ENUMERATING THE FAKE PROJECTIVE PLANES: ELIMINATING THE MATRIX ALGEBRA CASES

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Abstract. Prasad and Yeung [21] gave a short explicit list of real fields k (either $\mathbb{Q}$ or a quadratic extension of $\mathbb{Q}$), complex quadratic extensions ℓ of k, and central simple algebras $\mathcal{D}$ of degree 3 over its center ℓ, such that the fundamental group of any fake projective plane must be a torsion-free cocompact subgroup of a unitary group $PU(h)$ associated with $(k,\ell,\mathcal{D})$ and an essentially unique nondegenerate hermitian form h on $\mathcal{D}$. They produced a fake projective plane in each of the cases for which $\mathcal{D}$ is a division algebra, and we have subsequently found all the fake projective planes in those cases (see [7]), and shown that none can arise from the cases when $\mathcal{D}$ is a matrix algebra, as conjectured in [21]. The purpose of this paper is to explain how the matrix algebra cases were excluded.

1. Introduction

A *fake projective plane* (abbreviated *fpp* below) is a smooth compact complex surface M which is not the complex projective plane but has the same Betti numbers as the complex projective plane $\mathbb{P}^2(\mathbb{C})$ (namely 1, 0, 1, 0, 1, and thereafter 0). An fpp is known (see [21]) to have the form

$$M = B(\mathbb{C}^2)/\Pi, \tag{1.1}$$

where $B(\mathbb{C}^2) = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ is the unit ball in $\mathbb{C}^2$, and where Π is a torsion-free cocompact arithmetic subgroup of $PU(2,1)$, isomorphic to the fundamental group of M. The well-known action of $PU(2,1)$ on $B(\mathbb{C}^2)$ is described in Section 4.1 below.

Let $\varphi : SU(2,1) \to PU(2,1)$ be the natural surjection. Its kernel is equal to $\{\omega^\nu I : \nu = 0,1,2\}$, where $\omega = e^{2\pi i/3}$. Let $\tilde{\Pi} = \varphi^{-1}(\Pi)$.

As explained in [21], because $\tilde{\Pi}$ is an arithmetic subgroup of $SU(2,1)$, there is a pair (k,ℓ) of fields, with k totally real and ℓ a totally complex quadratic extension of k, and there is a central simple algebra $\mathcal{D}$ of degree 3 with center ℓ, and there is an involution ι of the second kind on $\mathcal{D}$ such that $k = \{x \in \ell : \iota(x) = x\}$ and so that $\tilde{\Pi}$ is commensurable with $\tilde{\Pi} \cap G(k)$, where

$$G(k) = \{\xi \in \mathcal{D}^\times : \xi\iota(\xi) = 1 \text{ and } \mathrm{Nrd}(\xi) = 1\}.$$

The G here is a simple simply connected algebraic k-group, with the property that $G(k_{v_0}) \cong SU(2,1)$ for one real place v_0 of k, and such that $G(k_v) \cong SU(3)$ for all other archimedean places v of k. These conditions determine G up to k-isomorphism. The commensurability can be described more precisely: there is a principal arithmetic subgroup Λ of $G(k)$ so that $\tilde{\Pi} \subset \Gamma$, where Γ is the normalizer of Λ in $SU(2,1)$, which satisfies $[\Gamma : \Lambda] < \infty$ and $[\Gamma : \tilde{\Pi}] < \infty$.

Prasad and Yeung in [21] showed that the k, ℓ and $\mathcal{D}$ here must come from a short list of possibilities. By the well-known classification of central simple algebras, $\mathcal{D}$ is either the matrix algebra $M_{3\times 3}(\ell)$ or is a division algebra (of dimension 9 over its center ℓ). They found at least one fpp for each $(k,\ell,\mathcal{D})$ in their list for which $\mathcal{D}$ is a division algebra. As we reported in [7], we have found all the fpps, and there are precisely 50 of them (up to homeomorphism; there are 100 up to biholomorphism). We did this by going through the list in [21] of possible $(k, \ell, \mathcal{D})$'s, determining all the possibilities for Λ, and by sometimes very lengthy computer-assisted calculations, determining all the possible fundamental groups of fpps that can arise from $(k, \ell, \mathcal{D})$. As conjectured in [21], all the fpps come from $(k, \ell, \mathcal{D})$ in the list in [21] for which $\mathcal{D}$ is a division algebra.

A significant part of the effort reported in [7] was to show that no fpps arise from the six $(k, \ell, \mathcal{D})$ in the list of [21] for which $\mathcal{D}$ is the matrix algebra $M_{3\times 3}(\ell)$. The six pairs (k, ℓ) are named $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_8$, $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$ in [21], and are as follows:

name	k	ℓ	defining polynomial for ℓ
$\mathcal{C}_1$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\zeta_5)$	$\zeta^4+\zeta^3+\zeta^2+\zeta+1$
$\mathcal{C}_3$	$\mathbb{Q}(\sqrt{5})$	$\mathbb{Q}(\sqrt{5}, i) \cong \mathbb{Q}(z)$	z^4+3z^2+1
$\mathcal{C}_8$	$\mathbb{Q}(\sqrt{2})$	$\mathbb{Q}(\sqrt{2}, i) \cong \mathbb{Q}(\zeta_8)$	ζ^4+1
$\mathcal{C}_{11}$	$\mathbb{Q}(\sqrt{3})$	$\mathbb{Q}(\sqrt{3}, i) \cong \mathbb{Q}(\zeta_{12})$	$\zeta^4-\zeta^2+1$
$\mathcal{C}_{18}$	$\mathbb{Q}(\sqrt{6})$	$\mathbb{Q}(\sqrt{6}, \zeta_3) \cong \mathbb{Q}(z)$	z^4-2z^2+4
$\mathcal{C}_{21}$	$\mathbb{Q}(\sqrt{33})$	$\mathbb{Q}(\sqrt{33}, \zeta_3) \cong \mathbb{Q}(z)$	$z^4-z^3-2z^2-3z+9$

Table 1.

The case $\mathcal{C}_3$ had been culled from the list in [21], but the argument in [21, Proposition 8.8] relies on the existence of elements of order 5 which were not explicitly given there. We shall exclude $\mathcal{C}_3$ by giving these elements below.

Particular properties of these six $\mathcal{C}_j$'s are given in Section 3. For now, note that each k is a real quadratic extension $\mathbb{Q}(r)$ of $\mathbb{Q}$, where $r^2 = 5$, 5, 2, 3, 6 and 33, respectively. The rings of integers in k and ℓ are denoted $\mathfrak{o}_k$ and $\mathfrak{o}_\ell$, respectively. For $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_{21}$, $\mathfrak{o}_k$ is $\mathbb{Z}[(r+1)/2]$, and $\mathfrak{o}_k = \mathbb{Z}[r]$ in the other three cases. Let V_∞ and V_f denote, respectively, the sets of archimedean and nonarchimedean places v of k. Then $V_\infty = \{v_0, v_0'\}$, where v_0 and v_0' correspond to the embeddings of k into $\mathbb{R}$ mapping r to the positive and negative square root of r^2, respectively. Let k_v denote the completion of k with respect to v, and for $v \in V_f$, let $\mathfrak{o}_v$ denote the valuation ring of k_v, and let q_v denote the order of the residue field of k_v.

In each case, we define a 3×3 symmetric matrix $F = F_{(\mathcal{C}_j, \emptyset)}$ with entries in $\mathfrak{o}_k$ and determinant 1 such that the hermitian form $h : \boldsymbol{x} \mapsto \boldsymbol{x}^* F \boldsymbol{x}$ on ℓ^3 is indefinite at v_0 and definite at v_0'. In terms of the numbers $x \in \mathfrak{o}_k$ in the table

	$\mathcal{C}_1$	$\mathcal{C}_3$	$\mathcal{C}_8$	$\mathcal{C}_{11}$	$\mathcal{C}_{18}$	$\mathcal{C}_{21}$
r^2	5	5	2	3	6	33
x	$(r+1)/2$	$(r+1)/2$	$r+1$	$r+1$	$r+2$	$(r+5)/2$

Table 2.

and for the cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_8$, respectively $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$, we define

$$F_{(\mathcal{C}_j,\emptyset)} = \begin{pmatrix} -x & 0 & 0 \\ 0 & -x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{respectively} \quad F_{(\mathcal{C}_j,\emptyset)} = \begin{pmatrix} -x & 1 & 0 \\ 1 & -2x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.2}$$

Each x is positive when r is taken as the positive square root of r^2, and negative when r is taken as the negative square root, so that h is indefinite at v_0 and definite at v_0'. In the cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_8$, x is invertible in $\mathfrak{o}_k$, with inverse $(r-1)/2$, $(r-1)/2$ and $r-1$, respectively. In the cases $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$, $2x^{-1}$ is in $\mathfrak{o}_k$, and equals $r-1$, $r-2$ and $(r-5)/2$, respectively. So the entries of $F_{(\mathcal{C}_j,\emptyset)}$ are all in $\mathfrak{o}_k$.

Writing $F = F_{(\mathcal{C}_j,\emptyset)}$, the map $g \mapsto F^{-1}g^*F$ is an involution of the second kind on $\mathcal{D} = M_{3\times 3}(\ell)$, and for the corresponding special unitary group $G = SU(h)$,

$$G(k) = \{g \in M_{3\times 3}(\ell) : g^*Fg = F \text{ and } \det(g) = 1\}.$$

Since the Euler-Poincaré characteristic $\chi(M)$ of a topological space M is the alternating sum of its Betti numbers, $\chi(M) = \chi(\mathbb{P}^2(\mathbb{C})) = 3$ when M is an fpp.

If Π is any torsion-free cocompact subgroup of $PU(2,1)$), define M by (1.1). Then for an appropriate normalization of the hyperbolic volume vol on $B(\mathbb{C}^2)$,

$$\chi(M) = 3\mathrm{vol}(\mathcal{F}_\Pi), \tag{1.3}$$

where $\mathcal{F}_\Pi \subset B(\mathbb{C}^2)$ is a fundamental domain for the action of Π on $B(\mathbb{C}^2)$. This is a result of Chern (or the Hirzebruch Proportionality theorem).

Applying (1.3) to the case an fpp M, we have $3 = \chi(M) = 3\mathrm{vol}(\mathcal{F}_\Pi)$, and so

$$\mathrm{vol}(\mathcal{F}_\Pi) = 1. \tag{1.4}$$

Starting from appropriately normalized Haar measure on $SU(2,1)$, invariant measures m can be defined on quotients $SU(2,1)/\Gamma$ of $SU(2,1)$ by cocompact discrete groups Γ, so that

(a) if $\Gamma_1 \subset \Gamma_2$, then $m(SU(2,1)/\Gamma_1) = [\Gamma_2 : \Gamma_1]m(SU(2,1)/\Gamma_2)$, and
(b) if $\omega I \in \Gamma$, then $m(SU(2,1)/\Gamma) = \frac{1}{3}\mathrm{vol}(\mathcal{F}_{\varphi(\Gamma)})$.

Applying (b) to the case $\Gamma = \tilde{\Pi}$, and using (1.4), and applying (a) to the inclusions $\Lambda \subset \Gamma$ and $\tilde{\Pi} \subset \Gamma$, where Γ is the normalizer of the principal arithmetic subgroup Λ of $G(k)$, as above, we obtain

$$m(SU(2,1)/\Lambda) = \frac{[\Gamma : \Lambda]}{3[\bar{\Gamma} : \Pi]}, \tag{1.5}$$

where $\bar{\Gamma} = \varphi(\Gamma)$, and we have used the simple fact that $[\Gamma : \tilde{\Pi}] = [\bar{\Gamma} : \Pi]$.

The principal arithmetic subgroup Λ has the form $G(k) \cap \prod_{v\in V_f} P_v$, where V_f is the set of non-archimedean places of k, and where each P_v is a parahoric subgroup of $G(k_v)$. By Prasad's Covolume Formula (equation (11) in [21, §2.11]), we have

$$m(SU(2,1)/\Lambda) = \frac{1}{D}\prod_{v\in\mathcal{T}} e'(P_v), \tag{1.6}$$

where D, the denominator of the rational number μ of [21, §8.2]), is as in the following table:

	$\mathcal{C}_1$	$\mathcal{C}_3$	$\mathcal{C}_8$	$\mathcal{C}_{11}$	$\mathcal{C}_{18}$	$\mathcal{C}_{21}$
D	600	32	128	864	48	12

Table 3.

and the numbers $e'(P_v)$ are integers defined in [21, §2.5], and where $\mathcal{T} = \mathcal{T}' \cup \mathcal{T}''$ for

$$\mathcal{T}' = \{v \in V_f : P_v \text{ is not maximal}\},$$

$$\mathcal{T}'' = \{v \in V_f : v \text{ is unramified in } \ell \text{ and } P_v \text{ is not hyperspecial}\}.$$

Comparing (1.5) and (1.6), and using the fact proved in [21, §5.4] that $[\Gamma : \Lambda] = 3$ in our situation (see Corollary 2.1 below), we have

$$[\bar{\Gamma} : \Pi]\prod_{v\in\mathcal{T}} e'(P_v) = D. \tag{1.7}$$

If p is a prime number divides the right hand side of (1.7), then $p \in \{2,3,5\}$. This strongly limits the possibilities for $\mathcal{T}$, and we shall see in Section 2 that $\mathcal{T}$ must be empty or a singleton set. Moreover, we show in Section 2 that P_v can be chosen maximal for each $v \in V_f$. When v splits in ℓ, any two maximal parahorics in $G(k_v)$

are conjugate by an element of $\overline{G}(k_v)$. When v does not split in ℓ, there are two $\overline{G}(k_v)$-conjugacy classes of maximal parahorics in $G(k_v)$, which we shall call *type 1* and *type 2*, respectively (see below). Then using [21, Proposition 5.3], we see that Λ may be assumed to be one of 13 possibilities — two for each $\mathcal{C}_j$, $j \neq 21$, and three for $\mathcal{C}_{21}$. These are indexed by $\mathcal{C}_j$ and

$$\mathcal{T}_1 = \{v \in V_f : v \text{ does not split in } \ell \text{ and } P_v \text{ is of type 2}\}. \tag{1.8}$$

These are the 13 possibilities:

$$\begin{aligned} &(\mathcal{C}_j, \emptyset) \text{ (for } j = 1,\ 3,\ 8,\ 11,\ 18 \text{ and } 21),\ \text{and } (\mathcal{C}_1, \{5\}),\ \text{and} \\ &(\mathcal{C}_j, \{2\}) \text{ (for } j = 3,\ 8,\ 11 \text{ and } 18), \quad \text{and } (\mathcal{C}_{21}, \{2+\}) \text{ and } (\mathcal{C}_{21}, \{2-\}). \end{aligned} \tag{1.9}$$

Here "5" denotes the unique 5-adic place of k in the $\mathcal{C}_1$ case, "2" denotes the unique 2-adic place of k in the cases $\mathcal{C}_j$ for $j = 3,\ 8,\ 11,\ 18$, and "2+" and "2−" denote the two 2-adic places of k in the $\mathcal{C}_{21}$ case. For each of these 13 possible $(\mathcal{C}_j, \mathcal{T}_1)$'s, using the matrices $c \in GL(3, \ell)$ given in Section 3 when $\mathcal{T}_1 \neq \emptyset$, and setting $c = I$ if $\mathcal{T}_1 = \emptyset$, we define $F_{(\mathcal{C}_j,\mathcal{T}_1)} = c^* F_{(\mathcal{C}_j,\emptyset)} c$. We shall show that

$$\bar{\Gamma} \cong \bar{\Gamma}_{(\mathcal{C}_j,\mathcal{T}_1)} = \{g \in M_{3\times 3}(\mathfrak{o}_\ell) : g^* F_{(\mathcal{C}_j,\mathcal{T}_1)} g = F_{(\mathcal{C}_j,\mathcal{T}_1)}\}/\mathcal{Z}, \tag{1.10}$$

where $\mathcal{Z} = \{tI : t \in \mathfrak{o}_\ell \text{ and } |t| = 1\}$. Equation (1.7) takes the form $[\bar{\Gamma} : \Pi] = D$ except in the cases $(\mathcal{C}_{11}, \{2\})$, $(\mathcal{C}_{18}, \{2\})$, $(\mathcal{C}_{21}, \{2+\})$ and $(\mathcal{C}_{21}, \{2-\})$, when it takes the form $[\bar{\Gamma} : \Pi] = D/3$. To show that there are no fpps arising from the matrix algebra case $\mathcal{C}_j$, it is enough to show that for each of the 2 or 3 possibilities $(\mathcal{C}_j, \mathcal{T}_1)$, there are no torsion-free subgroups Π of $\bar{\Gamma}_{(\mathcal{C}_j,\mathcal{T}_1)}$ satisfying this index condition. A basic tool for this is the following simple lemma:

Lemma 1.1. *Suppose that Π is a torsion-free subgroup of finite index in a group $\bar{\Gamma}$. Let K be a finite subroup of $\bar{\Gamma}$. Then $|K|$ divides $[\bar{\Gamma} : \Pi]$.*

Proof. There is an action $g\Pi \mapsto kg\Pi$ of K on the set $\bar{\Gamma}/\Pi$ of cosets. No $k \in K \setminus \{1\}$ can fix any $g\Pi$. For $kg\Pi = g\Pi$ implies that $g^{-1}kg \in \Pi$, contradicting the torsion-free hypothesis. So if $\bar{\Gamma}/\Pi$ is the union of s K-orbits, then $[\bar{\Gamma} : \Pi] = s|K|$. □

In eight of the 13 cases, we are able to produce a finite subgroup K such that $|K|$ does not divide the required index $[\bar{\Gamma} : \Pi]$, thus eliminating those cases.

