Fourier Series
Even and Odd Functions

Even and Odd functions

Fourier series take on simpler forms for Even and Odd functions

Even function

A function is Even if $f(x) = f(-x)$ for all x.

The graph of an even function is symmetric about the y-axis.

In this case $$\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$$

Examples: $x^2, \quad x^4, \quad \cos x, \ldots$

Even and Odd functions

Odd function

A function is Odd if $f(x) = -f(-x)$ for all x.

The graph of an odd function is skew-symmetric about the y-axis.

In this case $$\int_{-L}^{L} f(x)\,dx = 0$$

Examples: x , x^3, $\sin x, \ldots$

Even and Odd functions

Most functions are neither odd nor even

E.g. $x + x^2$, e^x

EVEN × EVEN = EVEN (+ × + = +)

ODD × ODD = EVEN (- × - = +)

ODD × EVEN = ODD (- × + = -)

Even and Odd functions

Any function $h(x)$ can be written as the sum of an even plus an odd function

$$h(x) = h_E(x) + h_O(x)$$

where

$$h_E(x) = \tfrac{1}{2}[h(x) + h(-x)] \quad \text{Even}$$

$$h_O(x) = \tfrac{1}{2}[h(x) - h(-x)] \quad \text{Odd}$$

Example: $h(x) = e^x$

$$h_E(x) = \tfrac{1}{2}[e^x + e^{-x}] = \cosh x$$

$$h_O(x) = \tfrac{1}{2}[e^x - e^{-x}] = \sinh x$$

$$e^x = \cosh x + \sinh x$$

Fourier series of an EVEN periodic function

Let $f(x)$ be even with period $2L$, then

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = 0$$

Fourier series of an EVEN periodic function

Thus the Fourier series of the even function is:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

This is called:

Half-range Fourier cosine series

Fourier series of an ODD periodic function

Let $f(x)$ be odd with period $2L$, then

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = 0, \qquad n = 0, 1, 2 \ldots.$$

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

This is called:

Half-range Fourier sine series

Odd and Even Extensions

Recall the temperature problem with the heat equation.

The function $f(x)$ is specified only on $0 \le x \le L$ and it is not necessarily odd.

However to satisfy the initial condition in the solution we end up with a Fourier Sine series,

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} e^{-\frac{n^2\pi^2 k}{L^2}t}$$

which gives an odd function for $-L \le x \le L$

Odd and Even Extensions

The solution to this is to extend the original function on to an odd function on $-L \le x \le L$

Example:

Find the even and odd extensions on $0 \le x \le L$ of the functions

$$f(x) = k, \qquad f(x) = x, \qquad f(x) = e^x$$

Example

Sketch the Fourier sine series of $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$
and determine its Fourier coefficients

Ans:

$$b_n = \frac{2}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$$

Summary of Fourier series

$f(x)$ neither ODD nor EVEN $\quad f(x) \sim a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + \mathrm{b_n sin}\frac{n\pi x}{L}\right)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$f(x)$ $EVEN$ $\quad f(x) \sim \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$ ***Fourier Cosine Series***

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \qquad b_n = 0$$

$f(x)$ ODD $\quad f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$ ***Fourier Sine Series***

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \qquad a_0 = a_n = 0$$

Fourier Convergence Theorem

Convergence of Fourier series

Piecewise smooth functions

A function is called piecewise smooth if it can be broken up into subintervals such that

$f(x)\ \ and\ \ f'(x)$

are continuous in each subinterval

(see the diagrams)

Fourier Convergence Theorem

Let $f(x)$ have period $2L$ and be piecewise smooth for $-L \le x \le L$ and let the Fourier series with coefficients given by the usual formulas.

$$a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + \mathrm{b_n} \sin\frac{n\pi x}{L}\right)$$

Then the series evaluated at any point x_0 in $-L \le x \le L$ converges to:

1) $f(x_0)$ if x_0 is a point of continuity
2) $\frac{1}{2}[f(x_0-) + f(x_0+)]$ if x_0 is a point of discontinuity