4.4 Noetherian Rings

Recall that a ring A is **Noetherian** if it satisfies the following three equivalent conditions:

(1) Every nonempty set of ideals of A has a maximal element (the **maximal condition**);

(2) Every ascending chain of ideals is stationary (the **ascending chain condition (a.c.c.)**);

(3) Every ideal of A is finitely generated.

Later in this section we will prove

Hilbert's Basis Theorem

which says that a polynomial ring in one indeterminate over a Noetherian ring is itself Noetherian.

In particular, by iteration, the polynomial ring $F[x_1, \ldots, x_n]$ over a field F is Noetherian.

It will follow quickly that all finitely generated rings are Noetherian.

But first we will prove that all proper ideals in Noetherian rings have primary decompositions,

and simplify the First Uniqueness Theorem concerning the uniqueness of associated prime ideals.

Call an ideal I of a ring A **irreducible** if, for all ideals J , K of A ,

$$I \;=\; J \cap K \quad \Longrightarrow \quad \big(I = J \quad \text{or} \quad I = K\big) \; .$$

Lemma: Every ideal of a Noetherian ring is a finite intersection of irreducible ideals.

Proof: Suppose the set

$$\Sigma \;=\; \{\, J \lhd A \mid J \text{ is not a finite intersection of irreducible ideals} \,\}$$

is nonempty. By the maximal condition, Σ has a maximal element M.

Certainly M is not irreducible, so $M = J \cap K$ for some ideals J, K such that

$$J \neq M \neq K \,.$$

But $M \subset J$ and $M \subset K$, so, by maximality of M in Σ ,

$$J \notin \Sigma \quad \text{and} \quad K \notin \Sigma \, .$$

Hence J and K are both finite intersections of irreducible ideals,

so $M = J \cap K$ also is such an intersection,

contradicting that $M \in \Sigma$.

Hence $\Sigma = \emptyset$, and the Lemma is proved.

Lemma: Every proper irreducible ideal of a Noetherian ring is primary.

Proof: Let A be a Noetherian ring. If I is a proper ideal then I is irreducible iff the zero ideal of A/I is irreducible.

It suffices then to suppose A is nonzero and $\{0\}$ is irreducible, and prove that $\{0\}$ is primary.

Let $x, y \in A$ such that $xy = 0$.

$$\boxed{\text{We show } x = 0 \text{ or } y^n = 0 \text{ for some } n \geq 1\,.}$$

Consider the chain of ideals

$$\text{Ann}\,(y) \subseteq \text{Ann}\,(y^2) \subseteq \ldots \subseteq \text{Ann}\,(y^n) \subseteq \ldots$$

which is stationary, since A is Noetherian. Hence

$$(\exists n \geq 1) \qquad \text{Ann}\,(y^n) = \text{Ann}\,(y^{n+1}) = \ldots$$

$$\boxed{\text{We show} \quad \langle x \rangle \cap \langle y^n \rangle \ = \ \{0\}\ .}$$

Suppose $z \ \in \ \langle x \rangle \cap \langle y^n \rangle$. Then

$$z \ = \ vx \ = \ wy^n \qquad (\exists v, w \in A)\ ,$$

so

$$wy^{n+1} \ = \ (wy^n)y \ = \ vxy \ = \ v0 \ = \ 0\ ,$$

so

$$w \in \mathsf{Ann}\ (y^{n+1}) \ = \ \mathsf{Ann}\ (y^n)\ ,$$

yielding

$$z \ = \ wy^n \ = \ 0\ .$$

This proves $\langle x\rangle \cap \langle y^n\rangle \ = \ \{0\}$. By irreducibility of $\{0\}$, we get

$$\langle x\rangle \ = \ \{0\} \quad \text{or} \quad \langle y^n\rangle \ = \ \{0\}\ ,$$

whence $x = 0$ or $y^n = 0$, and we are done.

The previous two lemmas prove:

Theorem: Every proper ideal of a Noetherian ring has a primary decomposition.

We can refine the First Uniqueness Theorem for primary decompositions, in this context, but first prove:

Proposition: Every ideal of a Noetherian ring contains a power of its radical.