In the remaining five cases, $(\mathcal{C}_1, \emptyset)$, $(\mathcal{C}_1, \{5\})$, $(\mathcal{C}_{11}, \emptyset)$, $(\mathcal{C}_{11}, \{2\})$ and $(\mathcal{C}_{18}, \emptyset)$, we show that no fpp's can arise by first finding further elements of $\bar{\Gamma}$. In fact, we can find a generating set for $\bar{\Gamma}$, and can find enough relations amongst these generators to get a presentation of $\bar{\Gamma}$, but we don't use this information, except in the case $(\mathcal{C}_{11}, \emptyset)$. While algebra packages such as Magma are able to find all (conjugacy classes of) subgroups of low index in finitely presented groups, in our five cases the required index $[\bar{\Gamma} : \Pi]$ is not nearly "low" enough. We wrote specialized C-programs, described below, to eliminate these cases. They showed that no torsion-free subgroup of the required index can exist in $\bar{\Gamma}$, except in the $(\mathcal{C}_{11}, \emptyset)$ case. We showed that there is (up to conjugation) a unique torsion-free subgroup Π of $\bar{\Gamma}_{(\mathcal{C}_{11},\emptyset)}$ having index $D = 864$. For the surface $M = B(\mathbb{C}^2)/\Pi$, we find that the Betti number b_1 is 2, not 0.

2. Restricting the possible Λ's

We start by verifying the hypothesis of [21, §5.4] in our situation.

Lemma 2.1. *Let q be an integer.*

(i) *If $q \geq 2$, then $q^2 + q + 1$ is divisible by a prime $p \notin \{2, 3, 5\}$.*
(ii) *If $q \geq 3$, then $q^3 + 1$ and $q^2 - q + 1$ are divisible by a prime $p \notin \{2, 3, 5\}$.*

Proof. It is easy to check that neither $q^2 + q + 1$ nor $q^2 - q + 1$ is divisible by 2, 5 or 9, though they may be divisible by 3. Since $q^2 + q + 1 \geq 7$ for $q \geq 2$ and $q^2 - q + 1 \geq 7$ for $q \geq 3$, the result follows. □

Corollary 2.1. *For each of the cases $\mathcal{C}_j$ under consideration, we have $[\Gamma : \Lambda] = 3$ and $\mathcal{T} \subset \{v \in V_f : v$ does not split in $\ell\}$.*

Proof. As explained in [21, §2.3], $[\Gamma : \Lambda]$ must be a power 3^α of 3, and it was shown in [21, §5.4] that $[\Gamma : \Lambda] = 3$ provided that $\{v \in \mathcal{T}' : v \text{ splits in } \ell\} = \emptyset$. From

$$\frac{3^{\alpha-1}}{[\bar{\Gamma} : \Pi]} = \frac{[\Gamma : \Lambda]}{3[\bar{\Gamma} : \Pi]} = m(SU(2,1)/\Lambda) = \frac{1}{D} \prod_{v \in \mathcal{T}} e'(P_v),$$

we see that $\prod_{v \in \mathcal{T}} e'(P_v)$ is a divisor of $3^{\alpha-1}D$. For each of the cases $\mathcal{C}_j$ under consideration, if a prime p divides $3^{\alpha-1}D$, then $p \in \{2, 3, 5\}$. So by Lemma 2.1(i), no number $q_v^2 + q_v + 1$ can divide $3^{\alpha-1}D$. This excludes there being any $v \in \mathcal{T}$ of type described in [21, §2.5(i)]. There are no $v \in \mathcal{T}$ of type described in [21, §2.5(ii)], because $\mathcal{T}_0 = \emptyset$ in this matrix algebra case (see [21, §5.1]). So $\mathcal{T}$ is contained in $\{v \in V_f : v \text{ does not split in } \ell\}$, and the hypothesis of [21, §5.4] is satisfied. □

Lemma 2.2. *With the notation of* (1.9),

(a) *in the case $\mathcal{C}_1$, $\mathcal{T}$ must be $\emptyset$ or $\{5\}$;*
(b) *for cases $\mathcal{C}_3$, $\mathcal{C}_8$, $\mathcal{C}_{11}$ and $\mathcal{C}_{18}$, $\mathcal{T}$ must be $\emptyset$ or $\{2\}$;*
(c) *in the case $\mathcal{C}_{21}$, $\mathcal{T}$ must be $\emptyset$, $\{2+\}$ or $\{2-\}$.*

Proof. By Corollary 2.1, we can use (1.7), and see that any $v \in \mathcal{T}$ is as described in [21, §2.5(iii) or §2.5(iv)]. By Lemma 2.1(ii), $q_v^2 - q_v + 1$ can only divide D if $q_v = 2$. Now $q_v = 4$ for the unique 2-adic place v of k in cases $\mathcal{C}_1$ and $\mathcal{C}_3$. While $q_v = 2$ holds for the unique 2-adic place v of k in case $\mathcal{C}_8$, for this v, $\ell_v = k_v \otimes_k \ell$ is a ramified extension of k_v. So if $v \in \mathcal{T}$ is of the type described in [21, §2.5(iii)], we must be in cases $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ or $\mathcal{C}_{21}$, with either v the unique 2-adic place in cases $\mathcal{C}_{11}$ and $\mathcal{C}_{18}$, or one of the two 2-adic places of k in the $\mathcal{C}_{21}$ case. Only in the $\mathcal{C}_{11}$ case is D divisible by 9, and so in case $\mathcal{C}_{21}$, the places 2+ and 2− cannot both be in $\mathcal{T}$, and only in case $\mathcal{C}_{11}$ might P_v be an Iwahori subgroup.

If v is a place of k of the type described in [21, §2.5(iv)], then v ramifies in ℓ, and so we must be in case $\mathcal{C}_1$, with v the unique 5-adic place of k, or in case $\mathcal{C}_3$ or $\mathcal{C}_8$, with v the unique 2-adic place of k. In these cases, v does not split in ℓ and $\ell_v = k_v \otimes_k \ell$ is a ramified extension of k_v, and so v will be in $\mathcal{T}' \subset \mathcal{T}$ if P_v is not maximal. □

We can restrict the choice of Λ, but first need to describe more concretely the parahoric subgroups P_v associated with G, as defined above for $F = F_{(\mathcal{C}_j, \emptyset)}$.

(a) If $v \in V_f$ and v splits in ℓ, then $\overline{G}(k_v) \cong PGL(3, k_v)$ and the vertices of the associated building X_v, which is if type $\tilde{A}_2$, can be viewed as homothety classes of $\mathfrak{o}_v$-lattices in k_v^3, where $\mathfrak{o}_v$ is the valuation ring of k_v. It is well-known that $PGL(3, k_v)$ acts transitively in the set of vertices of X_v, which means that any two maximal parahoric subgroups of $G(k_v)$ are conjugate by an element of $\overline{G}(k_v)$. We can choose the vertex to be the hypothety class of $\mathfrak{o}_v^3$, and the corresponding maximal parahoric subgroup of $G(k_v)$ is $SL(3, \mathfrak{o}_v)$.

(b)If $v \in V_f$ does not split in ℓ, denote also by v the unique place of ℓ over v. Then $\ell_v = k_v \otimes_k \ell$ is a quadratic extension of k_v, and the complex conjugation automorphism of ℓ extends to an automorphism of ℓ_v fixing the elements of k_v, and is still denoted $x \mapsto \bar{x}$. Then $\overline{G}(k_v) \cong \{g \in M_{3\times 3}(\ell_v) : g^* F g = F\}/\{tI : t \in \ell_v \text{ and } t\bar{t} = 1\}$. The associated building X_v is a tree, which is semi-homogeneous if ℓ_v is an unramified extension of k_v, and is a homogeneous tree if ℓ_v is a ramified extension of k_v. Writing $\mathfrak{O}_v$ for the valuation ring of ℓ_v, the vertices of X_v are $\mathfrak{O}_v$-lattices $\mathcal{L}$ in ℓ_v^3 of one of two types, which we now describe (see [6] for more details). Given a lattice $\mathcal{L}$, the lattice dual to $\mathcal{L}$ with respect to F is by definition

$$\mathcal{L}' = \{\boldsymbol{x} \in \mathfrak{O}_v^3 : \boldsymbol{y}^* F \boldsymbol{x} \in \mathfrak{O}_v \text{ for all } \boldsymbol{y} \in \mathcal{L}\}.$$

Let $\pi_v \in \mathfrak{O}_v$ be a uniformizer of ℓ_v, i.e., an element such that $\pi_v\mathfrak{O}_v$ is the unique prime ideal of $\mathfrak{O}_v$. A vertex of X_v of type 1 is a lattice $\mathcal{L}$ such that $\mathcal{L}' = \mathcal{L}$, while a vertex of type 2 is a lattice $\mathcal{M}$ such that $\pi_v\mathcal{M}' \subsetneqq \mathcal{M} \subsetneqq \mathcal{M}'$. The edges of the tree X_v are the pairs $\mathcal{L}, \mathcal{M}$ of lattices of types 1 and 2 respectively such that $\pi_v\mathcal{M}' \subset \mathcal{L} \subset \mathcal{M}'$. When v is unramified in ℓ, each type 1 vertex has $q_v^3 + 1$ type 2 neighbors, and each type 2 vertex has $q_v + 1$ type 1 neighbors. When v is ramified in ℓ, each vertex of one type has $q_v + 1$ neighbors of the other type. If $g \in M_{3\times 3}(\ell_v)$ and $g^*Fg = F$, then $(g\mathcal{L})' = g(\mathcal{L}')$, so if $\mathcal{L}$ is of type 1 or 2, then $g(\mathcal{L})$ is as well. It is well-known that $\overline{G}(k_v)$ acts transitively on the sets of vertices of each type. If $g \in GL(3, \mathfrak{O}_v)$, then $(g(\mathfrak{O}_v^3))' = (g^*F)^{-1}(\mathfrak{O}_v^3)$. So $\mathfrak{O}_v^3$ is a type 1 lattice if $F \in GL(3, \mathfrak{O}_v)$. Also $g(\mathfrak{O}_v^3)$ is a type 2 vertex if and only if g^*Fg and $\pi_v(g^*Fg)^{-1}$ have entries in $\mathfrak{O}_v$, but are not in $GL(3, \mathfrak{O}_v)$. Now $F = F_{(\mathcal{C}_j, \mathcal{T}_1)} \in SL(3, \mathfrak{o}_v) \subset SL(3, \mathfrak{O}_v)$, so that $\mathfrak{O}_v^3$ is a type 1 vertex of X_v, and the corresponding parahoric subgroup is $\{g \in SL(3, \mathfrak{O}_v) : g^*Fg = F\}$. For $i = 1, 2$, a maximal parahoric subgroup of $G(k_v)$ is called type i if it the stabilizer of a vertex of the type i.

Lemma 2.3. *Given the fundamental group $\Pi \subset PU(2,1)$ of an fpp, let $\tilde{\Pi}$ be the inverse image in $SU(2,1)$ of Π. Then in cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_8$, a principal arithmetic subgroup $\Lambda = G(k) \cap \prod_{v\in V_f} P_v$ of $G(k)$ such that $\tilde{\Pi}$ is contained in the normalizer of Λ in $SU(2,1)$ can be chosen so that $\mathcal{T} = \emptyset$.*

Proof. Assume that we are in case $\mathcal{C}_1$, as the proof for the cases $\mathcal{C}_3$ and $\mathcal{C}_8$ is similar. Let $\Lambda = G(k) \cap \prod_{v\in V_f} P_v$ be a principal arithmetic subgroup of $G(k)$ such that $\tilde{\Pi}$ is contained in the normalizer Γ of Λ in $SU(2,1)$. Then $\mathcal{T} = \emptyset$ or $\{5\}$, by Lemma 2.2. Suppose that $\mathcal{T} = \{5\}$. Then from the definition of $\mathcal{T}$, we see that the unique 5-adic place v_5 of k is in $\mathcal{T}'$, so that the parahoric subgroup P_{v_5} of $G(k_{v_5})$ is not maximal. Let $\tilde{P}_{v_5} \supset P_{v_5}$ be a maximal parahoric subgroup of $G(k_{v_5})$. Let $\tilde{P}_v = P_v$ for all other $v \in V_f$, and form $\tilde{\Lambda} = G(k) \cap \prod_{v\in V_f} \tilde{P}_v$, and let $\tilde{\Gamma}$ be the normalizer of $\tilde{\Lambda}$ in $SU(2,1)$. Then $\Lambda \subset \tilde{\Lambda}$, and the "$\mathcal{T}$" of $\tilde{\Lambda}$ is $\emptyset$. Applying Corollary 2.1 to both Λ and $\tilde{\Lambda}$, we have $[\Gamma : \Lambda] = 3$ and $[\tilde{\Gamma} : \tilde{\Lambda}] = 3$. But as $\omega \notin \ell$ in case $\mathcal{C}_1$ (and also in case $\mathcal{C}_3$ and $\mathcal{C}_8$), the diagonal matrix ωI belongs to Γ and $\tilde{\Gamma}$, but not to Λ or $\tilde{\Lambda}$. Hence $\Gamma = \Lambda Z$ and $\tilde{\Gamma} = \tilde{\Lambda} Z$, where Z is the group of order 3 generated by ωI. Thus $\tilde{\Pi} \subset \Gamma = \Lambda Z \subset \tilde{\Lambda} Z = \tilde{\Gamma}$. So we can replace Λ by $\tilde{\Lambda}$. □

Let us refer to the 5-adic place in k in the $\mathcal{C}_1$ case, the 2-adic place in k in the cases $\mathcal{C}_3$, $\mathcal{C}_8$, $\mathcal{C}_{11}$ and $\mathcal{C}_{18}$, and both 2-adic places in k in the case $\mathcal{C}_{21}$, as the **special places**. For each special place v, we fix a type 2 neighbor $c(\mathfrak{O}_v^3)$ of the type 1 vertex $\mathcal{L} = \mathfrak{O}_v^3$, where $c \in GL(3, \ell)$ has the form

$$c = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{33} \end{pmatrix}. \tag{2.1}$$

The matrices c, which are listed in Section 3, are chosen so that

(a) c^*Fc and $\pi_v(c^*Fc)^{-1}$ have entries in $\mathfrak{O}_v$, but are not in $GL(3, \mathfrak{O}_v)$, and
(b) c and $\pi_v c^{-1}$ have entries in $\mathfrak{O}_v$, but are not in $GL(3, \mathfrak{O}_v)$,
(c) $c \in GL(3, \mathfrak{O}_w)$ for each place w of ℓ other than v.

where, as before π_v is a uniformizer of ℓ_v. The conditions (a) ensure that $\mathcal{M} = c(\mathfrak{O}_v^3)$ satisfies $\pi_v\mathcal{M}' \subsetneqq \mathcal{M} \subsetneqq \mathcal{M}'$, so that $\mathcal{M}$ is a vertex of type 2 in X_v. The conditions (b) ensure that $\pi_v\mathcal{M}' \subsetneqq \mathcal{L} \subsetneqq \mathcal{M}'$, so that $\mathcal{L}$ and $\mathcal{M}$ are neighbors in X_v. Condition (c) will be used in the proof of Lemma 2.5 below.

We now show that to deal with all six cases $\mathcal{C}_j$ under consideration, we may assume that the principal arithmetic subgroup Λ is one of 13 possibilities, and give the value of the product $\prod_{v\in\mathcal{T}} e'(P_v)$ in each case.

Lemma 2.4. *Suppose that Π is the fundamental group of a fpp, and let $\tilde{\Pi}$ denote its inverse image under $\varphi : SU(2,1) \to PU(2,1)$. Then conjugating Π by an element of $\overline{G}(k)$ if necessary, we may choose the principal arithmetic subgroup $\Lambda = G(k) \cap \prod_{v \in V_f} P_v$ such that $\tilde{\Pi}$ is contained in the normalizer Γ of Λ in $SU(2,1)$ to have the following properties (where $F = F_{(\mathcal{C}_j,\emptyset)}$ is as in (1.2)):*

(i) $P_v = SL(3, \mathfrak{o}_v)$ for every $v \in V_f$ which splits in ℓ;

(ii) $P_v = \{g \in SL(3, \mathfrak{O}_v) : g^*Fg = F\}$ for every $v \in V_f$ which does not split in ℓ, except that for the above special places, P_v may instead be the following particular type 2 maximal parahoric subgroup:

$$\{g \in SL(3, \ell_v) : g^*Fg = F \text{ and } g(c(\mathfrak{O}_v^3)) = c(\mathfrak{O}_v^3)\}. \tag{2.2}$$

In cases $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_8$, $\mathcal{C}_{11}$ and $\mathcal{C}_{18}$ write $\Lambda = \Lambda_{(\mathcal{C}_j,\emptyset)}$ if P_v for the special place v is of type 1, and write $\Lambda = \Lambda_{(\mathcal{C}_j,\{v\})}$ if P_v for the special place v is as in (2.2). The same notation applies in the case $\mathcal{C}_{21}$, noting that at most one of the P_v's for the two special places can be of type 2. Thus $\Lambda = \Lambda_{(\mathcal{C}_j,\mathcal{T}_1)}$, in the notation of (1.8).

