THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

Solutions to Problem Sheet for Week 4

MATH1901: Differential Calculus (Advanced) Semester 1, 2017

Web Page: sydney.edu.au/science/maths/u/UG/JM/MATH1901/
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Material covered

- ☐ Definition of a function $f\colon A \to B$ and composites, domain, codomain and image/range of a function;
- ☐ Injective, surjective, and bijective functions; inverse functions.
- ☐ The concept of natural domain of a real valued function of a real variable.
- ☐ The graph of a function, and the horizontal line test for injectivity.
- ☐ The hyperbolic sine and cosine functions $\sinh x$ and $\cosh x$.

Outcomes

After completing this tutorial you should

- ☐ understand the concepts of domain, codomain and image/range of functions;
- ☐ be able to calculate the image/range of various functions;
- ☐ be able to prove whether given functions are injective, surjective or bijective and compute inverse functions;
- ☐ identify the natural domain of real valued functions of a real variable;
- ☐ work with the hyperbolic cosine and sine functions, and prove identities involving them.

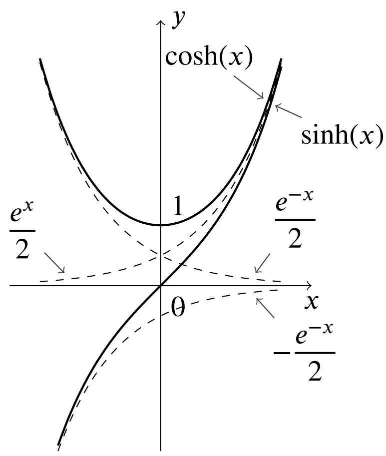
Summary of essential material

The hyperbolic sine and cosine. The *hyperbolic cosine* and *hyperbolic sine* functions are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

for all $x \in \mathbb{R}$. They share many properties with the cosine and sine functions as shown in some questions below.

The graph of the hyperbolic cosine function is the shape of a hanging cable or chain attached at two ends.



Functions. Let A and B be sets. A *function* $f\colon A \to B$ is a rule which assigns exactly one element of B to each element of A. We write $x \mapsto f(x)$ to indicate the value $f(x)$ assigned to x. The set A is called the *domain* of f, the set B the *codomain* of f. The *image* or *range* of f is $\operatorname{im}(f) = \{f(a) \mid a \in A\} \subseteq B$.

The function f is *surjective* or *onto* if $\operatorname{im}(f) = B$. To show that f is surjective one has to show that for every $y \in B$ there exists $x \in A$ such that $f(x) = y$.

The function f is *injective* or *one-to-one* if every point in the image comes from exactly one element in the domain. To show a function is injective prove

$$\big(x_1, x_2 \in A \text{ and } f(x_1) = f(x_2)\big) \Rightarrow x_1 = x_2$$

(the converse is obvious by definition of a function). The above means for all choices of x_1, x_2 with $f(x_1) = f(x_2)$ the implication has to be true.

The function f is *bijective* or *invertible* if it is both injective and surjective. In that case there exists an *inverse function* is the function $f^{-1}\colon B \to A$ defined by

$$f^{-1}(y) = \big(\text{the unique element } x \in A \text{ such that } f(x) = y\big).$$

In practice, to find f^{-1} we solve the equation $y = f(x)$ for $x \in A$.

Questions to complete during the tutorial

1. Let $f(x) = x^2$, considered as a function $f : A \to B$ for the various A and B listed below. In each case decide whether f is injective and whether f is surjective.

(a) $f : \mathbb{R} \to \mathbb{R}$

Solution: The range of the function is $[0, \infty)$, and so the function is not surjective. Furthermore, since $f(-1) = 1 = f(1)$ the function is not injective.

(b) $f : [-1, 2] \to [0, 4]$

Solution: Since the range of the function is $[0, 4]$, this function is surjective. Since $f(-1) = 1 = f(1)$ it is not injective.

(c) $f : [0, 1] \to [0, 1]$

Solution: The range is $[0, 1]$, and hence the function is surjective. If $f(x_1) = f(x_2)$ with $x_1, x_2 \in [0, 1]$ then $x_1^2 = x_2^2$, and so $x_1 = x_2$ (since $x_1, x_2 \geq 0$), and hence f is injective. Thus the function is bijective.

(d) $f : [0, \infty) \to [0, \infty)$

Solution: The range is $[0, \infty)$, and so the function is surjective. It is also injective (as in the previous part).

(e) $f : \mathbb{N} \to \mathbb{N}$

Solution: This function is injective (as in the previous parts). However it is not surjective, because the range is $\{n^2 \mid n \in \mathbb{N}\} = \{0, 1, 4, 9, 16, \dots\} \neq \mathbb{N}$.

(f) $f : \mathbb{Q} \to [0, \infty)$

Solution: This function is not injective (since $f(-1) = 1 = f(1)$). It is also not surjective, because, for example, 2 is not in the range of the function (for otherwise $\sqrt{2}$ would be rational, and it isn't!).

2. (a) Show that $\cosh^2 x - \sinh^2 x = 1$ for all $x \in \mathbb{R}$.

Solution: Directly from the definition of cosh and sinh in terms of the exponential function, we have

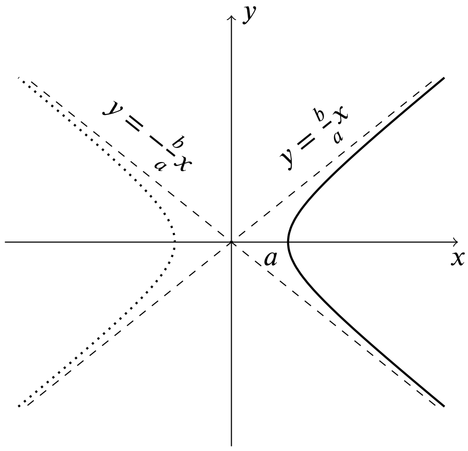
$$\cosh^2 x - \sinh^2 x = \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = 1.$$

(b) Let $a, b > 0$. Show that $x(t) = a\cosh(t)$, $y(t) = b\sinh(t)$ $(t \in \mathbb{R})$ is a parametric representation of one branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Solution: Since $\cosh^2 t - \sinh^2 t = 1$ we have

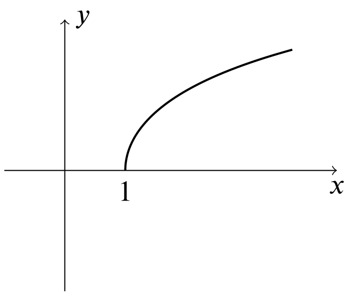
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{a^2\cosh^2(t)}{a^2} - \frac{b^2\sinh^2(t)^2}{b^2} = \cosh^2(t) - \sinh^2(t) = 1$$

from the previous part. As $x(t) = \cosh(t) > 0$ for all $t \in \mathbb{R}$ we only obtain the right branch of the hyperbola. The dashed lines are the asymptotes.

