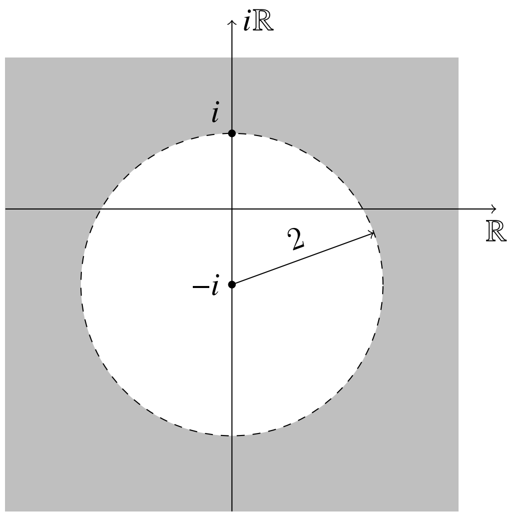
(b) $\{z \in \mathbb{C} \mid \frac{1}{2} \leq |z+i| < 1\}$

Solution: The set $\{z \in \mathbb{C} \mid \frac{1}{2} \leq |z+i| < 1\}$ is an annulus, as illustrated below:



(c) $\{ z \in \mathbb{C} \mid |z+i| > 2 \}$

Solution: The set $\{z \in \mathbb{C} \mid |z+i| > 2\}$ is the 'outside' of the circle of radius 2 centred at $-i$, not including the circle itself.

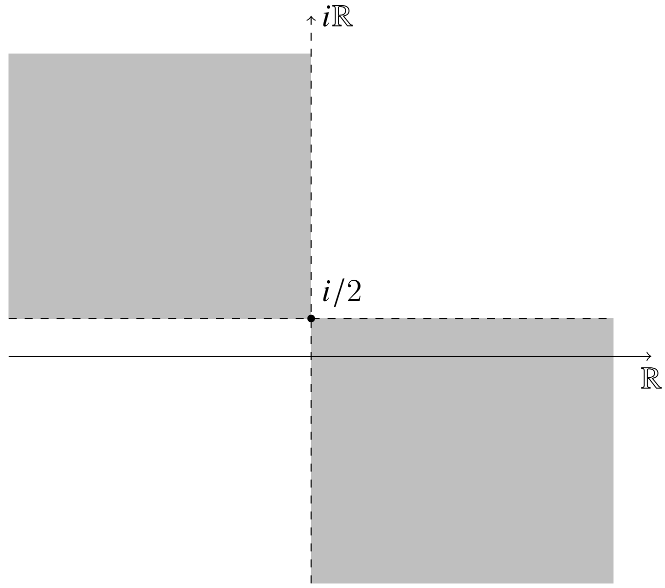


(d) $\{z \in \mathbb{C} \mid \operatorname{Im}(z^2) < \operatorname{Re} z\}$

Solution: Let $z = x + iy$. Then $z^2 = (x^2 - y^2) + 2xyi$, so the condition in the question becomes $2xy < x$ or equivalently $x(2y-1) < 0$. This is satisfied exactly when

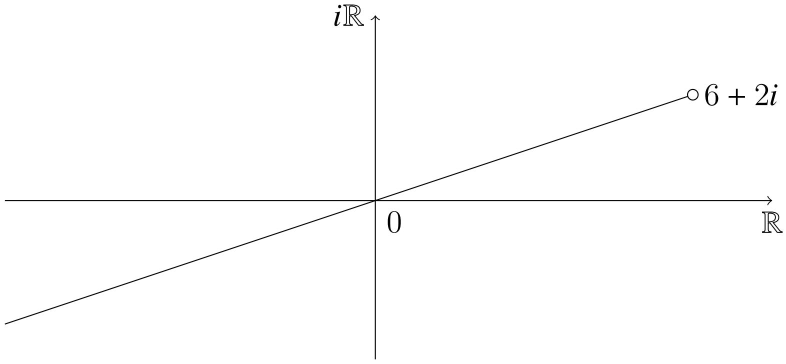
$$x < 0, \quad y > \frac{1}{2} \qquad \text{or} \qquad x > 0, \quad y < \frac{1}{2}.$$

So the set consists of the following two regions (dashed boundary lines not included):



(e) $\{ z \in \mathbb{C} \mid \operatorname{Im}(2z - \overline{z}(1+i)) = 0 \text{ and } \operatorname{Re}(2z - \overline{z}(1+i)) < 4 \}$

Solution: Writing $z = x + iy$ we see that $2z - \overline{z}(1+i) = x - y + i(3y - x)$. This has imaginary part zero if and only if $3y = x$. Thus the required complex numbers z are those numbers $z = 3y + iy = y(3+i)$ where $3y - y = 2y < 4$, that is, $y < 2$. This is the open half-line shown in the diagram.