The product $\prod_{v \in \mathcal{T}} e'(P_v)$ equals 1 except in the four cases $(\mathcal{C}_{11}, \{2\})$, $(\mathcal{C}_{18}, \{2\})$, $(\mathcal{C}_{21}, \{2+\})$ and $(\mathcal{C}_{21}, \{2-\})$, when $\prod_{v \in \mathcal{T}} e'(P_v) = 3$.

Proof. If $v \in V_f$ splits in ℓ, then $v \notin \mathcal{T}$ by Corollary 2.1. and so P_v is maximal (as $\mathcal{T}' \subset \mathcal{T}$). By the discussion before Lemma 2.3, P_v is conjugate by an element of $\overline{G}(k_v)$ to $SL(3, \mathfrak{o}_v)$.

Suppose now that $v \in V_f$ does not split in ℓ.

If we are in one of the cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_8$, then by Lemma 2.3, we may suppose that $\mathcal{T} = \emptyset$. As $\mathcal{T}'' \subset \mathcal{T}$, P_v must be hyperspecial except for the v which ramifies in ℓ, which is the special place in each of these three cases. By the discussion before Lemma 2.3, for this v, P_v is conjugate by an element of $\overline{G}(k_v)$ to either $\{g \in SL(3, \mathfrak{O}_v) : g^*Fg = g\}$ or to the group (2.2).

If we are in one of the cases $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$, then by Lemma 2.2, $\mathcal{T}$ is either empty or consists of a single 2-adic place of k. As no v ramifies in ℓ, and as $\mathcal{T}'' \subset \mathcal{T}$, either v is a 2-adic place or P_v is hyperspecial. In the latter case, P_v is again conjugate by an element of $\overline{G}(k_v)$ to $\{g \in SL(3, \mathfrak{O}_v) : g^*Fg = g\}$. When v is a 2-adic place, P_v is either maximal hyperspecial, maximal and non-hyperspecial, or Iwahori. In the first two of these three cases, P_v is conjugate by an element of $\overline{G}(k_v)$ to either $\{g \in SL(3, \mathfrak{O}_v) : g^*Fg = g\}$ or to the group (2.2). As we saw in the proof of Lemma 2.2, in cases $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$, P_v cannot be Iwahori, because 9 does not divide D.

If, for the 2-adic place of k in the $\mathcal{C}_{11}$ case, P_v is an Iwahori subgroup of $G(k_v)$, then P_v fixes both endpoints of some edge of the tree X_v. Now $\overline{G}(k_v)$ acts transitively on the edges of the tree X_v. So conjugating, we may assume that the edge fixed by P_v is the one with endpoints $\mathfrak{O}_v^3$ and $c(\mathfrak{O}_v^3)$. If P_v is not Iwahori, it is conjugate by an element of $\overline{G}(k_v)$ to either $\{g \in SL(3, \mathfrak{O}_v) : g^*Fg = g\}$ or to the group (2.2).

Since the class number of ℓ is 1, the number $h_{\ell,3}$ (see [21, §2.1]) is also 1, and so by [21, Proposition 5.3], there is a $g \in \overline{G}(k)$ which conjugates all the P_v, $v \in V_f \setminus \mathcal{T}$, to the above particular maximal parahorics. Replacing Π by $g\Pi g^{-1}$, we may assume that the P_v's are the above particular ones.

We can now identify the normalizer Γ of Λ in $SU(2,1)$. In the cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_8$ we have already seen in the proof of Lemma 2.3 that $\Gamma = \Lambda Z$, where Z is the group of order 3 generated by ωI. Assume that we are in of the cases $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$. Then $\omega \in \ell$ and so $Z \subset \Lambda$. Since $[\Gamma : \Lambda] = 3$, to find Γ, it is sufficient to show that

$$\gamma = t \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \quad \text{where} \quad \omega = e^{2\pi i/3} \text{ and } t = e^{-2\pi i/9}, \tag{2.3}$$

normalizes Λ, as γ is not in Λ because $t \notin \ell$. If $g = (g_{ij})$, then

$$\gamma g \gamma^{-1} = \begin{pmatrix} g_{11} & g_{12} & \omega^{-1} g_{13} \\ g_{21} & g_{22} & \omega^{-1} g_{23} \\ \omega g_{31} & \omega g_{32} & g_{33} \end{pmatrix}.$$

It is clear from this that γ normalizes any $SL(3, \mathfrak{o}_v)$, when v splits in ℓ, and normalizes $\{g \in SL(3, \mathfrak{O}_v) : g^* F g = F\}$ when v does not split in ℓ because γ commutes with any of our matrices $F_{(\mathcal{C}_j, \emptyset)}$. Moreover γ normalizes the group (2.2) because γ commutes with any c of the form (2.1).

We can now see that for the 2-adic place v of k in the $\mathcal{C}_{11}$ case, P_v can be chosen not to be Iwahori. For if P_v is the Iwahori subgroup fixing the edge from $\mathfrak{O}_v^3$ to $c(\mathfrak{O}_v^3)$, then $P_v \subset \tilde{P}_v = \{g \in SL(3, \mathfrak{O}_v) : g^* F g = F\}$. Let $\tilde{\Lambda}$ be the principal arithmetic subgroup obtained by replacing P_v by $\tilde{P}_v$, and leaving the other $P_{v'}$'s unchanged. Then the normalizer $\tilde{\Gamma}$ of $\tilde{\Gamma}$ is equal to $\tilde{\Lambda}\langle\gamma\rangle$, and so contains $\Lambda\langle\gamma\rangle = \Gamma$, and hence contains $\tilde{\Pi}$. So we can replace Λ by $\tilde{\Lambda}$.

Finally, the above shows that in formula 1.7, the product $\prod_{v \in V_f} e'(P_v)$ is 1 except for the four cases listed at the end of this lemma's statement, when it equals 3. □

Lemma 2.5. *Let $\Lambda = \Lambda_{(\mathcal{C}_j, \mathcal{T}_1)}$ denote the particular principal arithmetic subgroup of $G(k)$ described in Lemma 2.4. Then*

$$\Lambda_{(\mathcal{C}_j, \mathcal{T}_1)} \cong \{g \in SL(\mathfrak{o}_\ell) : g^* F_{(\mathcal{C}_j, \mathcal{T}_1)} g = F_{(\mathcal{C}_j, \mathcal{T}_1)}\}. \tag{2.4}$$

Proof. Let $g \in G(k)$. Then $g = (g_{ij}) \in M_{3\times 3}(\ell)$ and $g^* F g = F$ for $F = F_{(\mathcal{C}_j, \emptyset)}$. When v splits in ℓ, then we have $P_v \cong SL(3, \mathfrak{o}_v)$, and under this isomorphism, $g \in G(k) \cap P_v$ if and only if the entries g_{ij} are in the valuation ring $\mathfrak{O}_{v'}$ for both places v' over v. When v does not split in ℓ, and $P_v = \{g \in SL(3, \mathfrak{O}_v) : g^* F g = F\}$, then $g \in G(k) \cap P_v$ if and only if the g_{ij}'s are in $\mathfrak{O}_v$. Hence $g \in \Lambda_{(\mathcal{C}_j, \emptyset)}$ if and only if $g_{ij} \in \mathfrak{O}_w$ for all i and j and all places w of ℓ, and so if and only if the g_{ij}'s are in $\mathfrak{o}_\ell$. This completes the proof when $\mathcal{T}_1 = \emptyset$.

When $\mathcal{T}_1 = \{v\} \neq \emptyset$, then in the same way, $g \in \Lambda_{(\mathcal{C}_j, \mathcal{T}_1)}$ if and only if $g_{ij} \in \mathfrak{O}_w$ for all i and j and all places $w \neq v$ of ℓ, and also $c^{-1} g c$ has entries in $\mathfrak{O}_v$. But by the condition (c) imposed on the matrix c after 2.1, $c^{-1} g c$ also has entries in $\mathfrak{O}_w$ for each place $w \neq v$ of ℓ. Hence $c^{-1} g c$ has entries in $\mathfrak{o}_\ell$ and is unitary with respect to $c^* F_{(\mathcal{C}_j, \emptyset)} c = F_{(\mathcal{C}_j, \mathcal{T}_1)}$, and $g \mapsto c^{-1} g c$ gives the desired isomorphism. □

Lemma 2.6. *Let $\Lambda = \Lambda_{(\mathcal{C}_j, \mathcal{T}_1)}$ denote the particular principal arithmetic subgroup of $G(k)$ described in Lemma 2.4, and let Γ denote its normalizer in $SU(2,1)$, and $\bar{\Gamma}$ the image in $PU(2,1)$ of Γ under $\varphi : SU(2,1) \to PU(2,1)$. Then*

$$\bar{\Gamma} \cong \bar{\Gamma}_{(\mathcal{C}_j, \mathcal{T}_1)} = \{g \in M_{3\times 3}(\mathfrak{o}_\ell) : g^* F_{(\mathcal{C}_j, \mathcal{T}_1)} g = F_{(\mathcal{C}_j, \mathcal{T}_1)}\}/\mathcal{Z},$$

where $\mathcal{Z} = \{tI : t \in \mathfrak{o}_\ell : |t| = 1\}$.

Proof. Let $\Lambda'_{(\mathcal{C}_j, \mathcal{T}_1)}$ denote the group on the right in (2.4). The map $g \mapsto g\mathcal{Z}$ from $\Lambda'_{(\mathcal{C}_j, \mathcal{T}_1)}$ to $\bar{\Gamma}_{(\mathcal{C}_j, \mathcal{T}_1)}$ is an isomorphism for cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_8$, in which 3 does not divide $|\mathcal{Z}| = |\mathfrak{o}_\ell^1|$, so that $\omega \notin \ell$, in the notation of Lemma 4.3 below. In cases $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$, in which $\omega \in \ell$, the map has kernel $\{\omega^\nu I : \nu = 0, 1, 2\}$ and image a normal subgroup of index 3.

We also have an isomorphism $g \mapsto cgc^{-1}$, $\Lambda'_{(\mathcal{C}_j, \mathcal{T}_1)} \to \Lambda_{(\mathcal{C}_j, \mathcal{T}_1)}$, and an embedding $h \mapsto \Delta h \Delta^{-1}$ of $G(k)$ into $SU(2,1)$, where $\Delta = \Delta_{(\mathcal{C}_j, \emptyset)}$, as in (4.16) below. Write $\tilde{g} = \Delta c g c^{-1} \Delta^{-1}$. In the cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_8$ in which $\omega \notin \ell$, the normalizer Γ of Λ in $SU(2,1)$ is $\{\tilde{g}(\omega^\nu I) : g \in \Lambda'_{(\mathcal{C}_j, \mathcal{T}_1)}$ and $\nu = 0, 1, 2\}$, and so its image under φ is $\{\tilde{g}\mathcal{Z}_0 : g \in \Lambda'_{(\mathcal{C}_j, \mathcal{T}_1)}\}$, where $\mathcal{Z}_0 = \{tI : t \in \mathbb{C}$ and $|t| = 1\}$.

We also have an embedding $\Gamma_{(\mathcal{C}_j,\mathcal{T}_1)} \to PU(2,1)$ which maps $g\mathcal{Z}$ to $\tilde{g}\mathcal{Z}_0$. So we have seen that in the cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_8$, the image of this map is exactly the image under φ of Γ, proving the lemma in these cases.

In the cases $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$ in which $\omega \in \ell$, the normalizer Γ of Λ in $SU(2,1)$ is $\{\tilde{g}(\gamma^\nu I) : g \in \Lambda'_{(\mathcal{C}_j,\mathcal{T}_1)}$ and $\nu = 0,1,2\}$, where γ is as in (2.3), and so its image under φ is $\{\tilde{g}\gamma_1^\nu \mathcal{Z}_0 : g \in \Lambda'_{(\mathcal{C}_j,\mathcal{T}_1)}$ and $\nu = 0,1,2\}$, where γ_1 is the diagonal matrix with diagonal entries 1, 1 and ω. Noting that $\gamma_1\mathcal{Z} \in \bar{\Gamma}_{(\mathcal{C}_j,\mathcal{T}_1)}$, we see that again the image of $\bar{\Gamma}_{(\mathcal{C}_j,\mathcal{T}_1)}$ under the above embedding into $PU(2,1)$ is exactly the image under φ of Γ, proving the lemma also in these cases. □

3. Details about the six pairs (k,ℓ) of fields

The class number of both k and ℓ is 1 in each of these cases. Let d_k and d_ℓ denote the field discriminants of k and ℓ.

Complex conjugation induces an automorphism of ℓ, and so if $\alpha \in \mathfrak{o}_\ell$ then $|\alpha|^2 \in \mathfrak{o}_k$, and so in cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_{21}$ can be written $P_0(\alpha) + Q(\alpha)(r+1)/2$, where $P_0(\alpha), Q(\alpha) \in \mathbb{Z}$, and in cases $\mathcal{C}_8$, $\mathcal{C}_{11}$ and $\mathcal{C}_{18}$ can be written $P(\alpha) + Q(\alpha)r$, where $P(\alpha), Q(\alpha) \in \mathbb{Z}$. In cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_{21}$, it will be convenient to write $P(\alpha) = 2P_0(\alpha) + Q(\alpha)$, so that $|\alpha|^2 = (P(\alpha) + Q(\alpha)r)/2$.

Lemma 3.1. *For any $\alpha \in \mathfrak{o}_\ell$,*

(i) $P(\alpha) \geq 0$, *with* $P(\alpha) = 0$ *if and only if* $\alpha = 0$;

(ii) $|Q(\alpha)| \leq \frac{1}{r}P(\alpha)$;

Proof. Choose an automorphism ψ of ℓ mapping r to $-r$. This commutes with conjugation. Hence, in cases $\mathcal{C}_8$, $\mathcal{C}_{11}$ and $\mathcal{C}_{18}$, applying ψ to both sides of $|\alpha|^2 = P(\alpha) + Q(\alpha)r$, we get $|\psi(\alpha)|^2 = P(\alpha) - Q(\alpha)r$. Hence

$$P(\alpha) = \frac{1}{2}\big(|\alpha|^2 + |\psi(\alpha)|^2\big) \text{ and } Q(\alpha) = \frac{1}{2r}\big(|\alpha|^2 + |\psi(\alpha)|^2\big),$$

and these formulas clearly imply (i) and (ii) in those cases. In cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_{21}$, we apply ψ to both sides of $|\alpha|^2 = (P(\alpha) + Q(\alpha)r)/2$, and get

$$P(\alpha) = |\alpha|^2 + |\psi(\alpha)|^2 \text{ and } Q(\alpha) = \frac{1}{r}\big(|\alpha|^2 + |\psi(\alpha)|^2\big),$$

and again (i) and (ii) follow. □

We choose an integral basis v_1, v_2, v_3, v_4 for $\mathfrak{o}_\ell$, and writing $\alpha = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$, we calculate $P(\alpha)$ and $Q(\alpha)$. In the three cases $\mathcal{C}_1$, $\mathcal{C}_8$ and $\mathcal{C}_{11}$ when ℓ is a cyclotomic field $\mathbb{Q}(\zeta)$, we have $\mathfrak{o}_\ell = \mathbb{Z}[\zeta]$ (see [1, Theorem 46], for example), and so we take $v_1 = 1$, $v_2 = \zeta$, $v_3 = \zeta^2$ and $v_4 = \zeta^3$.

3.1. The case $\mathcal{C}_1$.

We realize the embedding of k into ℓ mapping r to $-2\zeta^2 - 2\zeta^3 - 1$ (this equals the positive square root of 5 when $\zeta = e^{2\pi i/5}$). With respect to the integral basis $v_1 = 1$, $v_2 = \zeta$, $v_3 = \zeta^2$ and $v_4 = \zeta^3$ of $\mathfrak{o}_\ell$, we have $P(\alpha) = 2P_0(\alpha) + Q(\alpha)$ for

$$P_0(\alpha) = a_1^2 - a_1a_2 + a_2^2 - a_2a_3 + a_3^2 - a_3a_4 + a_4^2 \quad \text{and}$$
$$Q(\alpha) = a_1a_2 - a_1a_3 - a_1a_4 + a_2a_3 - a_2a_4 + a_3a_4.$$

The smallest eigenvalue $\lambda_{\min}$ of the form associated with $P = 2P_0 + Q$ is $1/2$, and so $P(\alpha) \geq \frac{1}{2}\sum_j a_j^2$.

We have $d_k = 5$ and $d_\ell = 125$.

The only prime p which ramifies in k is 5, and $5\mathfrak{o}_k = \mathfrak{p}^2$ for $\mathfrak{p} = r\mathfrak{o}_k$.

The only place of k which ramifies in ℓ is the 5-adic one. In fact, the prime 5 ramifies totally in ℓ, and since $N_{\ell/\mathbb{Q}}(\zeta - 1) = 5$ we have $5\mathfrak{o}_\ell = \mathfrak{P}^4$ for $\mathfrak{P} = (\zeta - 1)\mathfrak{o}_\ell$, and $\zeta - 1$ is a uniformizer for the 5-adic place of ℓ.