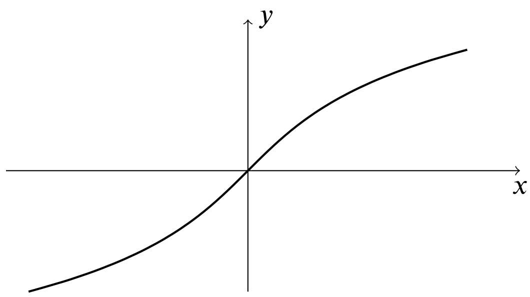


(c) Explain, using the graphs, why $\sinh : \mathbb{R} \to \mathbb{R}$ and $\cosh : [0, \infty) \to [1, \infty)$ are bijective. Sketch the graphs of the inverse functions.

Solution: We can deduce injectivity using the horizontal line test (note that we are restricting the domain of cosh, and so we are only considering the part of the graph with $x \geq 0$). Surjectivity is also clear from the graphs. The graphs are obtained by reflection at the line $y = x$, giving:



$\cosh^{-1} : [1, \infty) \to [0, \infty)$



$\sinh^{-1} : \mathbb{R} \to \mathbb{R}$

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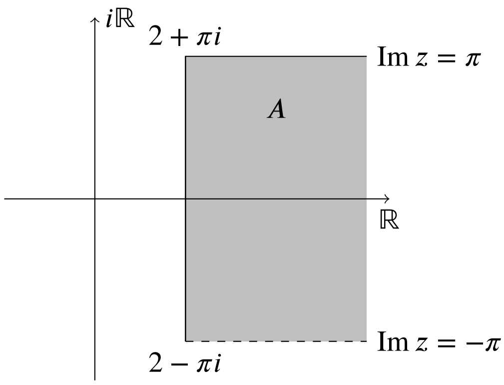
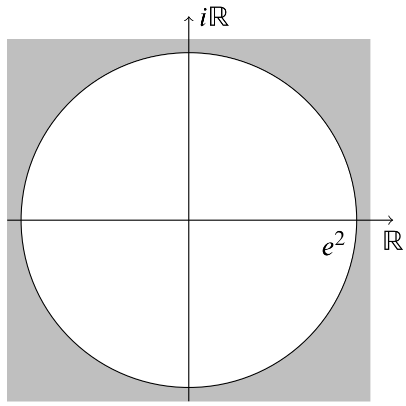
3. Let $A = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 2 \text{ and } -\pi < \mathrm{Im}(z) \leq \pi\}$, and let B be the image of A under $f(z) = e^z$.

(a) Sketch A and B, and show that $f : A \to B$ is bijective.

Solution:



z-plane



w-plane

This function is necessarily surjective (because the codomain is defined to be the range for this function). To prove injectivity, suppose that $z_1, z_2 \in A$ with $e^{z_1} = e^{z_2}$. Writing $z_1 = z_1 + iy_1$ and $z_2 = z_2 + iy_2$ we have $e^{x_1}e^{iy1} = e^{x_2}e^{iy_2}$. Equating the modulii of these complex numbers and using that $|e^{iy}| = 1$ gives $e^{x_1} = e^{x_2}$, and so $x_1 = x_2$. Hence $e^{iy_1} = e^{iy_2}$ and thus $y_1 = y_2 + 2k\pi$ for some $k \in \mathbb{Z}$, however since $-\pi < y_1 \leq \pi$ and $-\pi < y_2 \leq \pi$ we have $k = 0$, and thus $y_1 = y_2$. Thus $z_1 = z_2$ and the function is injective.

(b) Find a formula for the inverse function $f^{-1} : B \to A$.

Solution: If $w = e^z = e^x(\cos y + i \sin y)$ then $|w| = e^x$ and $\mathrm{Arg}(w) = y$ (because $-\pi < y \leq \pi$). Thus $x = \ln |w|$ and $y = \mathrm{Arg}(w)$, and so the inverse function is

$$f^{-1}(w) = \ln |w| + i \,\mathrm{Arg}(w).$$

4. A function $f : \mathbb{R} \to \mathbb{R}$ is called *strictly increasing* if $x_1 < x_2$ implies that $f(x_1) < f(x_2)$.

(a) Show that if f is strictly increasing then f is injective.

Solution: If $x_1 \neq x_2$ then either $x_1 < x_2$ or $x_2 < x_1$. In the first case we have $f(x_1) < f(x_2)$, and in the second case $f(x_2) < f(x_1)$. In both cases we have $f(x_1) \neq f(x_2)$, and so f is injective.

(b) Show that if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are strictly increasing, then the composition $g \circ f : \mathbb{R} \to \mathbb{R}$ is strictly increasing. Deduce that $g \circ f$ is injective.

Solution: Let $x_1 < x_2$. Since $f(x)$ is increasing we have $f(x_1) < f(x_2)$, and since $g(x)$ is increasing, we have $g(f(x_1)) < g(f(x_2))$. Thus $g \circ f$ is increasing. Thus $g \circ f$ is injective by the previous part.

(c) Using the result of the previous part, and the fact that e^x is strictly increasing, prove that $\cosh : [0, \infty) \to \mathbb{R}$ is strictly increasing, and hence injective.