Proof: Let I be an ideal of a Noetherian ring A, so

$$r(I) \;=\; \langle\, x_1\,,\;\ldots\;,\; x_k\,\rangle$$

for some $x_1,\ldots,x_k \in A$. Then

$$(\forall i=1,\ldots,k)(\exists n_i \geq 1) \qquad x_i^{n_i} \;\in\; I\;.$$

Put

$$m \;=\; \sum_{i=1}^{k}(n_i - 1) \;+\; 1\;.$$

Observe that

$$\big(r(I)\big)^m \;=\; \langle\, x_1^{j_1}\ldots x_k^{j_k} \;\mid\; \sum_{i=1}^{k} j_i \;=\; m \,\rangle\ .$$

But if $\displaystyle\sum_{i=1}^{k} j_i \;=\; m$ then $j_\ell \geq n_\ell$ for some $\ell \in \{1,\ldots,k\})$.

Hence each generator of $\big(r(I)\big)^m$ lies in I , so $\big(r(I)\big)^m \subseteq I$.

Corollary: The nilradical is nilpotent in a Noetherian ring.

Proof: If A is Noetherian, then

$$N = r(\{0\})$$

is the nilradical of A, so, by the previous Proposition,

$$N^m \subseteq \{0\} \text{ for some } m \geq 1 \,,$$

whence equality holds, which proves N is nilpotent.

Exercises: **(1)** Find an example of a Noetherian ring whose Jacobson radical does not equal the nilradical.

(2) Show that if a ring satisfies the d.c.c. on ideals then the nilradical and Jacobson radical are equal.

(3) Find an example of an ideal I of a ring A which does not contain a power of its radical $r(I)$ (so necessarily A is not Noetherian).

Theorem: Let A be Noetherian, Q and M ideals of A with M maximal. TFAE

(i) Q is M-primary;

(ii) $r(Q) = M$;

(iii) $M^n \subseteq Q \subseteq M \qquad (\exists n > 0)$.

Proof: (i) $\implies$ (ii): follows by definition.

(ii) $\implies$ (i): follows (regardless of whether A is Noetherian) by an earlier result (page 684).

(ii) $\implies$ (iii): follows from the previous Proposition.

(iii) $\implies$ (ii): if $M^n \subseteq Q \subseteq M$ for some $n > 0$ then

$$M = r(M^n) \subseteq r(Q) \subseteq r(M) = M ,$$

so $r(Q) = M$.

Our final observation about primary decompositions of ideals in Noetherian rings is a refinement of the First Uniqueness Theorem (page 701):

Theorem: Let A be a Noetherian ring and I a proper ideal. Then the prime ideals belonging to I are the prime ideals in

$$\{ \, (I : x) \;\mid\; x \in A \, \} \, .$$

Proof: Let $I = \bigcap_{i=1}^{n} Q_i$ be a minimal primary decomposition of I, and put

$$P_i \;=\; r(Q_i) \qquad (\forall i) \, .$$

Then, by the First Uniqueness Theorem,

$$\{ P_1 , \ldots , P_n \} \;=\; \{ \text{prime ideals } P \mid (\exists x \in A) \quad P = r(I : x) \}$$

If $x \in A$ and $(I : x)$ is prime, then

$$r(I : x) = (I : x)$$

is prime, so $(I : x) \in \{P_1, \ldots, P_n\}$.

Conversely, let $i \in \{1, \ldots, n\}$. By the earlier

Proposition (page 815),

$$(\exists m \geq 1) \qquad Q_i \supseteq P_i^m .$$

Put $R = \bigcap_{j \neq i} Q_j$. Then

$$R\, P_i^m \subseteq R \cap P_i^m \subseteq R \cap Q_i = I .$$

Let m_0 be the least integer such that

$$R\, P_i^{m_0} \subseteq I .$$

If $m_0 = 0$ then

$$R \subseteq I \subseteq R ,$$

so $I = R$, which contradicts minimality of the primary decomposition of I .

Hence $m_0 \geq 1$, and so we may choose some

$$x \in R\,P^{m_0-1} \setminus I .$$

In particular $x \in R \backslash I$, so $x \notin Q_i$. As in the proof of the First Uniqueness Theorem (see pages 706-707), we have

$$(I : x) \ = \ (Q_i : x)$$

and

$$r(I : x) \ = \ r(Q_i : x) \ = \ P_i \ .$$

Certainly then

$$(I : x) \ \subseteq \ r(I : x) \ \subseteq \ P_i \ .$$

Also,

$$P_i\, x \;\; \subseteq \;\; R\, P_i^{m_0 - 1} P_i \;\; = \;\; R\, P^{m_0} \;\; \subseteq \;\; I \; ,$$

so $P_i \subseteq (I : x)$, whence equality holds.