For the 5-adic place of k, $\zeta - 1$ is uniformizer of $\ell_v = \mathbb{Q}_5(\zeta)$, and we set

$$c = \begin{pmatrix} \zeta^3 - \zeta^2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then c has the form (2.1), and the properties (a) and (b) described after (2.1), as

$$(\zeta - 1)c^{-1} = \begin{pmatrix} \zeta^3 & \zeta^3 & 0 \\ 0 & 1 - \zeta & 0 \\ 0 & 0 & \zeta - 1 \end{pmatrix},$$

$$c^* F c = \begin{pmatrix} 2\zeta^3 + 2\zeta^2 + 1 & -\zeta^3 - \zeta^2 - 2\zeta - 1 & 0 \\ \zeta^3 + \zeta^2 + 2\zeta + 1 & 2\zeta^3 + 2\zeta^2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$(\zeta - 1)(c^* F c)^{-1} = \begin{pmatrix} -\zeta^3 - 2\zeta^2 - \zeta - 1 & -\zeta - 1 & 0 \\ \zeta + 1 & -\zeta^3 - 2\zeta^2 - \zeta - 1 & 0 \\ 0 & 0 & \zeta - 1 \end{pmatrix}.$$

3.2. **The case $\mathcal{C}_3$.** We realize the embedding of k into $\ell = \mathbb{Q}(z)$ and the field isomorphism $\mathbb{Q}(r, i) \cong \ell$ by mapping r to $3 + 2z^2$ and i to $2z + z^3$. Magma verifies that $v_1 = 1$, $v_2 = (r + 1)/2$, $v_3 = i$ and $v_4 = i(r + 1)/2$ is an integral basis of $\mathfrak{o}_\ell$, and we calculate that

$$P(\alpha) = a_1^2 + a_2^2 + a_3^2 + a_4^2 \quad \text{and} \quad Q(\alpha) = 2a_1a_2 + a_2^2 + 2a_3a_4 + a_4^2.$$

We have $d_k = 5$ and $d_\ell = 400 = 2^4 \times 5^2$. Note that the k of $\mathcal{C}_3$ equals that of $\mathcal{C}_1$.

Only the 2-adic place v of k ramifies in ℓ, and $2\mathfrak{o}_\ell = \mathfrak{P}^2$ for $\mathfrak{P} = (i + 1)\mathfrak{o}_\ell$. So $i + 1 = z^3 + 2z + 1$ is a uniformizer of $\ell_v = \mathbb{Q}_2(i)$, and we set

$$c = \begin{pmatrix} z^2 + z + 1 & z^2 + 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This c has the required properties, as

$$(i + 1)c^{-1} = \begin{pmatrix} z^2 + 2 & -1 & 0 \\ 0 & z^3 + 2z + 1 & 0 \\ 0 & 0 & z^3 + 2z + 1 \end{pmatrix},$$

$$c^* F c = \begin{pmatrix} -2z^2 - 2 & -z^2 + z - 1 & 0 \\ -z^2 - z - 1 & -2z^2 - 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$(i + 1)(c^* F c)^{-1} = \begin{pmatrix} -z^3 - z^2 - 3z - 2 & z^2 + 2 & 0 \\ z^3 + 3z & -z^3 - z^2 - 3z - 2 & 0 \\ 0 & 0 & z^3 + 2z + 1 \end{pmatrix}.$$

3.3. **The case $\mathcal{C}_8$.** We realize the embedding of k into $\ell = \mathbb{Q}(\zeta)$ and the isomorphism $\mathbb{Q}(r,i) \cong \ell$ by mapping r to $\zeta + \zeta^{-1} = \zeta - \zeta^3$ and i to ζ^2. Using the integral basis 1, ζ, ζ^2 and ζ^3, we calculate

$$P(\alpha) = a_0^2 + a_1^2 + a_2^2 + a_3^2 \quad \text{and} \quad Q(\alpha) = a_0a_1 - a_0a_3 + a_1a_2 + a_2a_3.$$

We have $d_k = 8$ and $d_\ell = 256$.The only prime p which ramifies in k is 2, and $2\mathfrak{o}_k = \mathfrak{p}^2$ for $\mathfrak{p} = r\mathfrak{o}_k$.

Only the 2-adic place v of k ramifies in ℓ, and $r\mathfrak{o}_\ell = \mathfrak{P}^2$ for $\mathfrak{P} = (\zeta-1)\mathfrak{o}_\ell$ because $r = (\zeta-1)^2(\zeta^3+\zeta^2-1)$, and $\zeta^3+\zeta^2-1$ has inverse $-\zeta^2+\zeta-1$, also in $\mathfrak{o}_\ell$. So $\zeta - 1$ is a uniformizer of $\ell_v = \mathbb{Q}_2(\zeta)$ for this v, and we set

$$c = \begin{pmatrix} \zeta-1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This c has the required properties, as

$$(\zeta-1)c^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & \zeta-1 & 0 \\ 0 & 0 & \zeta-1 \end{pmatrix}, \quad c^*Fc = \begin{pmatrix} \zeta^3-\zeta & \zeta^2+\zeta & 0 \\ -\zeta^3-\zeta^2 & 2\zeta^3-2\zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$(\zeta-1)(c^*Fc)^{-1} = \begin{pmatrix} -2\zeta^3-2\zeta^2 & -\zeta^3+\zeta+1 & 0 \\ -\zeta^2-\zeta-1 & -\zeta^3-\zeta^2 & 0 \\ 0 & 0 & \zeta-1 \end{pmatrix}.$$

3.4. **The case $\mathcal{C}_{11}$.** We realize the embedding of k into $\ell = \mathbb{Q}(\zeta)$ and the isomorphism $\mathbb{Q}(r,i) \cong \ell$ by mapping r to $\zeta + \zeta^{-1} = 2\zeta - \zeta^3$ and i to ζ^3. Using the integral basis 1, ζ, ζ^2 and ζ^3, we calculate

$$P(\alpha) = a_0^2 + a_0a_2 + a_1^2 + a_1a_3 + a_2^2 + a_3^2 \quad \text{and} \quad Q(\alpha) = a_0a_1 + a_1a_2 + a_2a_3.$$

Calculating eigenvalues, we find that $P(\alpha) \geq \frac{1}{2}\sum_j a_j^2$.

We have $d_k = 12$ and $d_\ell = 144$.The only primes p which ramify in k are 2 and 3. No $v \in V_f$ ramifies in ℓ. The 2-adic place of k is inert in ℓ, and in particular does not split in ℓ. The x of the Table 2, namely $x = r+1$, is a uniformizer of $\ell_v = \mathbb{Q}_2(z)$ (and so is $2x^{-1} = r-1$), and we set

$$c = \begin{pmatrix} 1 & 1 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This c has the required properties, as

$$xc^{-1} = \begin{pmatrix} x & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{pmatrix},$$

$$c^*Fc = \begin{pmatrix} -x & 0 & 0 \\ 0 & -x & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x(c^*Fc)^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & x \end{pmatrix}.$$

3.5. **The case $\mathcal{C}_{18}$.** We realize the embedding of k into $\ell = \mathbb{Q}(z)$ and the isomorphism $\mathbb{Q}(r,\zeta_3) \cong \ell$ by mapping r to $z^3/2 - 2z$ and ζ_3 to $-z^2/2$. Magma verifies that $v_1 = 1$, $v_2 = z$, $v_3 = z^2/2$ and $v_4 = z^3/2$ form an integral basis of $\mathfrak{o}_\ell$, and calculate that

$$P(\alpha) = a_1^2 + a_1a_3 + 2a_2^2 + 2a_2a_4 + a_3^2 + 2a_4^2 \quad \text{and} \quad Q(\alpha) = -(a_1a_2 + a_2a_3 + a_3a_4).$$

Calculating eigenvalues, we find that $P(\alpha) \geq \frac{1}{2}\sum_j a_j^2$.

We have $d_k = 24$ and $d_\ell = 576 = 2^6 \times 3^2$. The only primes p which ramify in k are 2 and 3.

No $v \in V_f$ ramifies in ℓ. In particular, the 2-adic place of k is inert in ℓ, while the 3-adic place of k splits in ℓ.

As for the $\mathcal{C}_{11}$ case, for the 2-adic place v of k, the x of the Table 2, namely $x = r+2$, is a uniformizer of $\ell_v = \mathbb{Q}_2(z)$ (as is $2x^{-1} = r-2$), and we define c as in the $\mathcal{C}_{11}$ case, except that $x = r+2$ now.

3.6. **The case $\mathcal{C}_{21}$.** We realize the embedding of k into $\ell = \mathbb{Q}(z)$ and the isomorphism $\mathbb{Q}(r,\zeta_3) \cong \ell$ by mapping r to $(2z^3 - 2z^2 - 10z - 3)/3$ and ζ_3 to $(-z^3 - 2z^2 + 2z + 3)/6$. Magma verifies that $v_1 = 1$, $v_2 = (r+1)/2$, $v_3 = \zeta_3$ and $v_4 = 1 - z$, form an integral basis of $\mathfrak{o}_\ell$, and we calculate that

$$\begin{aligned} P_0(\alpha) &= a_1^2 - a_1a_3 + a_1a_4 + 8a_2^2 + 8a_2a_4 + a_3^2 - a_3a_4 + 3a_4^2, \\ Q(\alpha) &= 2a_1a_2 + a_1a_4 + a_2^2 - a_2a_3 + 2a_2a_4 - a_3a_4 + a_4^2. \end{aligned}$$

We have $P(\alpha) = 2P_0(\alpha) + Q(\alpha) \geq \lambda_{\min}\sum_j a_j^2$ for $\lambda_{\min} = 0.772\ldots$.

We have $d_k = 132 = 2^2 \times 3 \times 11$ and $d_\ell = 1089 = 3^2 \times 11^2$. The 2-adic place of $\mathbb{Q}$ splits in k, and so there are two 2-adic places $2+$ and $2-$ of k, corresponding to the prime ideals $\frac{r+5}{2}\mathfrak{o}_k$ and $\frac{r-5}{2}\mathfrak{o}_k$ of $\mathfrak{o}_k = \mathbb{Z}[(r+1)/2]$, respectively. Indeed, $2 = \frac{r+5}{2} \times \frac{r-5}{2}$, with $\frac{r+5}{2}, \frac{r-5}{2} \in \mathbb{Z}[(r+1)/2]$ and $(r \pm 5)/(r \pm 5) = (29 \pm 5r)/4 \notin \mathbb{Z}[(r+1)/2]$. The 3-adic and 11-adic places of $\mathbb{Q}$ ramify in k. Explicitly, $3 = (6+r)(6-r)$, and $(6 \pm r)/(6 \mp r) = 23 \pm 4r$, while $11 = (2r-11)(2r+11)$, with $(2r \pm 11)/(2r \mp 11) = (23 \pm 4r)$.

No $v \in V_f$ ramifies in ℓ. In particular, the two 2-adic places of k are inert in ℓ, the 3-adic valuation on k splits in ℓ, and the 11-adic valuation on k is inert in ℓ.

Both 2-adic places $2+$ and $2-$ are inert in ℓ, and in particular do not split in ℓ. The x of the Table 2, i.e., $x = \frac{r+5}{2}, = x_+$, say, is a uniformizer of ℓ_{2+}, and $2x^{-1} = \frac{r-5}{2}, = x_-$, say, is a uniformizer of ℓ_{2-}, and we set

$$c_+ = \begin{pmatrix} 1 & 1 & 0 \\ 0 & x_+ & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad c_- = \begin{pmatrix} x_- & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then c_+ and c_- have the required properties, as

$$x_+c_+^{-1} = \begin{pmatrix} x_+ & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x_+ \end{pmatrix} \quad \text{and} \quad x_-c_-^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & x_- & 0 \\ 0 & 0 & x_- \end{pmatrix}, \quad \text{and}$$

$$c_\epsilon^* F c_\epsilon = \begin{pmatrix} -x_\epsilon & 0 & 0 \\ 0 & -x_\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad x_\epsilon(c_\epsilon^* F c_\epsilon)^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & x_\epsilon \end{pmatrix} \quad \text{for } \epsilon = \pm.$$

4. Finding elements of $\bar\Gamma$

So far, we have seen that the fundamental group Π of an fpp must be a torsion-free subgroup of the group $\bar\Gamma = \bar\Gamma_{(\mathcal{C}_j,\mathcal{T}_1)}$ defined in (1.10) of index D in 9 of 13 cases $(\mathcal{C}_j,\mathcal{T}_1)$, and of index $D/3$ in the other 4 cases. Our method of eliminating a case depends on finding enough elements of $\bar{\Gamma}_{(\mathcal{C}_j,\mathcal{T}_1)}$ to show that this group contains no torsion-free subgroup of that index.

4.1. $PU(2,1)$ **and its action on** $B(\mathbb{C}^2)$**.** Let

$$F_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{4.1}$$

and let $U(2,1)$ denote the group of complex 3×3 matrices g such that $g^*F_0g = F_0$, and let $P(2,1)$ denote the quotient of $U(2,1)$ by its center, $\{tI : |t| = 1\}$. Then $PU(2,1)$ acts on the unit ball $B(\mathbb{C}^2) = \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$ in $\mathbb{C}^2$. The action of the image of $g \in U(2,1)$ is given by

$$(z_1,z_2) \mapsto (w_1,w_2) \quad \text{if and only if} \quad g\begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \\ 1 \end{pmatrix} \quad \text{for some } \lambda.$$

This action preserves the hyperbolic metric d on $B(\mathbb{C}^2)$, which satisfies

$$\cosh^2(d(z,w)) = \frac{|1-\langle z,w\rangle|^2}{(1-|z|^2)(1-|w|^2)}, \tag{4.2}$$

(see [3, Page 310] for example) where $\langle z,w\rangle = z_1\bar{w}_1 + z_2\bar{w}_2$ and $|z| = \sqrt{|z_1|^2+|z_2|^2}$ for $z = (z_1,z_2)$ and $w = (w_1,w_2)$ in $B(\mathbb{C}^2)$.

In particular, writing 0 for the origin in $B(\mathbb{C}^2)$, and using $g.0 = (g_{13}/g_{33}, g_{23}/g_{33})$ for $g = (g_{ij}) \in U(2,1)$ and (4.5) below, we see that

$$\cosh^2(d(0,g.0)) = |g_{33}|^2. \tag{4.3}$$

If a matrix $g = (g_{ij})$ satisfies $g^*F_0g = F_0$, then it is invertible, and its inverse also satisfies the condition. So for $F = F_0$, the following three matrices are zero

$$M = g^*Fg - F, \quad R = gF^{-1}g^* - F^{-1} \quad \text{and} \quad N = \theta^{-1}g^{\mathrm{adj}} - F^{-1}g^*F \tag{4.4}$$

where $\theta = \det(g)$ and $g^{\mathrm{adj}} = \theta g^{-1}$ is the transpose of the cofactor matrix of g. While of course $F^{-1} = F$ when $F = F_0$, we shall also use the equations (4.4) for g satisfying $g^*Fg = F$ for the matrices $F = F_{(\mathcal{C}_j,\mathcal{T}_1)}$. Now (4.4) gives many conditions on the entries g_{ij} of g. In particular, when $F = F_0$, from $M_{33} = 0$ we see that

$$|g_{13}|^2 + |g_{23}|^2 = |g_{33}|^2 - 1, \tag{4.5}$$

and from $R_{11} = 0$ we see that

$$|g_{11}|^2 + |g_{12}|^2 = |g_{13}|^2 + 1. \tag{4.6}$$

Lemma 4.1. *Given complex numbers g_{11}, g_{12}, g_{13}, g_{23}, g_{33} and θ, so that $|\theta| = 1$ and the g_{ij}'s satisfy* (4.5) *and* (4.6)*, there is a unique matrix g satisfying $g^*F_0g = F_0$ and $\det(g) = \theta$ with the given entries g_{11}, g_{12}, g_{13}, g_{23}, g_{33} in its first row and third column.*

Proof. From $M_{31} = 0$ and $M_{32} = 0$, we must have

$$g_{31} = \frac{g_{11}\bar{g}_{13} + g_{21}\bar{g}_{23}}{\bar{g}_{33}} \quad \text{and} \quad g_{32} = \frac{g_{12}\bar{g}_{13} + g_{22}\bar{g}_{23}}{\bar{g}_{33}}, \tag{4.7}$$

respectively. From $N_{21} = 0$ and $N_{11} = 0$, we must have

$$g_{21} = \frac{\theta\bar{g}_{12}\bar{g}_{33} - g_{11}g_{23}\bar{g}_{13}}{|g_{23}|^2 - |g_{33}|^2} \quad \text{and} \quad g_{22} = -\frac{\theta\bar{g}_{11}\bar{g}_{33} - g_{12}g_{23}\bar{g}_{13}}{|g_{23}|^2 - |g_{33}|^2} \tag{4.8}$$

respectively. Notice that the denominators appearing here cannot be zero, since by by (4.5), $|g_{33}| \geq 1$ and $|g_{23}|^2 - |g_{33}|^2 = -(|g_{13}|^2 + 1) < 0$.