Solution: By definition, $\cosh x = g(e^x)$ where $g(x) = \frac{x+x^{-1}}{2}$. To apply the result of the previous part, we take $E = [0, \infty)$. Certainly e^x is increasing on E, and when restricted to this domain its range is $D = [1, \infty)$. So all we need to check is that $g(x)$ is increasing on D. In other words, we need to show that if $1 \leq x_1 < x_2$, then

$$\frac{x_1 + x_1^{-1}}{2} < \frac{x_2 + x_2^{-1}}{2}.$$

After multiplying by 2 and rearranging, this inequality becomes

$$(x_2 - x_1)(1 - x_1^{-1}x_2^{-1}) > 0,$$

which is true because both factors on the left-hand side are strictly positive. So we have proved that $\cosh x$ is increasing on $[0, \infty)$.

Remark: You could also use some calculus to check that $\cosh x : [0, \infty) \to \mathbb{R}$ is strictly increasing, because $f'(x) = \sinh x > 0$ for $x > 0$. This implies that $\cosh x : [0, \infty) \to \mathbb{R}$ is strictly increasing (the proof of this fact will come when we discuss the Mean Value Theorem in a few weeks!).

5. Each formula below belongs to a function $f : A \to B$ where we take $A \subseteq \mathbb{R}$ to be the natural domain of f, and we take the codomain B to be the image of the natural domain under f. Thus each function is automatically surjective. In each case find A, and decide if the function $f : A \to B$ is a bijection. If so, find a formula for the inverse function.

(a) $f(x) = \dfrac{x-2}{x+2}$,

Solution: The natural domain (also known as the domain of definition) of f is $\mathbb{R} \setminus \{-2\}$, that is, the set of all real numbers except -2. Observe that for any x, $\frac{x-2}{x+2} \neq 1$, and so 1 is not in the range. The number y is in the range if the equation $y = \frac{x-2}{x+2}$ has at least one solution for x in the domain. Rearranging this equation gives $x = \frac{2(1+y)}{1-y}$, showing that for any $y \neq 1$, there is one and only one x, namely $x = \frac{2(1+y)}{1-y}$, such that $y = \frac{x-2}{x+2}$. Hence the range of f is $\mathbb{R} \setminus \{1\}$ and the function

$$f : \mathbb{R} \setminus \{-2\} \to \mathbb{R} \setminus \{1\}, \qquad f(x) = \frac{x-2}{x+2}$$

is a bijection. The inverse function is

$$f^{-1} : \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{-2\}, \qquad f^{-1}(y) = \frac{2(1+y)}{1-y}.$$

(The graph of f is a hyperbola with vertical asymptote $x = -2$ and horizontal asymptote $y = 1$.)

(b) $f(x) = \sqrt{2+5x}$,

Solution: The natural domain is $[-\frac{2}{5}, \infty)$. The range of f is $[0, \infty)$. For any $y \geq 0$, there is one and only one x such that $y = \sqrt{2+5x}$, namely $x = \frac{1}{5}(y^2 - 2)$. Hence

$$f : [-\tfrac{2}{5}, \infty) \to [0, \infty)$$

is a bijection and the inverse function is

$$f^{-1} : [0, \infty) \to [-\tfrac{2}{5}, \infty), \qquad f^{-1}(y) = \tfrac{1}{5}(y^2 - 2).$$

(c) $f(x) = x|x| + 1$.

Solution: The natural domain is $\mathbb{R}$. When $x \geq 0$, $f(x) = x^2 + 1$ and $f(x)$ takes all values in $[1, \infty)$. When $x \leq 0$, $f(x) = 1 - x^2$ and $f(x)$ takes all values in $(-\infty, 1]$. Hence the range of f is $\mathbb{R}$. The function is injective as it is an increasing function on each of the intervals $[0, \infty)$ and $(-\infty, 0]$, and therefore increasing on the whole of $\mathbb{R}$. Hence $f : \mathbb{R} \to \mathbb{R}$ is a bijection and its inverse function is $f^{-1} : \mathbb{R} \to \mathbb{R}$, where

$$f^{-1}(y) = \begin{cases} -\sqrt{1-y}, & y < 1, \\ \sqrt{y-1}, & y \geq 1. \end{cases}$$

6. (a) The function $\cosh : [0, \infty) \to [1, \infty)$ is a bijection, so has an inverse $\cosh^{-1} : [1, \infty) \to [0, \infty)$. Show that $\cosh^{-1}(x) = \ln\big(x + \sqrt{x^2 - 1}\big)$.

Solution: To find $cosh^{-1}$ we have to solve $y = \cosh x = \frac{e^x + e^{-x}}{2}$ for x. Multiplying the equation by $2e^x$ we obtain $2y = e^x + e^{-x}$. Thus $e^{2x} - 2ye^x + 1 = 0$ which is reducible to a quatratic $\big(e^x\big)^2 - 2ye^x + 1 = 0$. Solving that quadratic using the usual formula yields

$$e^x = \frac{2y \pm \sqrt{4y^2 - 4}}{2} = y \pm \sqrt{y^2 - 1},$$

so that $x = \ln(y \pm \sqrt{y^2 - 1})$. We must choose the positive sign to ensure that $x \geq 0$, because $y - \sqrt{y^2 - 1} < 1$ for $y > 1$. Thus we get $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$.

(b) The function $\cosh : (-\infty, 0] \to [1, \infty)$ is also a bijection. Find a formula for its inverse function.

Solution: If $x \in (-\infty, 0)$, we would take the negative sign, to obtain $\cosh^{-1} x = \ln(x - \sqrt{x^2 - 1})$.

7. For what values of the constants a, b, c (with $b \neq 0$) is the function with formula

$$f(x) = \frac{x - a}{bx - c} \qquad \text{and domain } \{x \in \mathbb{R} \mid x \neq c/b\}$$

equal to its own inverse? (*Hint*: It may help to draw the graph.)