This shows

$$\{\, P_1 \, \ldots , P_n \,\} \;\; = \;\; \{ \text{ prime ideals } (I : x) \;\mid\; x \in A \,\} \; ,$$

and completes the proof of the Theorem.

We now investigate the preservation of the property of being Noetherian under certain natural constructions.

We have already observed (on page 766) that

homomorphic images of Noetherian rings are Noetherian,

and (on page 766) that

a finitely generated module over a Noetherian ring is Noetherian.

Theorem: Let A be a subring of a ring B .

Suppose that A is Noetherian and B is finitely generated as an A-module.

Then B is a Noetherian ring.

Proof: By the immediately preceding observation, B is a Neotherian A-module.

But all ideals of B are also A-submodules of B (though not necessarily conversely).

Since A-submodules satisfy the a.c.c., so do ideals of B, so B is a Noetherian ring.

Example: The ring $\mathbb{Z}[i]$ of Gaussian integers is a finitely generated $\mathbb{Z}$-module, and $\mathbb{Z}$ is Noetherian. By the previous Theorem, $\mathbb{Z}[i]$ is a Noetherian ring.

Theorem: Rings of fractions of Noetherian rings are Noetherian.

Proof: Let A be a Noetherian ring and S a multiplicatively closed subset.

Let $J \lhd S^{-1}A$, so

$$J = S^{-1}I \qquad (\exists I \lhd A) .$$

since all ideals of $S^{-1}A$ are extended. But A is Noetherian, so

$$I = \langle x_1 , \ldots , x_n \rangle$$

for some $x_1, \ldots\ x_n \in A$, whence

$$J \ = \ \langle\ x_1/1\ ,\ \ldots\ ,\ x_n/1\ \rangle.$$

Thus all ideals of $S^{-1}A$ are finitely generated, which shows $S^{-1}A$ is Noetherian.

Hilbert's Basis Theorem: Let A be a Noetherian ring. Then the polynomial ring $A[x]$ is Noetherian.

Proof: We prove that all ideals of $A[x]$ are finitely generated. Consider $\{0\} \neq J \lhd A[x]$, and put

$$I = \{ \text{ leading coefficients of polynomials in } J \} .$$

It is easy to show that $I \lhd A$, so

$$I = \langle a_1 , \ldots , a_n \rangle$$

for some $a_1, \ldots, a_n \in A$, since A is Noetherian.

Then, for each $i = 1, \ldots, n$, there is a polynomial

$$p_i(x) \;=\; a_i x^{d_i} \;+\; (\text{ terms of lower degree })$$

in J , for some $d_i \geq 0$. Put

$$J' \;=\; \langle\, p_1(x) \;,\; \ldots \;,\; p_n(x) \,\rangle$$

and

$$d \;=\; \max \{\, d_1 \;,\; \ldots \;,\; d_n \,\} \;.$$

Let

$$M = \{ \, q(x) \in A[x] \mid \text{degree of } q(x) \le d \, \} \, .$$

$$\boxed{\textbf{Claim:} \quad J = (J \cap M) + J' \, .}$$

Clearly $(J \cap M) + J' \subseteq J$. Conversely let $0 \neq p(x) \in J$. Then

$$p(x) = ax^m + (\text{ terms of lower degree })$$

for some $m \geq 0$.

$$\boxed{\text{If } m \leq d \text{ then } p(x) \in J \cap M \, .}$$

Suppose $m > d$. Since $a \in I$

$$a = \sum_{i=1}^{n} u_i \, a_i \qquad (\exists u_1, \ldots, u_n \in A) \, .$$

Put

$$q(x) = p(x) - \sum_{i=1}^{n} u_i \, x^{m-d_i} \, p_i(x) \, .$$

Then $q(x)$ is an element of J of degree $< m$.

By an inductive hypothesis,

$$q(x) \;\in\; (J \cap M) \;+\; J' \,,$$

so

$$p(x) \;=\; q(x) \;+\; \sum_{i=1}^{n} \; u_i \, x^{n-d_i} \, p_i(x)$$

$$\in \;\; (J \cap M) + J' \,.$$

This proves $J \subseteq (J \cap M) + J'$, whence equality holds, and the Claim is proved.