Defining first g_{21} and g_{22} using (4.8), then g_{31} and g_{32} using (4.7), we have a matrix g, and must check that $g^*F_0g = F_0$ and $\det(g) = \theta$. Write $r_1 = |g_{11}|^2 + |g_{12}|^2 - |g_{13|}|^2 - 1$ and $c_3 = |g_{13}|^2 + |g_{23}|^2 - |g_{33}|^2 + 1$. Then $M_{31} = M_{32} = 0$

and $M_{33} = c_3 = 0$, by (4.5). Also, $M_{11} = (r_1 + (1 - |g_{11}|^2)c_3)/(|g_{23}|^2 - |g_{33}|^2)$, $M_{21} = g_{11}\bar{g}_{12}c_3/(|g_{23}|^2 - |g_{33}|^2)$, and $M_{22} = (r_1 + (1 - |g_{12}|^2)c_3)/(|g_{23}|^2 - |g_{33}|^2)$, and $M_{ji} = \overline{M_{ij}}$. So $M = 0$ is a consequence of (4.5) and (4.6). We see that $\det(g)$ equals θ by writing $\det(g)/\theta = c_3 + (r_1 + c_3)(c_3 - 1)/(|g_{23}|^2 - |g_{33}|^2)$. □

4.2. **Column 3 and row 1 conditions for F.** Suppose that g is a 3×3 matrix satisfying $g^*Fg = F$, where F is one of the thirteen matrices $F_{(\mathcal{C}_j,\mathcal{T}_1)}$ defined above. Each such F has the form

$$F = \begin{pmatrix} f_{11} & f_{12} & 0 \\ f_{21} & f_{22} & 0 \\ 0 & 0 & f_{33} \end{pmatrix}. \tag{4.9}$$

Forming matrices M, R and N as in (4.4), from the equations $M_{33} = 0$ and $R_{11} = 0$, we find the following conditions on the entries in column 3 and row 1 of g:

$$(f_{11}f_{22} - f_{12}f_{21})|g_{13}|^2 + |f_{21}g_{13} + f_{22}g_{23}|^2 = -f_{22}f_{33}(|g_{33}|^2 - 1) \tag{4.10}$$

$$(f_{11}f_{22} - f_{12}f_{21})|g_{11}|^2 + |f_{12}g_{11} - f_{11}g_{12}|^2 = -\frac{f_{11}}{f_{33}}(f_{11}f_{22} - f_{12}f_{21})|g_{13}|^2 + f_{11}f_{22}. \tag{4.11}$$

In all our cases, we have $f_{33} = 1$, and for the cases $(\mathcal{C}_j,\mathcal{T}_1)$ with $\mathcal{T}_1 \neq \emptyset$, we have $f_{11}f_{22} - f_{12}f_{21} = |\delta|^2$ for $\delta = \det(c)$. Dividing both sides of the equations (4.10) and (4.11) by $|\delta|^2$, the equations become

$$|g_{13}|^2 + \Big|\frac{f_{21}}{\delta}g_{13} + \frac{f_{22}}{\delta}g_{23}\Big|^2 = -\frac{f_{22}}{|\delta|^2}(|g_{33}|^2 - 1) \tag{4.12}$$

and

$$|g_{11}|^2 + \Big|\frac{f_{12}}{\delta}g_{11} - \frac{f_{11}}{\delta}g_{12}\Big|^2 = -f_{11}|g_{13}|^2 + \frac{f_{11}f_{22}}{|\delta|^2}. \tag{4.13}$$

We find that $f_{21}/\delta = f_{12}/\bar{\delta}, f_{22}/\delta, f_{11}/\bar{\delta} \in \mathfrak{o}_k$ in each case. Equations (4.10) and (4.11) also have the form (4.12) and (4.13) for the cases $(\mathcal{C}_j,\emptyset)$, with $\delta = \det(c) = 1$. We list the equations (4.12) and (4.13) in the next two tables:

name	r	$\mathcal{T}_1$	Column 3 condition
$\mathcal{C}_1$	$\sqrt{5}$	$\emptyset$	$|g_{13}|^2 + |\frac{r-1}{2}g_{23}|^2 = \frac{r-1}{2}(|g_{33}|^2 - 1)$
$\mathcal{C}_3$	$\sqrt{5}$	$\emptyset$	$|g_{13}|^2 + |\frac{r-1}{2}g_{23}|^2 = \frac{r-1}{2}(|g_{33}|^2 - 1)$
$\mathcal{C}_8$	$\sqrt{2}$	$\emptyset$	$|g_{13}|^2 + |(r-1)g_{23}|^2 = (r-1)(|g_{33}|^2 - 1)$
$\mathcal{C}_{11}$	$\sqrt{3}$	$\emptyset$	$|g_{13}|^2 + |g_{13} - (r-1)g_{23}|^2 = (r-1)(|g_{33}|^2 - 1)$
$\mathcal{C}_{18}$	$\sqrt{6}$	$\emptyset$	$|g_{13}|^2 + |g_{13} - (r-2)g_{23}|^2 = (r-2)(|g_{33}|^2 - 1)$
$\mathcal{C}_{21}$	$\sqrt{33}$	$\emptyset$	$|g_{13}|^2 + |g_{13} - \frac{r-5}{2}g_{23}|^2 = \frac{r-5}{2}(|g_{33}|^2 - 1)$
$\mathcal{C}_1$	$\sqrt{5}$	$\{5\}$	$|g_{13}|^2 + |\frac{r+1}{2}g_{13} + (\zeta - \zeta^{-1})g_{23}|^2 = \frac{r+1}{2}(|g_{33}|^2 - 1)$
$\mathcal{C}_3$	$\sqrt{5}$	$\{2\}$	$|g_{13}|^2 + |g_{13} + (1 - 2z - z^3)g_{23}|^2 = \frac{r+1}{2}(|g_{33}|^2 - 1)$
$\mathcal{C}_8$	$\sqrt{2}$	$\{2\}$	$|g_{13}|^2 + |(r+1)g_{13} - 2(\zeta^2 + \zeta)g_{23}|^2 = 2(r+1)(|g_{33}|^2 - 1)$
$\mathcal{C}_{11}$	$\sqrt{3}$	$\{2\}$	$|g_{13}|^2 + |g_{23}|^2 = \frac{r-1}{2}(|g_{33}|^2 - 1)$
$\mathcal{C}_{18}$	$\sqrt{6}$	$\{2\}$	$|g_{13}|^2 + |g_{23}|^2 = \frac{r-2}{2}(|g_{33}|^2 - 1)$
$\mathcal{C}_{21}$	$\sqrt{33}$	$\{2+\}$	$|g_{13}|^2 + |g_{23}|^2 = \frac{r-5}{4}(|g_{33}|^2 - 1)$
$\mathcal{C}_{21}$	$\sqrt{33}$	$\{2-\}$	$|g_{13}|^2 + |g_{23}|^2 = \frac{r+5}{4}(|g_{33}|^2 - 1)$

name	r	$\mathcal{T}_1$	Row 1 condition
$\mathcal{C}_1$	$\sqrt{5}$	$\emptyset$	$\lvert g_{11}\rvert^2 + \lvert\frac{r+1}{2}g_{12}\rvert^2 = \frac{r+1}{2}\lvert g_{13}\rvert^2 + 1$
$\mathcal{C}_3$	$\sqrt{5}$	$\emptyset$	$\lvert g_{11}\rvert^2 + \lvert\frac{r+1}{2}g_{12}\rvert^2 = \frac{r+1}{2}\lvert g_{13}\rvert^2 + 1$
$\mathcal{C}_8$	$\sqrt{2}$	$\emptyset$	$\lvert g_{11}\rvert^2 + \lvert(r+1)g_{12}\rvert^2 = (r+1)\lvert g_{13}\rvert^2 + 1$
$\mathcal{C}_{11}$	$\sqrt{3}$	$\emptyset$	$\lvert g_{11}\rvert^2 + \lvert g_{11} + (r+1)g_{12}\rvert^2 = (r+1)\lvert g_{13}\rvert^2 + 2$
$\mathcal{C}_{18}$	$\sqrt{6}$	$\emptyset$	$\lvert g_{11}\rvert^2 + \lvert g_{11} + (r+2)g_{12}\rvert^2 = (r+2)\lvert g_{13}\rvert^2 + 2$
$\mathcal{C}_{21}$	$\sqrt{33}$	$\emptyset$	$\lvert g_{11}\rvert^2 + \lvert g_{11} + \frac{r+5}{2}g_{12}\rvert^2 = \frac{r+5}{2}\lvert g_{13}\rvert^2 + 2$
$\mathcal{C}_1$	$\sqrt{5}$	$\{5\}$	$\lvert g_{11}\rvert^2 + \lvert\frac{r+1}{2}g_{11} + (\zeta - \zeta^{-1})g_{12}\rvert^2 = r\lvert g_{13}\rvert^2 + \frac{r+5}{2}$
$\mathcal{C}_3$	$\sqrt{5}$	$\{2\}$	$\lvert g_{11}\rvert^2 + \lvert g_{11} - (1+2z+z^3)g_{12}\rvert^2 = (r-1)\lvert g_{13}\rvert^2 + 2$
$\mathcal{C}_8$	$\sqrt{2}$	$\{2\}$	$\lvert g_{11}\rvert^2 + \lvert(r+1)g_{11} - (\zeta^2+\zeta^3)g_{12}\rvert^2 = r\lvert g_{13}\rvert^2 + 2(r+2)$
$\mathcal{C}_{11}$	$\sqrt{3}$	$\{2\}$	$\lvert g_{11}\rvert^2 + \lvert g_{12}\rvert^2 = (r+1)\lvert g_{13}\rvert^2 + 1$
$\mathcal{C}_{18}$	$\sqrt{6}$	$\{2\}$	$\lvert g_{11}\rvert^2 + \lvert g_{12}\rvert^2 = (r+2)\lvert g_{13}\rvert^2 + 1$
$\mathcal{C}_{21}$	$\sqrt{33}$	$\{2+\}$	$\lvert g_{11}\rvert^2 + \lvert g_{12}\rvert^2 = \frac{r+5}{2}\lvert g_{13}\rvert^2 + 1$
$\mathcal{C}_{21}$	$\sqrt{33}$	$\{2-\}$	$\lvert g_{11}\rvert^2 + \lvert g_{12}\rvert^2 = \frac{r-5}{2}\lvert g_{13}\rvert^2 + 1$

These column 3 and row 1 conditions are equations of the form appearing in (i) and (ii) of the next result.

Lemma 4.2. *For $\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{33} \in \mathfrak{o}_\ell$, write $|\alpha_{ij}|^2 = p_{ij} + q_{ij}r$ if $\mathfrak{o}_k = \mathbb{Z}[r]$, and $|\alpha_{ij}|^2 = (p_{ij} + q_{ij}r)/2$ if $\mathfrak{o}_k = \mathbb{Z}[(r+1)/2]$, where $p_{ij}, q_{ij} \in \mathbb{Z}$, as in Lemma 3.1.*

(i) If $\alpha_{13}, \alpha_{23}, \alpha_{33}$ satisfy an equation

$$|\alpha_{13}|^2 + |\alpha_{23}|^2 = (c_0 + c_1 r)(|\alpha_{33}|^2 - 1)$$

where $c_0 + c_1 r \in k$ and $c_0 < c_1 r$, then $q_{33} = \lfloor \frac{1}{r} p_{33} \rfloor$.

(ii) If $\alpha_{11}, \alpha_{12}, \alpha_{13}$ satisfy an equation

$$|\alpha_{11}|^2 + |\alpha_{12}|^2 = (d_0 + d_1 r)|\alpha_{13}|^2 + e_0 + e_1 r,$$

where $d_0 + d_1 r, e_0 + e_1 r \in k$, $d_0 < d_1 r$ and $(e_0 - re_1)/(rd_1 - d_0) < 2r$ when $\mathfrak{o}_k = Z[r]$ and $(e_0 - re_1)/(rd_1 - d_0) < r$ when $\mathfrak{o}_k = Z[(r+1)/2]$, then $q_{13} \le \frac{1}{r} p_{13} < q_{13} + 2$.

Proof. Assume first that $\mathfrak{o}_k = \mathbb{Z}[r]$. Since $r^2 = N \in \mathbb{Z}$, we have

$$\begin{aligned} p_{33}c_1 + q_{33}c_0 &= q_{13} + q_{23} + c_1 \\ Nq_{33}c_1 + p_{33}c_0 &= p_{13} + p_{23} + c_0. \end{aligned} \tag{4.14}$$

By Lemma 3.1,

$$p_{33}c_1 + q_{33}c_0 \le \frac{1}{r}\big(p_{13} + p_{23}\big) + c_1 = \frac{1}{r}\big(Nq_{33}c_1 + p_{33}c_0 - c_0\big) + c_1$$

Rearranging, we have

$$p_{33}\big(c_1 - \frac{c_0}{r}\big) \le (rc_1 - c_0)q_{33} + c_1 - \frac{c_0}{r}.$$

By our assumption that $c_0 < c_1 r$, we can divide through by $rc_1 - c_0$ and get $\frac{1}{r}p_{33} \le q_{33} + \frac{1}{r}$, from which (i) follows. When $\mathfrak{o}_k = \mathbb{Z}[(r+1)/2]$, similar calculations lead to $\frac{1}{r}p_{33} \le q_{33} + \frac{2}{r}$, and again (i) follows, since $r^2 = 5$ or 33.

The equation in (ii) leads to equations

$$\begin{aligned} p_{13}d_1 + q_{13}d_0 + e_1 &= q_{11} + q_{12} \\ Nq_{13}d_1 + p_{13}d_0 + e_0 &= p_{11} + p_{12}. \end{aligned} \tag{4.15}$$

By Lemma 3.1 again,

$$p_{13}d_1 + q_{13}d_0 + e_1 \le \frac{1}{r}\big(p_{11} + P(\beta)\big) = \frac{1}{r}\big(r^2 q_{13}d_1 + p_{13}d_0 + e_0\big)$$

Rearranging, and using $rd_1 > d_0$, we have

$$\frac{1}{r}p_{13} \le q_{13} + \frac{e_0 - re_1}{r(rd_1 - d_0)}$$

By our assumption, we have $\frac{1}{r}p_{13} < q_{13} + 2$ and (ii) holds. In the case when $\mathfrak{o}_k = \mathbb{Z}[(r+1)/2]$, similar calculations lead to $\frac{1}{r}p_{13} < q_{13} + 2(e_0 - re_1)/(r(rd_1 - d_0))$ and again (ii) follows. ☐

4.3. **The action of $\bar\Gamma$ on $B(\mathbb{C}^2)$.** With x is as in the Table 2, for the cases $\mathcal{C}_1$, $\mathcal{C}_3$ and $\mathcal{C}_8$, respectively $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$, we define $\Delta_{(\mathcal{C}_j,\emptyset)}$ by

$$\Delta_{(\mathcal{C}_j,\emptyset)} = \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{x} \end{pmatrix}, \quad \text{respectively} \quad \Delta_{(\mathcal{C}_j,\emptyset)} = \begin{pmatrix} x & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{x} \end{pmatrix}. \tag{4.16}$$

In each case, $\Delta^*_{(\mathcal{C}_j,\emptyset)} F_0 \Delta_{(\mathcal{C}_j,\emptyset)} = -xF_{(\mathcal{C}_j,\emptyset)}$. For the seven singleton sets $\mathcal{T}_1$ listed in (1.9), and for the seven matrices c listed in Section 3, let $F_{(\mathcal{C}_j,\mathcal{T}_1)} = c^* F_\emptyset c$, as before. Then defining $\Delta_{(\mathcal{C}_j,\mathcal{T}_1)} = \Delta_{(\mathcal{C}_j,\emptyset)} c$, we have

$$\Delta^*_{(\mathcal{C}_j,\mathcal{T}_1)} F_0 \Delta_{(\mathcal{C}_j,\mathcal{T}_1)} = -xF_{(\mathcal{C}_j,\mathcal{T}_1)}, \tag{4.17}$$

so that (4.17) holds for all thirteen cases listed in (1.9). So if $g^* F_{(\mathcal{C}_j,\mathcal{T}_1)} g = F_{(\mathcal{C}_j,\mathcal{T}_1)}$, then $\tilde g = \Delta_{(\mathcal{C}_j,\mathcal{T}_1)} g \Delta^{-1}_{(\mathcal{C}_j,\mathcal{T}_1)}$ is in $U(2,1)$. So if $g \in M_{3\times 3}(\mathfrak{o}_\ell)$ and $g^* F_{(\mathcal{C}_j,\mathcal{T}_1)} g = F_{(\mathcal{C}_j,\mathcal{T}_1)}$, and $(z_1, z_2) \in B(\mathbb{C}^2)$, we write

$$(g\mathcal{Z}).(z_1,z_2) = (w_1,w_2) \quad \text{if and only if} \quad \tilde g \begin{pmatrix} z_1 \\ z_2 \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} w_1 \\ w_2 \\ 1 \end{pmatrix} \quad \text{for some } \lambda,$$

where $\mathcal{Z} = \{tI : t \in \mathfrak{o}_\ell \text{ and } |t| = 1\}$, as before. This defines an action of the group $\bar\Gamma_{(\mathcal{C}_j,\mathcal{T}_1)}$ of (1.10) on $B(\mathbb{C}^2)$.

Because of the block form (4.9) of $F_{(\mathcal{C}_j,\mathcal{T}_1)}$, and similarly that of $\Delta_{(\mathcal{C}_j,\mathcal{T}_1)}$, the $(3,3)$ entry of g is the same as that of $\tilde g$. Hence $\cosh^2(d(0,(g\mathcal{Z}).0)) = |g_{33}|^2$.