Solution: Using the fact that $b \neq 0$, we can rewrite the formula for $f(x)$ as follows:

$$f(x) = \frac{1}{b} + \frac{\frac{c-ab}{b^2}}{x - \frac{c}{b}}.$$

Notice that if $c = ab$, then this becomes the constant function $f(x) = \frac{1}{b}$, which is clearly not injective and hence has no inverse. So we must have $c \neq ab$. Then the graph of $f(x)$ is a hyperbola, with vertical asymptote at $x = \frac{c}{b}$, and horizontal asymptote at $y = \frac{1}{b}$. So the domain of f is $\mathbb{R} \setminus \{\frac{c}{b}\}$, and the range of f is $\mathbb{R} \setminus \{\frac{1}{b}\}$. Setting $y = f(x)$ and solving for x, we get a formula for the inverse function:

$$f^{-1}(y) = \frac{c}{b} + \frac{\frac{c-ab}{b^2}}{y - \frac{1}{b}},$$

which is another hyperbola, this time with domain $\mathbb{R} \setminus \{\frac{1}{b}\}$ and range $\mathbb{R} \setminus \{\frac{c}{b}\}$. In order to have $f = f^{-1}$, the domain of f must equal the domain of f^{-1} (which is the range of f), so we must have $c = 1$. Conversely, if $c = 1$ then the two formulas become the same, so f does equal its own inverse. To sum up, the answer to the question is that f is equal to its own inverse whenever $b \neq 0$, $c = 1$, and $a \neq 1/b$.

8. Prove the hyperbolic "sum of angles" formulae, for all $x, y \in \mathbb{R}$:

(a) $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Solution: We have

$$\begin{aligned}
&\cosh x \cosh y + \sinh x \sinh y \\
&\qquad = \left[\tfrac{1}{2}(e^x + e^{-x})\right]\left[\tfrac{1}{2}(e^y + e^{-y})\right] + \left[\tfrac{1}{2}(e^x - e^{-x})\right]\left[\tfrac{1}{2}(e^y - e^{-y})\right] \\
&\qquad = \tfrac{1}{4}\left[(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}) + (e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y})\right] \\
&\qquad = \tfrac{1}{4}(2e^{x+y} + 2e^{-x-y}) \\
&\qquad = \tfrac{1}{2}(e^{x+y} + e^{-(x+y)}) \\
&\qquad = \cosh(x+y)
\end{aligned}$$

(b) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.

Solution: We have

$$\begin{aligned}
&\sinh x \cosh y + \cosh x \sinh y \\
&\qquad = \left[\tfrac{1}{2}(e^x - e^{-x})\right]\left[\tfrac{1}{2}(e^y + e^{-y})\right] + \left[\tfrac{1}{2}(e^x + e^{-x})\right]\left[\tfrac{1}{2}(e^y - e^{-y})\right] \\
&\qquad = \tfrac{1}{4}\left[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})\right] \\
&\qquad = \tfrac{1}{4}(2e^{x+y} - 2e^{-x-y}) \\
&\qquad = \tfrac{1}{2}(e^{x+y} - e^{-(x+y)}) \\
&\qquad = \sinh(x+y)
\end{aligned}$$

Extra questions for further practice

9. Suppose that $f : A \to B$ is bijective. Define what is meant by the inverse function $f^{-1} : B \to A$, and explain why this definition makes sense.

Solution: Recall that the inverse function of a bijective function $f : A \to B$ is the function $f^{-1} : B \to A$ such that

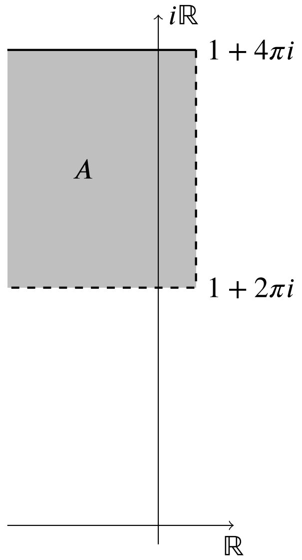
$$f^{-1}(b) = (\text{the unique } a \in A \text{ such that } f(a) = b).$$

To see that this makes sense, note that for each $b \in B$ there is at least one element $a \in A$ such that $f(a) = b$ (because $f : A \to B$ is *surjective*), and this element is unique because $f : A \to B$ is *injective*.

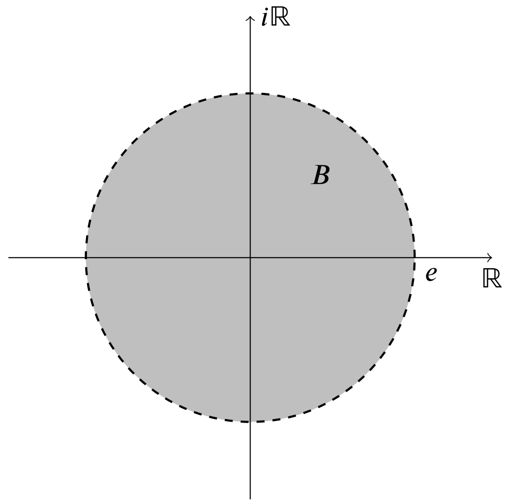
10. Let $A = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 1 \text{ and } 2\pi < \mathrm{Im}(z) \le 4\pi\}$, and let B be the image of A under $f(z) = e^z$.

(a) Sketch A and B, and show that $f : A \to B$ is bijective.

Solution:



z-plane



w-plane

This function is necessarily surjective (because the codomain is defined to be the range for this function). To prove injectivity, suppose that $z_1, z_2 \in A$ with $e^{z_1} = e^{z_2}$. Writing $z_1 = z_1 + iy_1$ and $z_2 = z_2 + iy_2$ we have $e^{x_1}(\cos y_1 + i\sin y_1) = e^{x_2}(\cos y_2 + i\sin y_2)$. Equating the modulii of these complex numbers gives $e^{x_1} = e^{x_2}$, and so $x_1 = x_2$. Also, $y_1 = y_2 + 2k\pi$ for some $k \in \mathbb{Z}$, however since $2\pi < y_1 \leq 4\pi$ and $2\pi < y_2 \leq 4\pi$ we have $k = 0$, and thus $y_1 = y_2$. Thus $z_1 = z_2$ and the function is injective.

(b) Find a formula for the inverse function $f^{-1}\colon B \to A$.