15. (a) Let $r > 2$. The set $\{z \in \mathbb{C} \mid |z+1| + |z-1| = r\}$ is a curve in the plane. Describe it and then find its equation in terms of x and y, where x, y are real and $z = x + iy$.

(b) Given $-2 < r < 2$, describe the curve $\{z \in \mathbb{C} \mid |z+1| - |z-1| = r\}$ and find its equation.

Solution: The condition which defines the set in part (a) can be interpreted geometrically as "the distance from z to -1 plus the distance from z to 1 is a constant which is greater than 2". Therefore the set in this part is an ellipse. The (real) equation of the ellipse is $\frac{x^2}{(\frac{r}{2})^2} + \frac{y^2}{(\frac{r}{2})^2 - 1} = 1$. The interpretation of the set in part (b) is done in a similar fashion. If $r = 0$ then we have $|z+1| = |z-1|$, which is the imaginary axis. If $r \neq 0$ we have a hyperbola with equation $\frac{x^2}{(\frac{r}{2})^2} - \frac{y^2}{1 - (\frac{r}{2})^2} = 1$.

These results can be obtained algebraically by setting $z = x + iy$ and transforming the given conditions into Cartesian equations in x and y. This process will be a test of your skill in algebraic manipulation, let us just go through the details for part (a). Writing $z = x + iy$ the condition $|z+1| + |z-1| = r$ gives

$$\sqrt{(x+1)^2 + y^2} = r - \sqrt{(x-1)^2 + y^2}.$$

Squaring both sides yields

$$x^2 + 2x + 1 + y^2 = r^2 - 2r\sqrt{(x-1)^2 + y^2} + x^2 - 2x + 1 + y^2.$$

Rearranging gives

$$2r\sqrt{(x-1)^2 + y^2} = r^2 - 4x,$$

and squaring this tells us that

$$4r^2\left(x^2 - 2x + 1 + y^2\right) = r^4 - 8xr^2 + 16x^2.$$

After tidying up we get

$$4(r^2-4)x^2 + 4r^2y^2 = r^2(r^2-4).$$

We are told that $r > 2$, and so $r^2 - 4 > 0$. Therefore we arrive at

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{where} \quad a = \frac{r}{2} \quad \text{and} \quad b = \sqrt{\frac{r^2}{4} - 1}.$$

Revision questions on complex numbers

The questions below are particularly relevant for those students who have not seen complex numbers at high school level.

16. Express the following complex numbers in Cartesian form:

(a) $(1+i)(1-i)$

Solution: $(1+i)(1-i) = (1+1) + (-1+1)i = 2$

(b) $(2+3i) - (4-5i)$

Solution: $(2+3i) - (4-5i) = -2 + 8i$

(c) $\dfrac{1+2i}{3-4i}$

Solution: $\dfrac{1+2i}{3-4i} = \dfrac{(1+2i)(3+4i)}{(3-4i)(3+4i)} = \dfrac{-5}{25} + \dfrac{10}{25}i$

(d) $(1+i)^2$

Solution: $(1+i)^2 = 1 + 2i + i^2 = 2i.$

(e) $(3-2i)\left(\dfrac{5}{2} - 7i\right)$

Solution: $(3-2i)\left(\dfrac{5}{2} - 7i\right) = \left(\dfrac{15}{2} - 14\right) + (-5-21)i = -\dfrac{13}{2} - 26i$

(f) $\dfrac{3i-5}{i+7}$

Solution: $\dfrac{3i-5}{i+7} = \dfrac{-5+3i}{7+i} \times \dfrac{7-i}{7-i} = -\dfrac{16}{25} + \dfrac{13}{25}i$

(g) i^{-1}

Solution: $i^{-1} = \dfrac{1}{i} = \dfrac{1}{i} \times \dfrac{i}{i} = -i.$

(h) i^9

Solution: $i^9 = i(i^2)^4 = i(-1)^4 = i.$

(i) $i^{123} - 4i^8 - 4i$.

Solution: $i^{123} - 4i^8 - 4i = i(i^2)^{61} - 4(i^2)^4 - 4i = -i - 4 - 4i = -4 - 5i$.

17. Find the principal argument of the following complex numbers.

(a) $-1 + i$

Solution: The principal argument is $3\pi/4$.

(b) $-3i$

Solution: The principal argument is $-\pi/2$.

(c) $-5e^{i7\pi/2}$

Solution: We have $-5e^{i7\pi/2} = -5(-i) = 5i$, and so the principal argument is $\pi/2$.