Lemma 4.3. *The group $\mathfrak{o}^1_\ell = \{t \in \mathfrak{o}_\ell : |t| = 1\}$ is cyclic, generated by $-\zeta_5$, i, ζ_8, ζ_{12}, $-\zeta_3$ and $-\zeta_3$ in the cases $\mathcal{C}_1$, $\mathcal{C}_3$, $\mathcal{C}_8$, $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$, respectively. Hence $|\mathcal{Z}| = |\mathfrak{o}^1|$ is equal to 10, 4, 8, 12, 6 and 6, respectively.*

Proof. If $\alpha \in \mathfrak{o}_\ell$ and $|\alpha|^2 = 1$, then $(P(\alpha), Q(\alpha)) = (1,0)$ if $\mathfrak{o}_k = \mathbb{Z}[r]$, and $(P(\alpha), Q(\alpha)) = (2,0)$ if $\mathfrak{o}_k = \mathbb{Z}[(r+1)/2]$. Write $\alpha = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$, where $v_1, \ldots, v_4$ is the integral basis of $\mathfrak{o}_\ell$ chosen in Section 3. Then $|a_j| \le 1/\sqrt{\lambda_{\min}}$ for each j if $\mathfrak{o}_k = \mathbb{Z}[r]$ and $|a_j| \le \sqrt{2/\lambda_{\min}}$ for each j if $\mathfrak{o}_k = \mathbb{Z}[(r+1)/2]$. So we run through all $(a_1, \ldots, a_4) \in \mathbb{Z}^4$ satisfying this condition and calculate $P(\alpha)$ and $Q(\alpha)$, counting the number of α's for which $(P(\alpha), Q(\alpha) = (1,0)$ if $\mathfrak{o}_k = \mathbb{Z}[r]$, and for which $(P(\alpha), Q(\alpha)) = (2,0)$ when $\mathfrak{o}_k = \mathbb{Z}[(r+1)/2]$. ☐

Lemma 4.4. *Let $K = K_{(\mathcal{C}_j,\mathcal{T}_1)}$ denote the group of $g\mathcal{Z} \in \bar\Gamma_{(\mathcal{C}_j,\mathcal{T}_1)}$ such that $(g\mathcal{Z}).0 = 0$. Then for the thirteen cases $(\mathcal{C}_j, \mathcal{T}_1)$, $|K|$ is as in the following tables, which also give the index (either D or $D/3$) which Π must be in $\bar\Gamma$:*

	$(\mathcal{C}_1,\emptyset)$	$(\mathcal{C}_3,\emptyset)$	$(\mathcal{C}_8,\emptyset)$	$(\mathcal{C}_{11},\emptyset)$	$(\mathcal{C}_{18},\emptyset)$	$(\mathcal{C}_{21},\emptyset)$	$(\mathcal{C}_1,\{5\})$
$\lvert K\rvert$	200	32	128	288	48	24	600
$[\bar\Gamma : \Pi]$	600	32	128	864	48	12	600

	$(\mathcal{C}_3,\{2\})$	$(\mathcal{C}_8,\{2\})$	$(\mathcal{C}_{11},\{2\})$	$(\mathcal{C}_{18},\{2\})$	$(\mathcal{C}_{21},\{2+\})$	$(\mathcal{C}_{21},\{2-\})$
$\lvert K\rvert$	96	128	288	72	72	72
$[\bar\Gamma:\Pi]$	32	128	288	16	4	4

Proof. Since $(g\mathcal{Z}).0 = (\tilde g_{13}/\tilde g_{33}, \tilde g_{23}/\tilde g_{33})$, we see that $(g\mathcal{Z}).0 = 0$ if and only if $\tilde g_{13} = \tilde g_{23} = 0$. Because $\Delta = \Delta_{(\mathcal{C}_j,\mathcal{T}_1)}$ has the form

$$\Delta = \begin{pmatrix}\delta_{11} & \delta_{12} & 0\\ \delta_{21} & \delta_{22} & 0\\ 0 & 0 & \sqrt{x}\end{pmatrix}, \tag{4.18}$$

(where $\delta_{ij} \in \mathfrak{o}_\ell$ for each i,j), $\tilde g_{13} = \tilde g_{23} = 0$ is equivalent to $g_{13} = g_{23} = 0$. Now with the M of (4.4), we see from $M_{33} = 0$ that $|g_{33}|^2 = 1$. Replacing g by tg_{33}, where $t = \bar g_{33}$, we may assume that $g_{33} = 1$. From $M_{31} = 0 = M_{32}$, we see that $g_{31} = g_{32} = 0$. Hence we may assume that

$$g = \begin{pmatrix}g_{11} & g_{12} & 0\\ g_{21} & g_{22} & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{4.19}$$

For N as in (4.4), using $F^* = F$ and assuming that $f_{22} \neq 0$, as we may, we can use $N_{12} = 0$ and $N_{11} = 0$ to express g_{21} and g_{22} in terms of g_{11} and g_{12}:

$$g_{21} = \frac{-f_{21}g_{11} + \theta f_{21}\bar g_{11} - \theta f_{11}\bar g_{12}}{f_{22}} \text{ and } g_{22} = \frac{\theta f_{22}\bar g_{11} - f_{21}g_{12} - \theta f_{12}\bar g_{12}}{f_{22}}. \tag{4.20}$$

Moreover, g_{11} and g_{12} must satisfy the equation

$$|g_{11}|^2 + \Big|\frac{f_{12}}{\bar\delta}g_{11} - \frac{f_{11}}{\bar\delta}g_{12}\Big|^2 = \frac{f_{11}f_{22}}{|\delta|^2}, \tag{4.21}$$

where $\delta = \det(c)$, which is just the row 1 condition (4.13) in the case $g_{13} = 0$.

The condition that the g_{ij}'s be in $\mathfrak{o}_\ell$ must be imposed. The simplest cases are $(\mathcal{C}_{11},\{2\})$, $(\mathcal{C}_{18},\{2\})$ and $(\mathcal{C}_{21},\{2\pm\})$, for which $f_{11} = f_{22}$ and $f_{12} = f_{21} = 0$. Then (4.20) implies that $g_{21} = -\theta\bar g_{12}$ and $g_{22} = \theta\bar g_{11}$, and the condition (4.21) is just $|g_{11}|^2 + |g_{12}|^2 = 1$. Thus $(P(g_{11}) + P(g_{12}), Q(g_{11}) + Q(g_{12}))$ equals $(1,0)$ in the cases $\mathcal{C}_{11}$ and $\mathcal{C}_{18}$, and equals $(2,0)$ in the cases $(\mathcal{C}_{21},\{2\pm\})$. Calculating the possible $(P(\alpha),Q(\alpha))$ with $P(\alpha) \le 2$, we see that $g_{12} = 0$ or $g_{11} = 0$, and so

$$\begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix} = \begin{pmatrix}t & 0\\ 0 & t'\end{pmatrix} \quad\text{or}\quad \begin{pmatrix}0 & t\\ t' & 0\end{pmatrix},$$

where $t,t' \in \mathfrak{o}_\ell$ and $|t| = |t'| = 1$. Thus $|K| = 2|\mathcal{Z}|^2$ in these four cases.

The next simplest cases are $(\mathcal{C}_1,\emptyset)$, $(\mathcal{C}_3,\emptyset)$ and $(\mathcal{C}_8,\emptyset)$, For these, (4.20) implies that $g_{21} = -\theta x^2\bar g_{12}$ and $g_{22} = \theta\bar g_{11}$, and the condition (4.21) is just $|g_{11}|^2+|xg_{12}|^2 = 1$, where x is as in the table of the Introduction, and is an invertible element of $\mathfrak{o}_k$ in these cases. Arguing as in the previous cases, we find that

$$\begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix} = \begin{pmatrix}t & 0\\ 0 & t'\end{pmatrix} \quad\text{or}\quad \begin{pmatrix}0 & tx^{-1}\\ t'x & 0\end{pmatrix}, \tag{4.22}$$

where $t,t' \in \mathfrak{o}_\ell$ and $|t| = |t'| = 1$. So also in these three cases, $|K| = 2|\mathcal{Z}|^2$.

For the cases $(\mathcal{C}_{11},\emptyset)$, $(\mathcal{C}_{18},\emptyset)$ and $(\mathcal{C}_{21},\emptyset)$, (4.20) implies that

$$g_{21} = \frac{x}{2}\big(g_{11} - \theta\bar g_{11} - x\theta\bar g_{12}\big) \quad\text{and}\quad g_{22} = \frac{1}{2}\big(2\theta\bar g_{11} + xg_{12} + x\theta\bar g_{12}\big),$$

and the condition (4.21) is just $|g_{11}|^2+|g_{11}+xg_{12}|^2 = 2$. For $\alpha,\beta \in \mathfrak{o}_\ell$, $|\alpha|^2+|\beta|^2 = 2$ holds if and only if that $(P(\alpha)+P(\beta), Q(\alpha)+Q(\beta))$ equals $(2,0)$ in cases $\mathcal{C}_{11}$ and $\mathcal{C}_{18}$, and equals $(4,0)$ in case $\mathcal{C}_{21}$.

(i) In case $(\mathcal{C}_{11},\emptyset)$, for the $\alpha \in \mathfrak{o}_\ell$ such that $P(\alpha) \le 2$, we have $(P(\alpha),Q(\alpha)) = (0,0)$, $(1,0)$, $(2,0)$, $(2,1)$ or $(2,-1)$. Hence $(P(\alpha),Q(\alpha),P(\beta),Q(\beta)) = (2,0,0,0)$,

$(1,0,1,0)$ or $(0,0,2,0)$, giving $(\alpha,\beta)=(\zeta^\nu(\zeta^3+1),0)$, $(\zeta^\nu,\zeta^\lambda)$ or $(0,\zeta^\nu(\zeta^3+1))$. Setting $g_{11}=\alpha$, $g_{12}=(\beta-\alpha)/x$ and $\theta=\zeta^\mu$, substituting these into the above formula for g and running through the possible α, β and θ, checking that the entries of g are all in $\mathfrak{o}_\ell$, we find that K has 288 elements. In fact, it is generated by the elements $u\mathcal{Z}$ and $v\mathcal{Z}$, where

$$u=\begin{pmatrix}\zeta^3+\zeta^2-\zeta & 1-\zeta & 0\\ \zeta^3+\zeta^2-1 & \zeta-\zeta^3 & 0\\ 0 & 0 & 1\end{pmatrix}\quad\text{and}\quad v=\begin{pmatrix}\zeta^3 & 0 & 0\\ \zeta^3+\zeta^2-\zeta-1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\tag{4.23}$$

which satisfy

$$u^3=I,\ v^4=I,\ \text{and}\ (uv)^2=(vu)^2.$$

Magma shows that an abstract group with presentation $\langle u,v : u^3=v^4=1,\ (uv)^2=(vu)^2\rangle$ has order 288, and so K has this presentation.

(ii) In case $(\mathcal{C}_{18},\emptyset)$, for the $\alpha\in\mathfrak{o}_\ell$ such that $P(\alpha)\le 2$, we have $(P(\alpha),Q(\alpha))=(0,0)$, $(1,0)$ or $(2,0)$. Hence $(P(\alpha),Q(\alpha),P(\beta),Q(\beta))=(2,0,0,0)$, $(1,0,1,0)$ or $(0,0,2,0)$, giving $(\alpha,\beta)=((-\omega)^\nu z,0)$, $((-\omega)^\nu,(-\omega)^\lambda)$ or $(0,(-\omega)^\nu z)$. Setting $g_{11}=\alpha$, $g_{12}=(\beta-\alpha)/x$ and $\theta=(-\omega)^\mu$, substituting these into the above formula for g and running through the possible α, β and θ, checking that the entries of g are all in $\mathfrak{o}_\ell$, we find that K has 48 elements. In fact, it is generated by the elements $u\mathcal{Z}$ and $w\mathcal{Z}$, where

$$u=\begin{pmatrix}\frac{z^3-2z}{2} & \frac{z^3+z^2-2z-4}{2} & 0\\ \frac{z^3-z^2-2z+4}{2} & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\quad\text{and}\quad v=\begin{pmatrix}-1 & 0 & 0\\ -(r+2) & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\tag{4.24}$$

They satisfy the relations $u^{24}=v^2=1$, $vuv^{-1}=u^{-5}$. These generators and relations give a presentation of K.

(ii) In case $(\mathcal{C}_{21},\emptyset)$, if $\alpha\in\mathfrak{o}_\ell$ and $P(\alpha)\le 4$, then $(P(\alpha),Q(\alpha))=(0,0)$ or $(2,0)$. Hence $(P(\alpha),Q(\alpha),P(\beta),Q(\beta))=(2,0,2,0)$, and $(\alpha,\beta)=((-\omega)^\nu,(-\omega)^\lambda)$ for some $\nu,\lambda\in\{0,\dots,5\}$. Setting $g_{11}=\alpha$, $g_{12}=(\beta-\alpha)/x$ and $\theta=(-\omega)^\mu$, substituting these into the above formula for g and running through the possible α, β and θ, checking that the entries of g are all in $\mathfrak{o}_\ell$, we find that K has 24 elements. It is generated by the elements $u\mathcal{Z}$, $v\mathcal{Z}$, $d\mathcal{Z}$ for

$$u=\begin{pmatrix}1 & -\frac{r-5}{2} & 0\\ \frac{r+5}{2} & -1 & 0\\ 0 & 0 & 1\end{pmatrix},\quad v=\begin{pmatrix}1 & 0 & 0\\ \frac{r+5}{2} & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\quad\text{and}\quad d=\begin{pmatrix}-\omega & 0 & 0\\ 0 & -\omega & 0\\ 0 & 0 & 1\end{pmatrix},$$

and (omitting the $\mathcal{Z}$'s) a presentation for K is given by the generators u,v and d and the relations $u^4=v^2=(vu)^2=1$, $du=ud$, $dv=vd$ and $d^3=u^2$.

In the case $(\mathcal{C}_1,\{5\})$, we use (4.20) to express g_{21} and g_{22} in terms of g_{11} and g_{12}, which must satisfy

$$|g_{11}|^2+\Big|\frac{r+1}{2}g_{11}+(\zeta-\zeta^{-1})g_{12}\Big|^2=\frac{r+5}{2}.$$

If $\alpha,\beta\in\mathfrak{o}_\ell$ and $|\alpha|^2+|\beta|^2=\frac{r+5}{2}$, then $P(\alpha)+P(\beta)=5$ and $Q(\alpha)+Q(\beta)=1$. We find that $(P(\alpha),Q(\alpha),P(\beta),P(\beta))$ must equal $(5,1,0,0)$, $(3,1,2,0)$, $(2,0,3,1)$ or $(0,0,5,1)$. The α for which $(P(\alpha),Q(\alpha))=(2,0)$ are the $(-\zeta)^\nu$, $\nu=0,\dots,9$. Those for which $(P(\alpha),Q(\alpha))=(3,1)$ are $(-\zeta)^\nu(\zeta+1)$, and those for which $(P(\alpha),Q(\alpha))=(5,1)$ are $(-\zeta)^\nu(\zeta^2-1)$. Running through the possibilities for α, β and θ, and checking when the g_{ij}'s are in $\mathfrak{o}_\ell$, we find that K has 600 elements, and is generated by the elements $u\mathcal{Z}$ and $v\mathcal{Z}$ for

$$u=\begin{pmatrix}-1 & \zeta^3 & 0\\ -\zeta^2 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\quad\text{and}\quad v=\begin{pmatrix}\zeta^4 & \zeta & 0\\ 0 & \zeta^3 & 0\\ 0 & 0 & 1\end{pmatrix},$$

which satisfy the relations $u^3 = v^5 = (uv^2)^5 = 1$ and $u(vu^2v) = (vu^2v)u$, which give a presentation for K.

In the case $(\mathcal{C}_3, \{2\})$, we use (4.20) to express g_{21} and g_{22} in terms of g_{11} and g_{12}, which must satisfy $|g_{11}|^2+\big|g_{11}-(1+2z+z^3)g_{12}\big|^2 = 2$. If $\alpha, \beta \in \mathfrak{o}_\ell$ and $|\alpha|^2+|\beta|^2 = 2$, then $P(\alpha)+P(\beta) = 4$ and $Q(\alpha)+Q(\beta) = 0$. We find that $(P(\alpha), Q(\alpha), P(\beta), P(\beta))$ must equal $(4,0,0,0)$, $(2,0,2,0)$ or $(0,0,4,0)$. The α for which $(P(\alpha), Q(\alpha)) = (2,0)$ are the i^ν, $\nu = 0,\dots,3$, while those for which $(P(\alpha), Q(\alpha)) = (4,0)$ are $i^\nu(i+1)$, $\nu = 0,\dots,3$. Running through the possibilities for α, β and θ, and checking when the g_{ij}'s are in $\mathfrak{o}_\ell$, we find that K has 96 elements, and is generated by the elements $u\mathcal{Z}$ and $v\mathcal{Z}$ corresponding to the matrices

$$u = \begin{pmatrix} 0 & i & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.25}$$

These satisfy $v^6 = 1 = (vu)^4$, $u^2v = vu^2$ and $v^3 = u^4$, and these generators and relations give a presentation for K.