Solution: If $w = e^z = e^x(\cos y + i\sin y)$ with $z = x + iy \in A$ then $|w| = e^x$. Moreover, $\mathrm{Arg}(w) = y - 3\pi$, because y is an argument of w, and since $2\pi < y \leq 4\pi$ we have $-\pi < y - 3\pi \leq \pi$, thus $y - 3\pi$ is the principal argument of w. Thus $x = \ln|w|$ and $y = \mathrm{Arg}(w) + 3\pi$, and so the inverse function is

$$f^{-1}(w) = \ln|w| + i(3\pi + \mathrm{Arg}(w)).$$

You should compare the with Question 3.

11. Let $f(x) = x^3$, considered as a function $f\colon A \to B$ for the various A and B listed below. In each case decide whether f is injective and weather f is surjective.

(a) $f\colon \mathbb{R} \to \mathbb{R}$

Solution: Bijective.

(b) $f\colon \mathbb{Z} \to \mathbb{Z}$

Solution: Injective, but not surjective.

(c) $f\colon \mathbb{Q} \to \mathbb{Q}$

Solution: Injective, but not surjective.

(d) $f\colon \{-1, 0, 2\} \to \{-1, 0, 8\}$

Solution: Bijective.

(e) $f\colon [0, 1] \to [-1, 1]$

Solution: Injective, not surjective.

(f) $f\colon [0, \infty) \to [0, \infty)$

Solution: Bijective.

12. Explain why the functions given by the formulas and domains below are injective. Find their ranges and formulas for their inverses.

(a) $f(x) = x^2 + x, \ x \geq -\frac{1}{2}$.

Solution: Because $x \mapsto x^2$ is an increasing function on $[0, \infty)$, $f(x) = (x + \frac{1}{2})^2 - \frac{1}{4}$ is an increasing function on the domain $[-1/2, \infty)$. It is therefore injective. As x runs over $[-1/2, \infty)$, $(x + \frac{1}{2})^2$ runs over $[0, \infty)$, so $f(x)$ runs over $[-1/4, \infty)$. Thus the range is $[-1/4, \infty)$.

Solving the equation $y = x^2 + x$ for x gives $x = -\frac{1}{2} \pm \sqrt{y + \frac{1}{4}}$. As we are only interested in the case that $x \geq -\frac{1}{2}$, we take the positive square root. Thus we get the following rule for the inverse function:

$$f^{-1}(y) = -\frac{1}{2} + \sqrt{y + \frac{1}{4}}.$$

(b) $g(x) = \sqrt[4]{x}, \ x \geq 0$.

Solution: $g(x)$ is injective by definition, because it is defined to be the inverse of the bijection $f\colon [0, \infty) \to [0, \infty)$ given by $f(y) = y^4$. (To spell out the proof of injectivity for this particular case, if we have $\sqrt[4]{x_1} = \sqrt[4]{x_2}$ then we must have $x_1 = x_2$, just by raising both sides to the fourth power.) So the range of $g(x)$ is $[0, \infty)$ and its inverse is

$$g^{-1}(y) = y^4.$$

(c) $h(x) = \dfrac{1+e^x}{1-e^x}$, $x \neq 0$.

Solution: Note that $h(x) = -1 + \frac{2}{1-e^x}$. Because $x \mapsto e^x$ is an increasing function, $x \mapsto 1-e^x$ is a decreasing function, so h is increasing on $(-\infty, 0)$ and also on $(0, \infty)$. It is not, however, increasing on the whole domain $\mathbb{R} \setminus \{0\}$: for example, $h(-1) > h(1)$. Nevertheless, h is injective, because it takes positive values on $(-\infty, 0)$ and negative values on $(0, \infty)$, so no horizontal line cuts the graph more than once. In fact, as x runs over $(-\infty, 0)$, $h(x)$ takes all values in $(1, \infty)$; and as x runs over $(0, \infty)$, $h(x)$ takes all values in $(-\infty, -1)$. So the range of h is $(-\infty, -1) \cup (1, \infty) = \mathbb{R}\backslash[-1, 1]$. To find the inverse function, set $y = h(x) = -1 + \frac{2}{1-e^x}$. Rearranging gives $e^x = 1 - \frac{2}{y+1}$, so $x = \ln(1 - \frac{2}{y+1}) = \ln(\frac{y-1}{y+1})$. Thus, for $y < -1$ or $y > 1$, we get:

$$h^{-1}(y) = \ln\left(\frac{y-1}{y+1}\right).$$

(d) $f(x) = \ln(3 + \sqrt{x-4})$, $x \geq 5$.

Solution: f is injective because it is strictly increasing on the given domain;this is because $\sqrt{x}$ is an increasing function and so is $\ln(x)$. As x runs over $[5, \infty)$, $\sqrt{x-4}$ runs over $[1, \infty)$, so $\ln(3 + \sqrt{x-4})$ runs over $[\ln(4), \infty)$. Thus the range is $[\ln(4), \infty)$. Solving the equation $y = \ln(3 + \sqrt{x-4})$ for x gives $x = (e^y - 3)^2 + 4$, so the formula for the inverse function is

$$f^{-1}(y) = (e^y - 3)^2 + 4.$$

13. Is the following statement true or false? "A function $f : \mathbb{R} \to \mathbb{R}$ is injective if and only if f is either strictly increasing or strictly decreasing." If you think it is true, give a proof. If you think it is false, give a counterexample.

Solution: It is true to say that if f is strictly increasing or strictly decreasing, then f is injective. For if x_1 and x_2 are distinct elements in the domain such that $x_1 < x_2$, then either $f(x_1) > f(x_2)$ (if f is decreasing) or $f(x_1) < f(x_2)$ (if f is increasing). In both cases, $f(x_1) \neq f(x_2)$. This shows that distinct inputs produce distinct outputs. However, the converse is not true. As a counter-example, consider the function f with domain $\mathbb{R}$ given by the formula

$$f(x) = \begin{cases} x, & x < 0, \\ 1, & x = 0, \\ x, & 0 < x < 1, \\ 0, & x = 1, \\ x, & x > 1. \end{cases}$$

This is an injective function by the horizontal line test, but is not an increasing function, as $0 < 1$ but $f(0) \nless f(1)$.