(d) $6 - 5i$.

Solution: $-\tan^{-1}(5/6) \approx -0.6947$ radians.

18. Write the following complex numbers in polar form.

(a) $1 + i$

Solution: $\sqrt{2}e^{i\frac{\pi}{4}}$

(b) $1 + \sqrt{3}i$

Solution: $2e^{i\frac{\pi}{3}}$

(c) $3\sqrt{3} + 3i$

Solution: $6e^{i\frac{\pi}{6}}$

(d) $1 + i$

Solution: $1 + i = \sqrt{2}e^{i\pi/4}$.

(e) $-1 + \sqrt{3}i$

Solution: $-1 + \sqrt{3}i = 2e^{i2\pi/3}$

(f) -5

Solution: $-5 = 5e^{i\pi}$

(g) i

Solution: $i = e^{i\pi/2}$

(h) $5 - 7i$

Solution: $|5 - 7i| = \sqrt{5^2 + 7^2} = \sqrt{74}$, and $\theta = -\tan(7/5)$. Thus $5 - 7i = \sqrt{74}\,\mathrm{cis}\,(-\tan(7/5))$.

19. Find the following, expressing your final answers first in polar form, and then in Cartesian form.

(a) $(1 + i)^{11}$

Solution: Using part (a), $32\sqrt{2}e^{i\frac{11\pi}{4}} = 32(-1 + i) = -32 + 32i$

(b) $(1 + \sqrt{3}i)^7$

Solution: Using part (a), $128e^{i\frac{7\pi}{3}} = 64 + 64i\sqrt{3}$

(c) $(3\sqrt{3}+3i)^3$

Solution: Using part (a), $216e^{\frac{\pi}{2}} = 216i$

(d) $\dfrac{1+i}{1+\sqrt{3}i}$

Solution: Using part (a), the polar form is $\dfrac{1}{\sqrt{2}}e^{-i\frac{\pi}{12}}$. To find the Cartesian form it is easiest to make a direct calculation (unless you remember exact formulae for $\cos(\pi/12)$ and $\sin(\pi/12)$). We have

$$\frac{1+i}{1+\sqrt{3}i} = \frac{(1+i)(1+\sqrt{3}i)}{(1+\sqrt{3}i)(1-\sqrt{3}i)} = \frac{1-\sqrt{3}}{4} + \frac{1+\sqrt{3}}{4}i$$

By the way, comparing with $\dfrac{1}{\sqrt{2}}e^{-i\pi/12} = \dfrac{1}{\sqrt{2}}(\cos(\pi/12) - i\sin(\pi/12))$ we see that $\cos(\pi/12) = \sqrt{2}(\sqrt{3}+1)/4$ and $\sin(\pi/12) = \sqrt{2}(\sqrt{3}-1)/4$.

(e) $\dfrac{3\sqrt{3}+3i}{1+i}$

Solution: Using part (a), the polar form is $\dfrac{6}{\sqrt{2}}e^{i-\frac{\pi}{12}}$, and direct calculation gives the Cartesian form:

$$\frac{3\sqrt{3}+3i}{1+i} = \frac{3(\sqrt{3}+i)(1-i)}{(1+i)(1-i)} = \frac{3(\sqrt{3}+1)}{2} - \frac{3(\sqrt{3}-1)}{2}i.$$

(f) $\dfrac{1+\sqrt{3}i}{3\sqrt{3}+3i}$

Solution: Using part (a), $\dfrac{1}{3}e^{i\frac{\pi}{6}} = \dfrac{1}{2\sqrt{3}} + \dfrac{1}{6}i$

Challenge questions (optional)

***20.** Use the binomial expansion $(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$ and de Moivre's Theorem to express $\cos 5\theta$ and $\sin 5\theta$ in terms of $\cos\theta$ and $\sin\theta$, respectively. Hence show that

$$\cos(\pi/5) = \frac{1+\sqrt{5}}{4} \quad \text{and} \quad \sin(\pi/5) = \frac{\sqrt{2(5-\sqrt{5})}}{4}.$$

Solution: Expanding $(\cos\theta + i\sin\theta)^5$ yields

$$\begin{aligned}(\cos\theta + i\sin\theta)^5 &= (\cos\theta)^5 + 5(\cos\theta)^4 i\sin\theta + 10(\cos\theta)^3(i\sin\theta)^2 \\ &\quad + 10(\cos\theta)^2(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5 \\ &= (\cos\theta)^5 + 5(\cos\theta)^4\sin\theta\, i - 10(\cos\theta)^3(\sin\theta)^2 \\ &\quad - 10(\cos\theta)^2(\sin\theta)^3\, i + 5\cos\theta(\sin\theta)^4 + (\sin\theta)^5\, i.\end{aligned}$$

On the other hand, $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$.