In the case $(\mathcal{C}_8, \{2\})$, we use (4.20) to express g_{21} and g_{22} in terms of g_{11} and g_{12}, which must satisfy $|g_{11}|^2 + \big|(r+1)g_{11} - (\zeta^2+\zeta^3)g_{12}\big|^2 = 2(r+2)$. If $\alpha, \beta \in \mathfrak{o}_\ell$ and $|\alpha|^2 + |\beta|^2 = 2(r+2)$, then $(P(\alpha), Q(\alpha), P(\beta), Q(\beta))$ equals either $(4,2,0,0)$, $(3,2,1,0)$, $(2,1,2,1)$, $(1,0,3,2)$ or $(0,0,4,2)$. The $\alpha \in \mathfrak{o}_\ell$ for which $(P(\alpha), Q(\alpha))$ equals $(1,0)$, $(2,1)$, $(3,2)$ and $(4,2)$ are the elements $\zeta^\nu\alpha_0$, for α_0 equal to 1, $1+\zeta$, $r+1$ and $1+\zeta+\zeta^2+\zeta^3$, respectively. Running through the possibilities for α, β and θ, and checking when the g_{ij}'s are in $\mathfrak{o}_\ell$, we find that K has 128 elements, and is generated by the elements $u\mathcal{Z}$ and $v\mathcal{Z}$ corresponding to the matrices

$$u = \begin{pmatrix} \zeta & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} \zeta^3+\zeta^2+\zeta & 2-\zeta^3 & 0 \\ \zeta+1 & -\zeta^3-\zeta^2-\zeta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.26}$$

These satisfy $u^8 = v^{16} = 1$, $uv^2 = v^2u$ and $uv^{-3}u = v^3$, and these generators and relations give a presentation for K. □

At this point, we can show that in five of the thirteen cases, there cannot be a torsion-free subgroup Π of $\bar\Gamma$ of the index needed for Π to be the fundamental group of an fpp. Indeed, for the cases $(\mathcal{C}_{21}, \emptyset)$, $(\mathcal{C}_3, \{2\})$, $(\mathcal{C}_{18}, \{2\})$, $(\mathcal{C}_{21}, \{2+\})$ and $(\mathcal{C}_{21}, \{2-\})$, $|K|$ is bigger than the required $[\bar\Gamma : \Pi]$, and so by Lemma 1.1, these cases cannot give rise to an fpp.

In three more of the cases $(\mathcal{C}_j, \mathcal{T}_1)$, we shall produce an element g of $\bar\Gamma$ of finite order n which does not divide the required $[\bar\Gamma : \Pi]$. Applying Lemma 1.1 to $K = \langle g\rangle$ shows these cases cannot give rise to an fpp.

4.4. **Method for finding all the $g \in \bar\Gamma$ with $d(0, g.0) \le C$.**

Lemma 4.5. *In each case, K contains $d_t\mathcal{Z}$ and $k_w\mathcal{Z}$ for the matrices*

$$d_t = \begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad k_w = \begin{pmatrix} -f_{21}/\delta & -f_{22}/\delta & 0 \\ f_{11}/\bar\delta & f_{12}/\bar\delta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.27}$$

for any $t \in \mathfrak{o}_\ell$ such that $|t| = 1$, and where $\delta = \det(c)$ is as in (4.12) and (4.13).

Proof. For $g = d_t$, it is clear from the form (4.9) of $F = F_{(\mathcal{C}_j, \mathcal{T}_1)}$ that $g^*Fg = F$. For $g = k_w$, we use (4.9) together with $f_{11}f_{22} - f_{12}f_{21} = |\delta|^2$, and $\bar\alpha = \alpha$ for $\alpha = f_{11}$, f_{22}, f_{21}/δ and $f_{12}/\bar\delta$ to see that $g^*Fg = F$. □

Corollary 4.1. *If $g \in \bar\Gamma = \bar\Gamma_{(\mathcal{C}_j, \mathcal{T}_1)}$, let $\alpha = g_{13}$ and $\beta = \big(1/\delta\big)\big(f_{21}g_{13} + f_{22}g_{23}\big)$ be as in (4.12), and for $g' = k_wg$, define α' and β' similarly. Then $(\alpha', \beta') = (-\beta, \alpha)$.*

Similarly, let $\alpha = g_{11}$ and $\beta = \big(1/\bar{\delta}\big)\big(f_{12}g_{11} - f_{11}g_{12}\big)$ be as in (4.13), and for $g' = gk_w$, define α' and β' similarly. Then $(\alpha', \beta') = (-\beta, \alpha)$.

Let $K = K_{(\mathcal{C}_j,\mathcal{T}_1)}$ be as in Lemma 4.4. We wish to find representatives of the double cosets KgK of all the $g \in \bar{\Gamma}$ for which $d(0, g.0)$ is less than a chosen bound. Let us write down the details of our method for the cases when $\mathfrak{o}_k = \mathbb{Z}[r]$. With small modifications, it can be used in the case $\mathfrak{o}_k = \mathbb{Z}[(r+1)/2]$ too.

Step 1. For the chosen integral basis $v_1, \dots, v_4$ of $\mathfrak{o}_\ell$ as in Section 3, we find all $t \in \mathfrak{o}_\ell$ such that $|t| = 1$ by calculating $P(\alpha)$ and $Q(\alpha)$ for $\alpha = a_1v_1 + \dots + a_4v_4$ for all a_j's satisfying $|a_j| \le \sqrt{1/\lambda_{\min}}$.

Step 2. Having chosen a bound B, we form a list of all pairs (p, q) of integers such that $0 \le p \le B$, and $p = P(\alpha)$ and $q = Q(\alpha)$ for some $\alpha \in \mathfrak{o}_\ell$, by calculating $P(\alpha)$ and $Q(\alpha)$ for $\alpha = a_1v_1 + \dots + a_4v_4$ for all a_j's satisfying $|a_j| \le \sqrt{B/\lambda_{\min}}$.

Step 3. We choose a set $\mathcal{R}_B$ of equivalence class representatives for the α's of Step 2, where $\alpha \sim \beta$ if $\beta = t\alpha$ for some $t \in \mathfrak{o}_\ell$ with $|t| = 1$.

Step 4. Form a list of the 10-tuples $(p_{11}, q_{11}, p_{12}, q_{12}, p_{13}, q_{13}, p_{23}, q_{23}, p_{33}, q_{33})$ for which (i) (p_{ij}, q_{ij}) is in the list of Step 2 for each of the five (i, j)'s here, (ii) the first six of these ten numbers satisfy, for $N = r^2$, (cf. (4.15))

$$\begin{aligned} p_{13}d_1 + q_{13}d_0 + e_1 &= q_{11} + q_{12}, \\ Nq_{13}d_1 + p_{13}d_0 + e_0 &= p_{11} + p_{12}; \end{aligned}$$

(iii) the last six of these ten numbers satisfy, for $N = r^2$, (cf. (4.14))

$$\begin{aligned} p_{33}c_1 + q_{33}c_0 &= q_{13} + q_{23} + c_1, \\ Nq_{33}c_1 + p_{33}c_0 &= p_{13} + p_{23} + c_0. \end{aligned}$$

(iv) $p_{13} \le p_{23}$ and $p_{11} \ge p_{12}$.

Step 5. For each 10-tuple $(p_{11}, q_{11}, p_{12}, q_{12}, p_{13}, q_{13}, p_{23}, q_{23}, p_{33}, q_{33})$ from Step 4, and any $\alpha_{11}, \alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{33} \in \mathcal{R}_B$ so that $|\alpha_{ij}|^2 = p_{ij} + q_{ij}r$, and any $s, t \in \mathfrak{o}_\ell$ such that $|s| = |t| = 1$, set $g_{11} = \alpha_{11}$, $g_{12} = (f_{12}\alpha_{11} - s\bar{\delta}\alpha_{12})/f_{11}$, $g_{13} = \alpha_{13}$, $g_{23} = (t\delta\alpha_{23} - f_{21}\alpha_{13})/f_{22}$ and $g_{33} = \alpha_{33}$. Discard any such $(g_{11}, g_{12}, g_{13}, g_{23}, g_{33})$ if g_{12} or g_{23} are not in $\mathfrak{o}_\ell$.

Step 6. For any $\theta \in \mathfrak{o}_\ell$ such that $|\theta|^2 = 1$, we can form a unique matrix $g = (g_{ij}) \in M_{3\times3}(\ell)$ so that $g^*Fg = g$ for $F = F_{(\mathcal{C}_j,\mathcal{T}_1)}$ and so that $\det(g) = \theta$, and so that g_{11}, g_{12}, g_{13}, g_{23} and g_{33} are as in Step 5. This is done by solving the equations $M_{31} = 0$, $M_{32} = 0$, $N_{11} = 0$ and $N_{12} = 0$, for g_{31}, g_{32}, g_{21} and g_{22}, respectively, where M and N are as in (4.4). We retain g only if the g_{ij}'s so found are all in $\mathfrak{o}_\ell$.

Step 7. We choose representatives for the double cosets KgK of the g's found using Steps 1 to 6. The union of these double cosets is the set of all $g\mathcal{Z}$ in $\bar{\Gamma}_{(\mathcal{C}_j,\mathcal{T}_1)}$ for which $|g_{33}|^2 = p_{33} + q_{33}r$, with p_{33} satisfying the constraints in (4.28) below.

Here are some comments on these steps:

Steps 1 and 2. We are using $P(\alpha) \ge \lambda_{\min}\sum_j a_j^2$ here. When $\mathfrak{o}_k = \mathbb{Z}[(r+1)/2]$, we must use the bounds $\sqrt{2/\lambda_{\min}}$ and $\sqrt{2B/\lambda_{\min}}$ on the $|a_j|$'s.

Step 4. When $\mathfrak{o}_k = \mathbb{Z}[(r+1)/2]$, the terms e_1 and e_0 on the left of the equations in (ii) are replaced by $2e_1$ and $2e_0$, respectively. The second equation in (ii) implies that $p_{11} + p_{12} \le (d_0 + d_1r)p_{13} + e_0$, and the second equation in (iii) implies that $p_{13} + p_{23} \le (c_0 + c_1r)p_{33} - c_0$, with "$+e_0$" and "$-c_0$" replaced by "$+2e_0$" and "$-2c_0$", respectively, when $\mathfrak{o}_k = \mathbb{Z}[(r+1)/2]$. Note that $p_{13} \le \frac{1}{2}(p_{13} + p_{23})$. To ensure that all the p_{ij}'s appearing satisfy $p_{ij} \le B$ (and thus appear in the list of

Step 2), we need p_{33} to satisfy

$$\begin{aligned} p_{33} &\le B, \\ (c_0 + c_1 r)p_{33} - c_0 &\le B \quad \text{and} \\ \frac{1}{2}(d_0 + d_1 r)\big((c_0 + c_1 r)p_{33} - c_0\big) + e_0 &\le B, \end{aligned} \tag{4.28}$$

replacing "$+e_0$" and "$-c_0$" replaced by "$+2e_0$" and "$-2c_0$", respectively, when $\mathfrak{o}_k = \mathbb{Z}[(r+1)/2]$.

The restrictions in (iv) of Step 4 may be imposed because of Corollary 4.1.

Step 5. We may suppose that g_{11}, g_{13} and $g_{33} \in \mathcal{R}_B$ because firstly, we can replace g by tg for some $t \in \mathfrak{o}_\ell$ with $|t| = 1$ to arrange that $g_{33} \in \mathcal{R}_B$. This doesn't change $g\mathcal{Z}$. Then we can replace g by $k_w g$ if necessary to ensure that $|g_{13}| \le |(1/\delta)(f_{21}g_{13} + f_{22}g_{23})|$, and then replace g by $d_t g$ if necessary to ensure that $g_{13} \in \mathcal{R}_B$. Finally, replacing g by gk_w if necessary, we may arrange that $|g_{11}| \ge |(1/\bar\delta)(f_{12}g_{11} - f_{11}g_{12})|$, then replace g by gd_t to ensure that $g_{11} \in \mathcal{R}_B$.

5. Eliminating three more cases $(\mathcal{C}_j, \mathcal{T}_1)$

5.1. **The case** $(\mathcal{C}_3, \emptyset)$**.** To eliminate this case, we simply have to check that the matrix

$$g = \begin{pmatrix} 0 & 1 & 1 \\ -x & 0 & 0 \\ 0 & 1 & x \end{pmatrix} \tag{5.1}$$

where $x = (r+1)/2$ is as in the Table 2, satisfies $g^*Fg = F$ for $F = F_{(\mathcal{C}_3,\emptyset)}$ and $g^5 = I$. Applying Lemma 1.1 to the finite subgroup $\langle g\mathcal{Z}\rangle$ of $\bar\Gamma_{(\mathcal{C}_3,\emptyset)}$, we see that $\bar\Gamma_{(\mathcal{C}_3,\emptyset)}$ cannot contain a torsion-free subgroup Π of the index 32 required for Π to be the fundamental group of an fpp.

Although the case $(\mathcal{C}_3, \{2\})$ has already been eliminated, let us mention here that also in that case, $\bar\Gamma$ contains an element of order 5. For the method of Section 4.4 yields the following element of $\bar\Gamma$:

$$a = \begin{pmatrix} 1 & z^2+3 & z^2+3 \\ 0 & -2z^3 - z^2 - 5z - 2 & -2z^3 - z^2 - 5z - 3 \\ 0 & -z^3 - z^2 - 3z - 2 & -z^3 - z^2 - 3z - 3 \end{pmatrix}.$$

We can show that in this case, $\bar\Gamma$ is generated by u and v (as given in (4.25)) and by a, but we omit the proof. The element $g = auv^2uauvuauv^2uauv^2auv^2au^2v^2$ of $\bar\Gamma$ has order 5.

5.2. **The case** $(\mathcal{C}_8, \emptyset)$**.** To eliminate this case, we simply have to check that the matrix

$$g = \begin{pmatrix} -\zeta - 1 & 0 & \zeta^3 \\ 2\zeta^2 + 3\zeta + 2 & \zeta^3 - 1 & -2\zeta^3 - \zeta^2 + \zeta + 2 \\ \zeta^3 + 2\zeta^2 + 2\zeta + 1 & -1 & -\zeta^3 + \zeta + 2 \end{pmatrix} \tag{5.2}$$

satisfies $g^*Fg = F$ for $F = F_{(\mathcal{C}_8,\emptyset)}$ and $g^3 = I$. Applying Lemma 1.1 to the finite subgroup $\langle g\mathcal{Z}\rangle$ of $\bar\Gamma_{(\mathcal{C}_8,\emptyset)}$, we see that $\bar\Gamma_{(\mathcal{C}_3,\emptyset)}$ cannot contain a torsion-free subgroup Π of the index 128 required for Π to be the fundamental group of an fpp.

5.3. **The case** $(\mathcal{C}_8, \{2\})$**.** To eliminate this case, we simply have to check that the matrix

$$g = \begin{pmatrix} 2\zeta^2 + 2\zeta + 1 & -2\zeta^3 - 2\zeta^2 + 1 & \zeta^2 + 2\zeta + 1 \\ -\zeta^3 + \zeta + 1 & -\zeta^2 - \zeta & -\zeta^3 + \zeta + 1 \\ -\zeta^2 - \zeta & \zeta^3 - 1 & -\zeta^2 - \zeta - 1 \end{pmatrix} \tag{5.3}$$

satisfies $g^*Fg = F$ for $F = F_{(\mathcal{C}_8,\{2\})}$ and $g^3 = I$. Applying Lemma 1.1 to the finite subgroup $\langle g\mathcal{Z}\rangle$ of $\bar\Gamma_{(\mathcal{C}_8,\{2\})}$, we see that $\bar\Gamma_{(\mathcal{C}_3,\{2\})}$ cannot contain a torsion-free subgroup Π of the index 128 required for Π to be the fundamental group of an fpp.

6. Eliminating the case $(\mathcal{C}_{18}, \emptyset)$

We start by applying Lemma 4.2. In this case, the second equation in (4.14) is

$$6q_{33} = p_{13} + p_{23} + 2p_{33} - 2,$$

and so if α_{13} and α_{23} are not both 0 (or equivalently, if $p_{13} + p_{23} \geq 1$), then $|\alpha_{33}| > 1$, so that $p_{33} \geq 1$ and therefore $q_{33} > 0$. Performing Step 2 of the procedure of Section 4.4, with $B = 3$, say, we find that the smallest integer p such that $p = P(\alpha)$ for some $\alpha \in \mathfrak{o}_\ell$ with $Q(\alpha) > 0$ is $p = 3$, and by Step 3 of that procedure, we find that $|\alpha|^2 = 3 + r$ for $\alpha = t(z-1)$ or $\alpha = t(\bar{z}-1)$ for some $t \in \mathfrak{o}_\ell$ with $|t| = 1$. Completing the procedure, we find that the $g \in M_{3\times 3}(\mathfrak{o}_\ell)$ with $g^*Fg = F$ for $F = F_{(\mathcal{C}_{18},\emptyset)}$ and $|g_{33}|^2 = 3 + r$ are the elements of the double cosets KaK and $Ka^{-1}K$ for

$$a = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2}(-z^2+2z-2) & 1-z & \frac{1}{2}(-z^2+2z-2) \\ \frac{1}{2}(-z^2+2z-2) & -z & \frac{1}{2}(-z^2+2z) \end{pmatrix} \tag{6.1}$$

(recall that $-z^2/2$ is in $\mathfrak{o}_\ell$, being a cube root of 1).