14. Give an example of functions $f : A \to B$ and $g : B \to C$ such that g is surjective yet the composition function $g \circ f : A \to C$ is not surjective.

Solution: Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$, and let $g : \mathbb{R} \to \mathbb{R}$ be the function $g(x) = x$. Then g is surjective, yet $g \circ f : \mathbb{R} \to \mathbb{R}$ is the function $(g \circ f)(x) = x^2$ is not surjective.

15. Last week you proved a closed formula for $1 + 2\cos x + 2\cos 2x + \cdots + 2\cos nx$. Find a corresponding 'hyperbolic' version for $1 + 2\cosh x + 2\cosh 2x + \cdots + 2\cosh nx$.

Solution: Since $\cosh x = \dfrac{e^x + e^{-x}}{2}$ we have, using the formula for the summation of a geometric series,

$$\begin{aligned} 1 + 2\sum_{k=1}^{n} \cosh kx &= 1 + \sum_{k=1}^{n} (e^{kx} + e^{-kx}) = 1 + \sum_{k=1}^{n} (e^x)^k + \sum_{k=1}^{n} (e^{-x})^k \\ &= 1 + \frac{e^x - e^{(n+1)x}}{1 - e^x} + \frac{e^{-x} - e^{-(n+1)x}}{1 - e^{-x}}, \end{aligned}$$

where we have used the geometric sum formula. In the first fraction we multiply the numerator and denominator by $e^{-x/2}$ and in the second fraction we multiply the numerator and denominator by $e^{x/2}$, giving

$$\begin{aligned} 1 + 2\cosh x + \cdots + 2\cosh nx &= 1 + \frac{-e^{x/2} + e^{(n+\frac{1}{2})x} + e^{-x/2} - e^{-(n+\frac{1}{2})x}}{e^{x/2} - e^{-x/2}} \\ &= \frac{e^{(n+\frac{1}{2})x} - e^{-(n+\frac{1}{2})x}}{e^{x/2} - e^{-x/2}} \\ &= \frac{\frac{1}{2}\left(e^{(n+\frac{1}{2})x} - e^{-(n+\frac{1}{2})x}\right)}{\frac{1}{2}\left(e^{x/2} - e^{-x/2}\right)} = \frac{\sinh(n+\frac{1}{2})x}{\sinh\frac{x}{2}}. \end{aligned}$$

This identity if valid for all $x \neq 0$. Alternatively, noting that $\cosh x = \cos(ix)$ we could have used the formula from last week to derive the above identity.

16. Show that if $f : A \to B$ is bijective, then the inverse function $f^{-1} : B \to A$ is also bijective.

Solution: To see that $f^{-1} : B \to A$ is injective, suppose that $f^{-1}(b_1) = f^{-1}(b_2)$. Then $f(f^{-1}(b_1)) = f(f^{-1}(b_2))$, and so $b_1 = b_2$ (since $f(f^{-1}(x)) = x$). To see that $f^{-1} : B \to A$ is surjective, note that if $a \in A$ then taking $b = f(a)$ we have $f^{-1}(b) = f^{-1}(f(a)) = a$, and so f^{-1} is surjective. Thus the inverse function is bijective.

17. Let A, B and C be sets and let $f : A \to B$ and $g : B \to C$ be functions.

(a) Show that if f and g are injective then the composition $g \circ f : A \to C$ is also injective.

Solution: Suppose that $g(f(a)) = g(f(b))$. Then $f(a) = f(b)$ (because g is injective) and then $a = b$ (because f is injective). So the composition is injective.

(b) Show that if f and g are surjective then the composition $g \circ f : A \to C$ is also surjective.

Solution: Let $c \in C$. Since $g : B \to C$ is surjective there is $b \in B$ with $g(b) = c$. Then, since $f : A \to B$ is surjective, there is $a \in A$ with $f(a) = b$. Thus $g(f(a)) = g(b) = c$, and so $g \circ f : A \to C$ is surjective.

(c) Deduce that the composition of bijections is again a bijection, and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Solution: This is immediate from the previous two parts.

Challenge questions (optional)

18. We say that the set A has the same *cardinality* as the set B if there exists a bijection $f : A \to B$.

(a) Show that if A has the same cardinality as B, then B has the same cardinality as A. That is, show that if there is a bijection $f : A \to B$ then there is a bijection $g : B \to A$.

Solution: If A and B have the same cardinality then there is a bijection $f : A \to B$, and then by Question 16 the inverse function $f^{-1} : B \to A$ is a bijection, and so B has the same cardinality as A.

(b) Show that if A has the same cardinality as B, and B has the same cardinality as C, then A has the same cardinality as C.

Solution: If A has the same cardinality as B then there is a bijection $f : A \to B$, and if B has the same cardinality as C then there is a bijection $g : B \to C$, and hence $g \circ f : A \to C$ is a bijection (by Question 17), and hence A has the same cardinality as C.

(c) Show that if A and B have finitely many elements then A and B have the same cardinality if and only if A and B have the same number of elements.

Solution: Suppose that A and B are finite sets, and write $|A|$ and $|B|$ for the number of elements in each set.

If A and B have the same cardinality, then there is a bijection $f : A \to B$. Since $f : A \to B$ is injective we have $|A| \leq |B|$ (because the elements $f(a)$ with $a \in A$ give $|A|$ distinct elements of B). Since $f : A \to B$ is surjective, we have $|A| \geq |B|$ (because the elements $f(a)$ with $a \in A$ give all of the elements of B). Thus $|A| = |B|$.

On the other hand, suppose that $|A| = |B| = n$, say. Let the elements of A be $a_1, \dots, a_n$ and the elements of B be $b_1, \dots, b_n$. Define $f : A \to B$ by $f(a_i) = b_i$ for $i = 1, \dots, n$. This is a bijection, and so A and B have the same cardinality.

19. We say that a set A has the same *cardinality* as the set $\mathbb{N}$ of natural numbers if there is a bijection $f : \mathbb{N} \to A$. In this case we say that A is *countable*. This means that we can write all of the elements of A in a list in which every element occurs exactly once:

$$a_0, a_1, a_2, \dots,$$

where $f(j) = a_j$ is a bijection $f : \mathbb{N} \to A$. Thus, morally, A has the "same size" as $\mathbb{N}$, because the elements of A are paired-up bijectively with the elements of $\mathbb{N}$.

