THE UNIVERSITY OF SYDNEY
SCHOOL OF MATHEMATICS AND STATISTICS

An Introduction to MAGMA

MagmaMondays: 9 October 2023 — Semester 2, 2023

Web Page: `https://sites.google.com/view/magma-mondays/`
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1. Suppose that *letters* is a sequence of letters. The following code produces all 'words' made from these letters.

```
[&*[ letters[i^p] : i in [1..n]] : p in Sym(n)] where n is #letters;
```

If you first type

```
letters := ElementToSequence("aact");
```

and then use the code above you will see that some 'words' appear twice.

(a) Write a few lines of code that produce a sequence of words without duplicates.

(b) Change the code so that it produces a sequence of three letter 'words'.

2. Write a function expression CatNum := ***func***< *n* | . . . > such that CatNum(n) returns the nth Catalan number.

3. Here is the CatSeq function from the lecture.

```
CatSeq := function(n);
    if n eq 0 then seq := [1];
    elif n eq 1 then seq := [1,1];
    else
        seq := $$(n-1);
        Append(~seq, &+[Integers()| seq[k+1]*seq[n-k] : k in [0..n-1]]);
    end if;
    return seq;
end function;
```

Rewrite CatSeq as a function expression using **select**.

4. A *hyperoval* in a projective plane of even order q is a set of $q+2$ points, no three of which are on a line.

(a) Find an example of a hyperoval in the 21-point projective plane. You can begin with the command

```
plane, points, lines := FiniteProjectivePlane(4);
```

Hint 1. What are *points*.1 and *points*.2? What is *lines*.3?

Hint 2. Exclude(~*S*, *v*) removes the element *v* from the set *S*. If you want to remove a representative from *S* and assign it to a variable *x*, use ExtractRep(~*S*, ~*x*).

(b) Write a function IsHyperoval(*P*, *X*) to test whether *X* is a hyperoval in a projective plane *P*.

(c) Find all the hyperovals in the 21-point projective plane.

(d) Find the orbits of the groups PGL(3, 4) and PSL(3, 4) on the set of hyperovals.

5. The points and lines of the 21-point plane can be identified with the 1- and 2-dimensional subspaces of a vector space of dimension 3 over the field of 4 elements. In this representation an example of a hyperoval is the set of singular points of a quadratic form together with its radical. You can use the following code to construct the form and the quadratic space.

```
P<x,y,z> := PolynomialRing(GaloisField(4),3);
f := x*y + z^2;
V := QuadraticSpace(f);
```

Find 6 *vectors* that represent the points of the hyperoval. Check that they do indeed form a hyperoval. (Hint. Radical(V) is the radical of V and QuadraticNorm(v) is the value of the quadratic form at the vector v.)

6. Let G be a group. Write a function that returns exactly one representative of $\{x, x^{-1}\}$ for all $x \in G$. Test your function on the cyclic groups of orders 2,3,4, and 5 and the dihedral groups of orders 6, 8, 10 and 12.

7. A non-empty subset S of a group G is *product-free* if $ab \notin S$ for all $a, b \in S$.

Using the functions

```
prodfree := func< S | forall{<a,b> : a,b in S | a*b notin S } >;
checkmax_1 := function(G)
  for a in G do
    if a eq One(G) then continue; end if;
    found := true;
    for b in G do
      if b eq One(G) or b eq a then continue; end if;
      if prodfree({a,b}) then found := false; continue; end if;
    end for;
    if found then return true, a; end if;
  end for;
  return false, _;
end function;
```

defined in the lecture find the groups in the Small Groups Database that contain a *maximal* product-free set of size 1.

8. Write a function *checkmax*$_2$ that can be used to find the groups in the Small Groups Database that contain a *maximal* product-free set of size 2.

Make a conjecture about the classification of all finite group with a maximal product-free set of size 2.