Equating real and imaginary parts of each expression yields

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

and

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$
$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$

Plugging $\theta = \pi/5$ into the formula for $\sin 5\theta$ we arrive at

$$0 = 16\sin^5(\pi/5) - 20\sin^3(\pi/5) + 5\sin(\pi/5)$$
$$= \sin(\pi/5)\big(16\sin^4(\pi/5) - 20\sin^2(\pi/5) + 5\big).$$

Since $\sin(\pi/5) \neq 0$ we may divide by it to obtain

$$16\sin^4(\pi/5) - 20\sin^2(\pi/5) + 5 = 0.$$

This is a quadratic equation in $\sin^2(\pi/5)$, and the quadratic formula gives

$$\sin^2(\pi/5) = \frac{5 \pm \sqrt{5}}{8}.$$

Should we take the + or − sign? Note that $\sin^2(\pi/5) \leq \sin^2(\pi/4) = 0.5$, and since $(5+\sqrt{5})/8 \geq (5+2)/8 = 7/8 > 0.5$ we conclude that the − sign is the right choice. Therefore

$$\sin(\pi/5) = \pm\frac{\sqrt{5-\sqrt{5}}}{2\sqrt{2}} = \pm\frac{1}{4}\sqrt{2(5-\sqrt{5})}.$$

Again we must choose the sign correctly, however clearly $\sin(\pi/5) > 0$, and thus

$$\sin(\pi/5) = \frac{1}{4}\sqrt{2(5-\sqrt{5})}.$$

Using $\cos(\pi/5) = \sqrt{1-\sin^2(\pi/5)}$ gives

$$\cos(\pi/5) = \frac{\sqrt{6+2\sqrt{5}}}{4}.$$

This last formula actually simplifies a little:

$$\sqrt{6+2\sqrt{5}} = \sqrt{1+2\sqrt{5}+(\sqrt{5})^2} = \sqrt{(1+\sqrt{5})^2} = 1+\sqrt{5},$$

and so we get the neater formula

$$\cos(\pi/5) = \frac{1+\sqrt{5}}{4}.$$

As a concluding remark for interest only, note that our formulae for $\cos(\pi/5)$ and $\sin(\pi/5)$ are built up from integers using only the operations of addition, subtraction, multiplication, division, and square roots. It is a remarkable fact that if $p > 2$ is a prime number, then there are formulae like those above for $\cos(\pi/p)$ and $\sin(\pi/p)$ if and only if $p = 2^{2^k} + 1$ for some integer k. You can learn about this in the third year course *MATH3962: Rings, fields, and Galois theory*. Prime numbers of the form $p = 2^{2^k} + 1$ are called *Fermat primes*, and the only known examples are given by $k = 0, 1, 2, 3, 4$ (nobody knows if there are any more examples!). The case $k = 1$ gives $p = 5$, and if $k = 2$ then $p = 17$. So there must be formulae like those above for $\cos(\pi/17)$ and $\sin(\pi/17)$. However these formulae aren't so nice – the exact formula for $\cos(2\pi/17)$ is given by

$$\frac{1}{16}\left[-1+\sqrt{17}+\sqrt{34-2\sqrt{17}}+\sqrt{68+12\sqrt{17}-16\sqrt{34+2\sqrt{17}}-2(1-\sqrt{17})\sqrt{34-2\sqrt{17}}}\right],$$

and the formulae for $\cos(\pi/17)$ and $\sin(\pi/17)$ follow from double angle formulae.

***21.** Prove the following property of the real numbers: Given any two numbers $a < b$ there is a rational number r and an irrational number s such that $a < r < b$ and $a < s < b$. You may use that $\sqrt{2}$ is irrational.

Solution: If $a < b$ then $b-a > 0$, and so we can choose $n \in \mathbb{N}$ large enough so that $n(b-a) > 1$. Thus there is an integer k with $na < k < nb$ (because nb and na differ by more than 1). Taking $r = \frac{k}{n}$ we have $a < r < b$, and so there is a rational number between the irrational numbers a and b.

We now show that there is an irrational number between a and b. Since $a < b$ also $a-\sqrt{2} < b-\sqrt{2}$. By what we have just shown there exists a rational number r_1 such that $a-\sqrt{2} < r_1 < b-\sqrt{2}$. Setting $s := r_1 + \sqrt{2}$ we have $a < s < b$. As r_1 is rational and $\sqrt{2}$ is irrational it follows that s is irrational.