(a) Show that $\mathbb{Z}$ is countable.

Solution: We list the elements of $\mathbb{Z}$ as

$$0, 1, -1, 2, -2, 3, -3, 4, -4, 5, -5, \dots.$$

Thus we have a bijection $f : \mathbb{N} \to \mathbb{Z}$ given by $f(0) = 0$, $f(1) = 1$, $f(2) = -1$, $f(3) = 2$, $f(4) = -2$, and so on. The general formula is

$$f(n) = \begin{cases} -n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

(b) Show that the set $\mathbb{N} \times \mathbb{N} = \{(m, n) \mid m \in \mathbb{N} \text{ and } n \in \mathbb{N}\}$ is countable.

Solution: We list the elements of $\mathbb{N} \times \mathbb{N}$ according to the sum of their entries:

$$\begin{array}{lllll}
(0,0) & & & & \\
(0,1) & (1,0) & & & \\
(0,2) & (1,1) & (2,0) & & \\
(0,3) & (1,2) & (2,1) & (3,0) & \\
(0,4) & (1,3) & (2,2) & (3,1) & (4,1) \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

Every element of $\mathbb{N} \times \mathbb{N}$ will occur in this list exactly once (specifically, the element (m, n) will occur as the $(m+1)$-th entry in the $(m+n+1)$-th row). This gives a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by setting:

$$\begin{array}{lllll}
f(0,0) = 0 & & & & \\
f(0,1) = 1 & f(1,0) = 2 & & & \\
f(0,2) = 3 & f(1,1) = 4 & f(2,0) = 5 & & \\
f(0,3) = 6 & f(1,2) = 7 & f(2,1) = 8 & f(3,0) = 9 & \\
f(0,4) = 10 & f(1,3) = 11 & f(2,2) = 12 & f(3,1) = 13 & f(4,1) = 14 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}$$

(c) Show that if A and B are countable then the set $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$ is also countable.

Solution: If A and B are countable then there is a bijection $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$. Let $h : A \times B \to \mathbb{N} \times \mathbb{N}$ be the function

$$h(a, b) = (f(a), g(b)).$$

We claim that h is a bijection. Firstly, h is injective, for:

$$\begin{aligned} h(a, b) = h(a', b') \quad &\Rightarrow \quad (f(a), g(b)) = (f(a'), g(b')) \\ &\Rightarrow \quad f(a) = f(a') \text{ and } g(b) = g(b') \\ &\Rightarrow \quad a = a' \text{ and } b = b' \text{ (as } f \text{ and } g \text{ injective)} \\ &\Rightarrow \quad (a, b) = (a', b'). \end{aligned}$$

Next, h is surjective, for if $(m, n) \in \mathbb{N} \times \mathbb{N}$ then by the surjectivity of $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$ there are elements $a \in A$ and $b \in B$ with $f(a) = m$ and $g(b) = n$. Therefore $h(a, b) = (f(a), g(b)) = (m, n)$.

Thus $h : A \times B \to \mathbb{N} \times \mathbb{N}$ is a bijection, and so $A \times B$ has the same cardinality as $\mathbb{N} \times \mathbb{N}$. Then by the previous part $\mathbb{N} \times \mathbb{N}$ has the same cardinality as $\mathbb{N}$, and so by Question 18(c) the set $A \times B$ has the same cardinality as $\mathbb{N}$, and hence is countable.

(d) Show that the set $X = \mathbb{Q} \cap [0, 1)$ of all rational numbers in the interval $[0, 1)$ is countable.

Solution: We list the rational numbers $r = \frac{m}{n}$ with $0 \leq r < 1$ according to the size of their denominators:

$$\begin{array}{ccccccc} 0/1 & & & & & & \\ 0/2 & 1/2 & & & & & \\ 0/3 & 1/3 & 2/3 & & & & \\ 0/4 & 1/4 & 2/4 & 3/4 & & & \\ 0/5 & 1/5 & 2/5 & 3/5 & 4/5 & & \\ 0/6 & 1/6 & 2/6 & 3/6 & 4/6 & 5/6 & \\ 0/7 & 1/7 & 2/7 & 3/7 & 4/7 & 5/7 & 6/7 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Unlike in part (b), there are repetitions in this list resulting from the fact that the representation of rational numbers is not unique, for example $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \cdots$. Crossing out these duplicates we obtain a list in which every rational number r with $0 \leq r < 1$ occurs exactly once:

$$\begin{array}{ccccccc} 0/1 & & & & & & \\ & 1/2 & & & & & \\ & 1/3 & 2/3 & & & & \\ & 1/4 & & 3/4 & & & \\ & 1/5 & 2/5 & 3/5 & 4/5 & & \\ & 1/6 & & & & 5/6 & \\ & 1/7 & 2/7 & 3/7 & 4/7 & 5/7 & 6/7 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

This produces a bijection $f : X \to \mathbb{N}$, and so X is countable. (Note: Another way of thinking of the "crossing out" of duplicates is to insist that when constructing the list above we only include expressions m/n with m and n having no factors in common. This will avoid the duplication).

(e) Deduce that the set $\mathbb{Q}$ of all rational numbers is countable.
Remark: This is rather surprising, since intuitively $\mathbb{Q}$ feels a lot "bigger" than $\mathbb{N}$.

Solution: Let X be as in the previous part. We first observe that the function

$$f : \mathbb{Z} \times X \to \mathbb{Q} \quad \text{with} \quad f(n, r) = n + r$$

is a bijection. To check that f is injective, note that if $f(n_1, r_1) = f(n_2, r_2)$ then $n_1 + r_1 = n_2 + r_2$, and so $r_1 - r_2 = n_2 - n_1$. Thus $r_1 - r_2$ is an integer, which forces $r_1 = r_2$ (because $0 \leq r_1 < 1$ and $0 \leq r_2 < 1$ implies that $-1 < r_1 - r_2 < 1$). Thus $n_2 - n_1 = 0$ also, and so $(n_1, r_1) = (n_2, r_2)$.
To check that f is surjective, note that each rational number can be written as m/n with $n > 0$. Then by the division-remainder theorem for integers we can write

$$m = qn + r \quad \text{for some } q, r \in \mathbb{Z} \text{ with } 0 \leq r < n.$$

Thus

$$\frac{m}{n} = q + \frac{r}{n} = f(q, r/n).$$

Thus $\mathbb{Z} \times X$ and $\mathbb{Q}$ have the same cardinality. Since both X and $\mathbb{Z}$ are countable (by parts (a) and (d)) it follows from part (c) that $\mathbb{Q}$ is also countable.