> Regarded as an A-module, M is finitely generated, by 1 , x , $\ldots$, x^d .

But A is Noetherian, so M is a Noetherian A-module (by the Theorem on page 766).

But then $J \cap M$, being an A-submodule of M

must be finitely generated as an A-module by, say,

$$q_1(x)\ ,\ \ldots\ ,\ q_k(x)\ .$$

The **ideal** of $A[x]$ generated by

$$q_1(x)\ ,\ \ldots\ ,\ q_k(x)\ ,\ p_1(x)\ ,\ \ldots\ ,\ p_n(x)$$

therefore contains $J \cap M$ and J' , and is contained in J ,

so equals $(J \cap M) + J' = J$.

Thus J is finitely generated, completing the proof that $A[x]$ is Noetherian.

Corollary: If A is Noetherian then so is $A[x_1, \dots, x_n]$ for every $n \geq 1$.

Proof: immediate by iterating Hilbert's Basis Theorem.

Corollary: If A is a Noetherian ring and B is a finitely generated A-algebra, then B is a Noetherian ring.

In particular, every finitely generated ring, and every finitely generated algebra over a field, is Noetherian.

Proof: If A is Noetherian and B is generated as an A-algebra by

$$b_1 \; , \; \ldots \; , \; b_n \; ,$$

then B is a homomorphic image of the polynomial ring $A[x_1, \ldots, x_n]$ under the map

$$p(x_1, \ldots, x_n) \mapsto p(b_1, \ldots, b_n) \ ,$$

so must be Noetherian,

since $A[x_1, \ldots, x_n]$ is Noetherian (by the previous Corollary),

and homomorphic images of Noetherian rings are Noetherian.

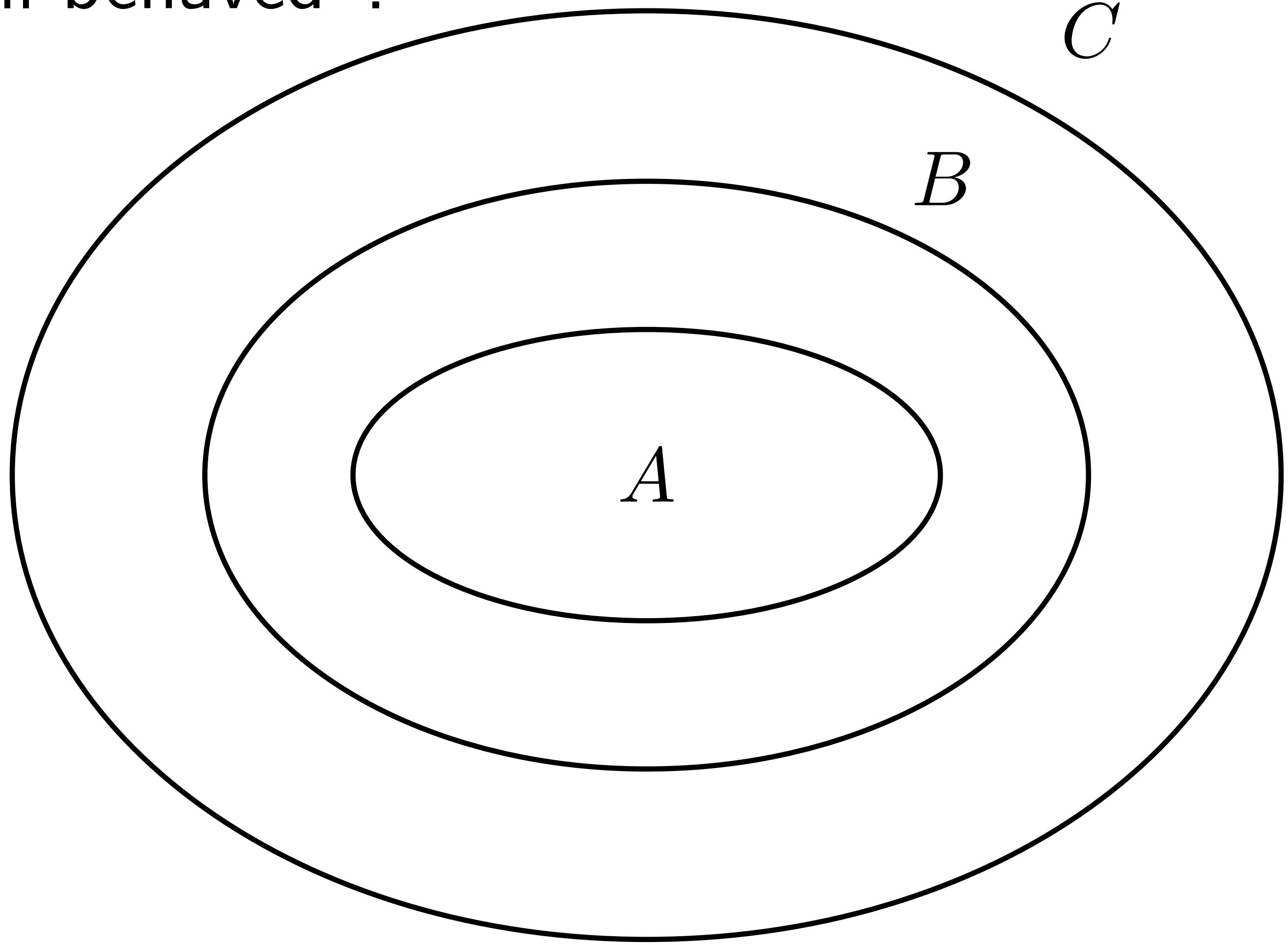
The last statement of the Corollary follows because every field is Noetherian,