(f) So perhaps every infinite set is countable? No: Show that the set of real numbers in the interval $[0, 1]$ is *not* countable.
Note: This is tough if you haven't seen something like it before!

Solution: Suppose, for a contradiction, that $[0, 1]$ is countable. Thus we can list all of the numbers on $[0, 1]$ as $r_1, r_2, r_3, r_4, \ldots$ such that every number in $[0, 1]$ appears exactly once in this list. Writing the decimal expansions of these numbers we have a list:

$$\begin{aligned} r_1 &= 0.a_{11}a_{12}a_{13}a_{14}\cdots \\ r_2 &= 0.a_{21}a_{22}a_{23}a_{24}\cdots \\ r_3 &= 0.a_{31}a_{32}a_{33}a_{34}\cdots \\ r_4 &= 0.a_{41}a_{42}a_{43}a_{44}\cdots \\ &\vdots \end{aligned}$$

We need to be a tiny bit careful here, because decimal expansions are not quite unique, for example

$$0.1499999999\cdots = 0.150000000\cdots$$

(prove this!). So in the decimal expansions above we stipulate that we do always take the representative that ends in $\cdots 99999 \cdots$, and so, for example, we take $0.1499999\cdots$ rather than $0.1500000\cdots$. With this proviso, the decimal expansions given above for $r_1, r_2, \ldots$ are uniquely determined, and by the assumption that $[0, 1]$ is countable, every decimal expansion $0.a_1a_2a_3a_4\cdots$ with no trailing zeros must occur on our list.
Now we construct a number $r \in [0, 1]$ and show that it is not on our list. This contradicts the assumption that our list was complete, and hence $[0, 1]$ and $\mathbb{N}$ do not have the same cardinality. We construct r as

$$r = 0.a_1a_2a_4a_4\cdots \quad \text{where} \quad a_i = \begin{cases} a_{ii} - 1 & \text{if } a_{ii} \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \\ a_{ii} + 1 & \text{if } a_{ii} = 0. \end{cases}$$

In other words: We construct a number $r = 0.a_1a_2a_3a_4\cdots$ such that the i-th digit in its decimal expansion is different from the number a_{ii}, where a_{ii} is the i-th digit in the decimal expansion of r_i. This number should appear somewhere on our list, since we assumed the list is complete. But it doesn't! To see this, note that $r \neq r_i$, because r and r_i differ in their i-th digits (by construction of r). Thus $[0, 1]$ is not countable.

(g) The *power set* of a set A is the set $\mathcal{P}(A) = \{B \mid B \subseteq A\}$ consisting of all subsets of A. For example, if $A = \{1, 2, 3\}$ then $\mathcal{A} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$, where $\emptyset = \{\}$ is the 'empty set'. Show that for any set A the set $\mathcal{P}(A)$ does not have the same cardinality as A. Hence deduce that there is a set 'bigger' than $\mathbb{R}$, and that in fact there is an infinite number of growing 'sizes' of infinite sets.

Solution: Suppose that there is a bijection $f : A \to \mathcal{P}(A)$. Thus to each $a \in A$ the function f gives a subset $f(a)$ of A. Consider the set

$$B = \{a \in A \mid a \notin f(a)\}.$$

Since B is a subset of A we have that $B \in \mathcal{P}(A)$. Thus, since $f : A \to \mathcal{P}(A)$ is surjective, there is an element $b \in A$ such that $B = f(b)$. Is the element b in B? If $b \in B$ then by the definition of B we have $b \notin f(b)$, but since $f(b) = B$ this says that $b \notin B$, a contradiction. On the other hand, if $b \notin B$ then since $B = f(b)$ we have $b \notin f(b)$, and thus by the definition of B we have $b \in B$, again a contradiction!

Thus no bijection $f : A \to \mathcal{P}(A)$ exists, and so A and $\mathcal{P}(A)$ do not have the same cardinality. Hence the cardinality of $\mathcal{P}(A)$ is a 'larger' cardinality than A (it certainly is not 'smaller', because there is an injective function $f : A \to \mathcal{P}(A)$ given by $f(a) = \{a\}$).

Applying this to $\mathbb{R}$ we see that the power set $\mathcal{P}(\mathbb{R})$ has a strictly 'larger' cardinality than $\mathbb{R}$. So $\mathcal{P}(\mathbb{R})$ is a very very large set indeed! But it does not stop here, we can take the power set of the power set of $\mathbb{R}$ (that is, $\mathcal{P}(\mathcal{P}(\mathbb{R}))$), and then the power set of the power set of the power set of $\mathbb{R}$, and so on. Each is larger in cardinality than the last, and so we get a growing chain of larger and larger cardinalities. Infinity certainly is an interesting beast.

Remark: The topics covered in this exercise were first discovered by the German mathematician Georg Cantor around 1870. He is generally considered as the father of set theory, and as the first person to really understand the complexity of infinity. His work shook the foundations of mathematics. Indeed many prominent mathematicians at the time rejected his ideas – for example, Henri Poincaré said that Cantor's theory was a "grave disease" infecting mathematics. However over time people had to accept the strangeness of infinity. In fact the complexity of infinity has lead to a far more beautiful world of mathematics. Sadly the hostility directed towards Cantor during his life only exacerbated the severe bouts of depression that plagued his later years, and he died in relative poverty in 1918 before the end of World War I.

There are some nice books in which you can discover more. For example, Brian Clegg's *A Brief History of Infinity* is a great read. You can also go directly to the source and have a go at reading Cantor's surprisingly accessible book *Contributions to the Founding of the Theory of Transfinite Numbers*.