and every ring is an algebra over its subring generated by 1 ,

which is isomorphic to either $\mathbb{Z}_n$, for some n , or $\mathbb{Z}$,

both of which are Noetherian.

The next (tricky) result says that, under certain conditions, an intermediate ring B, sandwiched between "well-behaved" rings A and C is itself "well-behaved".



Theorem: Suppose $A \subseteq B \subseteq C$ is a chain of subrings and that

(i) A is Noetherian;

(ii) C is finitely generated **as an** A-**algebra**;

(iii) C is finitely generated **as a** B-**module**.

Then B is finitely generated as an A-algebra.

Proof: Let

$x_1 , \ \ldots \ , \ x_m$ generate C as an A-algebra,

and

$y_1 , \ \ldots \ , \ y_n$ generate C as a B-module.

Then

$$\boxed{\begin{gathered}(\forall i = 1, \ldots, m)\,(\exists\, b_{i1} , \ldots \ , b_{in} \ \in \ B \ ) \\ x_i \ = \ \sum_{k=1}^{n} b_{ik}\, y_k \end{gathered}}$$

and

$$\boxed{\begin{gathered}(\forall i, j \in \{1, \ldots, m\})\,(\exists\, c_{ij1}\,, \ldots\,, c_{ijn} \,\in\, B\,) \\ y_i\, y_j \;=\; \sum_{k=1}^{n} c_{ijk}\, y_k\,.\end{gathered}}$$

Let B_0 be the A-subalgebra of B generated by

$$\left\{\, b_{ik}\,,\; c_{ijk} \;\middle|\; i, j \in \{1, \ldots, m\}\,,\; k \in \{1, \ldots, n\} \,\right\}.$$

By the previous Corollary, B_0 is a Noetherian ring.

Further

$$A \subseteq B_0 \subseteq B \subseteq C$$

is a chain of subrings.

> **Claim:** C is generated by $y_1 , \ldots , y_n$ as a B_0-module.

Each element c of C can be expressed as a **polynomial** in $x_1, \ldots, x_m$ with coefficients from A .

But each x_i and each product $y_i y_j$ is a **linear combination** of $y_1, \ldots, y_n$ with coefficients from B_0 ,

so that, after multiplying out, we can express c as a **linear combination** of $y_1, \ldots, y_n$ with coefficients from B_0 , which proves the Claim.

By the Claim and the observation that B_0 is Noetherian, we conclude that

$$C \text{ is a Noetherian } B_0\text{-module.}$$

But B is a B_0-submodule of C, so

$$B \text{ is a Noetherian } B_0\text{-module,}$$

so is finitely generated.

But B_0 is finitely generated as an A-algebra, so, finally,

$$B \text{ is also finitely generated as an } A\text{-algebra,}$$

and the Theorem is proved.