Let u and v be the generators of $K = K_{(\mathcal{C}_{18},\emptyset)}$ given in (4.24). The above shows that the $g \in \bar{\Gamma}$ for which $d(0, g.0) > 0$ is minimal are the elements of $KaK \cup Ka^{-1}K$. While we do not need this here, we can show that $\bar{\Gamma}$ is generated by u, v and a. Relations satisfied (mod $\mathcal{Z}$) by these elements include

$$u^{24} = v^2 = 1,\ vuv^{-1} = u^{-5}, \quad av = va,\ (au^4)^4 = v,\ a^3 = (au^{-1})^3 = 1. \tag{6.2}$$

We do not claim that these relations give a presentation of $\bar{\Gamma}$. One can show that if we add the relations $(au^5a^{-1}u^2)^3 = (au^5a^{-1}u^3)^3 = 1$ to those listed in (6.2), we do get a presentation of $\bar{\Gamma}$, but we do not need this here.

To show that there are no fpps in the case $(\mathcal{C}_{18}, \emptyset)$, it is sufficient to prove the following result:

Proposition 6.1. *The group $\bar{\Gamma} = \bar{\Gamma}_{(\mathcal{C}_{18},\emptyset)}$ does not have any torsion-free subgroups of index 48.*

Proof. If Π is a torsion-free subgroup of $\bar{\Gamma}$, then the 48 cosets $k\Pi$, $k \in K$, are distinct, and so $[\bar{\Gamma} : \Pi] \geq 48$. Assume that Π is torsion-free and $[\bar{\Gamma} : \Pi] = 48$. The elements of K form a transversal for Π. Each element of K may be written $v^\epsilon u^\alpha$, where $\epsilon \in \{0, 1\}$ and $\alpha \in \{0, \ldots, 23\}$. Write $au^j\Pi = v^{\epsilon(j)}u^{\alpha(j)}\Pi$ in this way. Then $avu^j\Pi = v^{\epsilon(j)+1}u^{\alpha(j)}\Pi$ because $av = va$. If $\alpha(j) = \alpha(j')$, then

$$au^j\Pi = v^{\epsilon(j)}u^{\alpha(j)}\Pi = v^{\epsilon(j)}u^{\alpha(j')}\Pi = av^{\epsilon(j)-\epsilon(j')}u^{j'}\Pi$$

because v and a commmute. Hence $j' = j$. So α is a permutation of $\{0, \ldots, 23\}$. If $\alpha(j) = j$, then $u^{-j}v^{-\epsilon(j)}au^j \in \Pi$, contradicting the torsion-free property of Π. So α has no fixed points.

Applying the formula $a(v^\delta u^j\Pi) = v^{\epsilon(j)+\delta}u^{\alpha(j)}\Pi$ three times, we find that

$$v^\delta u^j\Pi = a^3v^\delta u^j\Pi = a(a(a(v^\delta u^j\Pi))) = v^{\delta'}u^{j'}\Pi,$$

where

$$\delta' = \epsilon(\alpha(\alpha(j))) + \epsilon(\alpha(j)) + \epsilon(j) + \delta \quad \text{and} \quad j' = \alpha(\alpha(\alpha(j))).$$

So the permutation α has order 3, and therefore has cycle type 3^8. Moreover, the sum of $\epsilon(j)$ is zero (mod 24) over the j's in each of the eight cycles, though we do not need this below.

We next use the relation $(au^4)^4 = v$. Let τ_n be the permutation $j \mapsto j + n$ (mod 24) of $\{0, \ldots, 23\}$, and write $\alpha_n(j) = \alpha(\tau_n(j))$. Now u^4 is in the center of K, and so

$$(au^4)(v^\beta u^j\Pi) = v^{\epsilon(j+4)+\beta}u^{\alpha(j+4)}\Pi = v^{\epsilon_4(j)+\beta}u^{\alpha_4(j)}\Pi \tag{6.3}$$

where $\epsilon_4(j) = \epsilon(\tau_4(j))$. The element au^4 has order 8, and commutes with v. As Π is torsion-free, $\alpha_4(j)$ can never equal j. Similarly, $\alpha_4(\alpha_4(j))$ can never equal j. Applying (6.3) four times, we get $v(v^\delta u^j \Pi) = (au^4)^4(v^\delta u^j \Pi) = v^{\delta'} u^{j'} \Pi$ for

$$\delta' = \epsilon_4(\alpha_4(\alpha_4(\alpha_4(j)))) + \epsilon_4(\alpha_4(\alpha_4(j))) + \epsilon_4(\alpha_4(j)) + \epsilon_4(j) + \delta$$

and $j' = \alpha_4(\alpha_4(\alpha_4(\alpha_4(j))))$. So the permutation α_4 has order 4, and as neither α_4 nor $\alpha_4 \circ \alpha_4$ has a fixed point, α_4 has cycle type 4^6. Moreover, the sum of $\epsilon_4(j)$ is zero (mod 24) over the j's in any of the six cycles of α_4.

The number of permutations α of $\{0, 1, \ldots, 23\}$ such that α has cycle type 3^8 and α_4 has cycle type 4^6 may be calculated using a standard formula from the character theory of finite groups, which we state below as Lemma 6.1. Such calculations can be done in Magma using its `SymmetricCharacterValue` function, for example. This finds that the number of such α's is 1 649 021 328, and so these α's form a very small proportion of the roughly 6×10^{23} permutations of $\{0, 1, \ldots, 23\}$. The group of permutations of $\{0, \ldots, 23\}$ commuting with τ_4 has order $4! \times 6^4 = 31\,104$ and acts by conjugation of this set of α's.

We now use the relation $(au^{-1})^3 = 1$. Firstly, using $u^{-1}v = vu^5$, we have

$$(au^{-1})(u^j\Pi) = v^{\epsilon(j-1)} u^{\alpha(j-1)} \Pi$$
$$(au^{-1})(vu^j\Pi) = v^{\epsilon(j+5)+1} u^{\alpha(j+5)} \Pi$$

It is routine to show that for each j, $u^j\Pi = (au^{-1})((au^{-1})((au^{-1})(u^j\Pi)))$ equals $v^{\epsilon'} u^{j'} \Pi$, where $\epsilon' \in \{0, 1\}$ and $j' = \beta(j)$ for one of the following four permutations β:

$$\alpha_{-1} \circ \alpha_{-1} \circ \alpha_{-1}, \quad \alpha_{-1} \circ \alpha_5 \circ \alpha_{-1}, \quad \alpha_5 \circ \alpha_{-1} \circ \alpha_{-1} \quad \text{and} \quad \alpha_5 \circ \alpha_5 \circ \alpha_{-1}. \tag{6.4}$$

Thus $\epsilon' = 0$, and one of these β's must fix j.

Similarly, $vu^j\Pi = (au^{-1})((au^{-1})((au^{-1})(vu^j\Pi)))$, equals $v^{\epsilon''} u^{j''} \Pi$, where $\epsilon'' \in \{0, 1\}$ and $j'' = \beta(j)$ for one of the following four permutations β:

$$\alpha_{-1} \circ \alpha_{-1} \circ \alpha_5, \quad \alpha_{-1} \circ \alpha_5 \circ \alpha_5, \quad \alpha_5 \circ \alpha_{-1} \circ \alpha_5 \quad \text{and} \quad \alpha_5 \circ \alpha_5 \circ \alpha_5. \tag{6.5}$$

Thus $\epsilon'' = 1$, and one of these β's must fix j. Thus the permutation α must have the three properties (1) α has cycle type 3^8; (2) α_4 has cycle type 4^6, and (3) for each j, one of the permutations in (6.4) and one of the permutations in (6.5) fixes j.

We wrote a C-program which ran through the set of α's satisfying (1) and (2), organized into orbits under the above action of the centralizer of τ_4, and checked that none of them satisfied property (3). This search was made more efficient using the fact that the group of order 48 generated by τ_1 and the permutation $j \mapsto -5j$ (mod 24) acts by conjugation on the set of α's satisfying all three properties.

This proved the proposition. □

Let G be a finite group. Let $\widehat{G}$ denote a full set of pairwise inequivalent irreducible representations of G. For $\pi \in \widehat{G}$, let $\chi_\pi(x) = \mathrm{Trace}(\pi(x))$ and $d_\pi = \chi_\pi(1)$ denote the character and degree of π. If C is a conjugacy class in G, write $\chi_\pi(C)$ for the constant value taken by $\chi_\pi(x)$ for $x \in C$.

Lemma 6.1. *Let C, D and E be conjugacy classes in G. Then for any $d \in D$,*

$$\sharp\{c \in C : cd \in E\} = \frac{|C|\,|E|}{|G|} \sum_{\pi \in \widehat{G}} \frac{\chi_\pi(C)\chi_\pi(D)\overline{\chi_\pi(E)}}{d_\pi} \tag{6.6}$$

7. Eliminating the case $(\mathcal{C}_1, \emptyset)$

We use the diagonal form $F = F_{(\mathcal{C}_1, \emptyset)}$ given in (1.2), where here $x = (r+1)/2$ and $r^2 = 5$. As we saw in the proof of Lemma 4.4, the stabilizer K of 0 in $\bar{\Gamma}$ consists of the 200 elements with matrix representatives (4.19), where (4.22) holds. Since

$\{t \in \mathfrak{o}_\ell : |t| = 1\}$ consists of the 10 elements $(-\zeta)^j$, $j = 0, \dots, 9$, we see that K is generated by the elements

$$d_1 = \begin{pmatrix} -\zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\zeta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 & x^{-1} & 0 \\ x & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.1}$$

and, with respect to these generators, has the presentation

$$d_1^{10} = d_2^{10} = 1 = w^2,\ d_1 d_2 = d_2 d_1, \text{ and } w d_1 w^{-1} = d_2. \tag{7.2}$$

Using the method described in Section 4.4, we find the following element of $\bar\Gamma$:

$$a = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -x & x \\ 0 & -1 & x \end{pmatrix}. \tag{7.3}$$

One may check that, mod $\mathcal{Z}$,

$$d_1 a = a d_1,\ a^2 = (a d_1^4 d_2^3)^3 = (a d_1^5 w)^5 = (a d_2^2 w a d_2^{-2} w)^3 = 1. \tag{7.4}$$

We can show that (mod $\mathcal{Z}$), the matrices d_1, d_2, w and a generate $\bar\Gamma$, and that they, together with the relations in (7.2) and (7.4), form a presentation of $\bar\Gamma$. It is not necessary to know this in order to eliminate there being any torsion-free subgroup Π of $\bar\Gamma$ of index $N = 600$. All we need to know is that a, d_1, d_2 and w belong to $\bar\Gamma$, and satisfy the above relations (we do not in fact need the relation $(a d_2^2 w a d_2^{-2} w)^3 = 1$).

For any set T, let $\mathrm{Perm}(T)$ denote the group of permutations of T. We shall use the following refinement of Lemma 1.1.

Lemma 7.1. *Suppose that Π is a torsion-free subgroup of finite index N in a group $\bar\Gamma$. Let T denote the set of cosets $g'\Pi$ of Π in $\bar\Gamma$. Then there is a homomorphism $\varphi : \bar\Gamma \to \mathrm{Perm}(T)$ such that*

(a) $(g, t) \mapsto \varphi(g)(t)$ *is a transitive action of* $\bar\Gamma$ *on* T, *and*
(b) *if* $g \in \bar\Gamma \setminus \{1\}$ *is of finite order, the permutation* $\varphi(g)$ *fixes no points of* T.

Conversely, if T is any set of size N, and if $\varphi : \bar\Gamma \to \mathrm{Perm}(T)$ is a homomorphism satisfying (a) and (b), then for any $t_0 \in T$, $\{g \in \bar\Gamma : g.t_0 = t_0\}$ is a torsion-free subgroup of $\bar\Gamma$ of index N.

Proof. Define φ by $\varphi(g)(g'\Pi) = gg'\Pi$. Then (a) clearly holds, and so does (b), for if $g \in \bar\Gamma \setminus \{1\}$ has finite order, and $\varphi(g)$ fixes $g'\Pi$, then $gg'\Pi = g'\Pi$, so that ${g'}^{-1} g g' \in \Pi$, contradicting the torsion-free hypothesis on Π.

Given $\varphi : \bar\Gamma \to \mathrm{Perm}(T)$ satisfying (a) and (b), $\{g \in \bar\Gamma : g.t_0 = t_0\}$ has index N because of (a), and is torsion-free because of (b). □

Note that in the context of Lemma 7.1, if $g \in \bar\Gamma$ has order n, then n must divide N, and $\varphi(g)$ has cycle type n^d, where $d = N/n$. That is, the cycle decomposition of $\varphi(g)$ consists of d cycles of length n.

In the present situation, if Π is a torsion-free subgroup of index $N = 600$ in $\bar\Gamma$, and if $g' \in \bar\Gamma$, then the 200 cosets $kg'\Pi$, $k \in K$, are distinct. So we can choose a transversal $T = T_{600}$ of Π in $\bar\Gamma$ of the form

$$T_{600} = \{k t_\alpha : k \in K \text{ and } \alpha \in \{0, 1, 2\}\},$$

for suitable $t_0, t_1, t_2 \in \bar\Gamma$. So identifying the set $\bar\Gamma/\Pi$ of cosets with T_{600}, the above transitive action of $\bar\Gamma$ on $\bar\Gamma/\Pi$ gives us a homomorphism $\varphi : \bar\Gamma \to \mathrm{Perm}(T_{600})$ with the property that $\varphi(k)(k' t_\alpha) = (kk') t_\alpha$ for $k, k' \in K$ and $\alpha \in \{0, 1, 2\}$.

We can write

$$T_{600} = \bigcup_{i=0}^{9} d_1^i T_{60} \quad \text{where} \quad T_{60} = \{d_2^j w^\epsilon t_\alpha : j \in \{0, 1, \dots, 9\} \text{ and } \epsilon \in \{0, 1\}\}.$$

The action φ of $\bar{\Gamma}$ on T_{600} induces an action φ' of the subgroup $C_{\bar{\Gamma}}(d_1) = \{g \in \bar{\Gamma} : gd_1 = d_1 g\}$ on T_{60}. For $t, t' \in T_{60}$,

$$\varphi'(g)(t) = t' \quad \text{if and only if} \quad \varphi(g)(t) = d_1^i t' \quad \text{for some } i.$$

Note that a, d_1 and d_2 are all in $C_{\bar{\Gamma}}(d_1)$. The action of d_1 on T_{60} is trivial, and the action of d_2 on T_{60} is simply $d_2^j w^\epsilon t_\alpha \mapsto d_2^{j+1} w^\epsilon t_\alpha$. Let $A, D_2 \in \mathrm{Perm}(T_{60})$ denote $\varphi'(a)$ and $\varphi'(d_2)$, respectively. If $g \in C_{\bar{\Gamma}}(d_1)$ has finite order, and is not in $\{d_1^i : i = 0, \ldots, 9\}$ then $\varphi'(g)$ fixes no point of T_{60}. Now $b = ad_1^4 d_2^3 \in C_{\bar{\Gamma}}(d_1)$ has order 3. Then $B = \varphi'(b)$ equals AD_2^3. Hence the permutations A, B and D_2 of T_{60} have cycle types 2^{30}, 3^{20} and 10^6, respectively, and $A = BD_2^{-3}$.

Let

$$\mathcal{B} = \{B \in \mathrm{Perm}(T_{60}) : B^3 = id = (BD_2^{-3})^2 \text{ and } B,\ BD_2^{-3} \text{ have no fixed points}\}$$

and

$$\mathcal{C} = \{C \in \mathrm{Perm}(T_{60}) : CD_2 = D_2 C\}.$$

Note that $\mathcal{C}$ acts on $\mathcal{B}$ by conjugation. It has $6! \times 10^6$ elements, consisting of the permutations

$$d_2^j t \mapsto d_2^{j+\tau_t} p(t), \tag{7.5}$$

where p is a permutation of $T_6 = \{w^\epsilon t_\alpha : \epsilon \in \{0,1\} \text{ and } \alpha \in \{0,1,2\}\}$, and where $\tau_t \in \{0, 1, \ldots, 9\}$ for each $t \in T_6$.

Lemma 7.2. *There are exactly* $77\,826\,756 \times 10^5$ *permutations* $B \in \mathcal{B}$*, and* $\mathcal{B}$ *is the union of exactly 12212 distinct* $\mathcal{C}$*-orbits.*

Proof. A full set $\{B_1, \ldots, B_N\}$ of orbit representatives was found using a back-track computer search, and N was found to be 12 212. The reader may find the B_i's in the file ".../gpc1_empty_blist.txt". For each i, the centralizer $\mathcal{C}_i$ in $\mathcal{C}$ of B_i was found. It turned out that the centralizer sizes $|\mathcal{C}_i|$ were as in the following table:

Centralizer size	1	2	3	4	6	8	10	12	20	36	60	72	200
Number of i's	9658	2106	59	244	72	11	5	36	11	3	3	3	1

Since $|\mathcal{C}| = 6! \times 10^6$, we find that

$$|\mathcal{B}| = 6! \times 10^6 \times (9658 + 2106/2 + 59/3 + 244/4 + 72/6 + 11/8 + 5/10 + 36/12 + 11/20 + 3/36 + 3/60 + 3/72 + 1/200) = 77\,826\,756 \times 10^5.$$

The value of $|\mathcal{B}|$ was confirmed independently by Lemma 6.1, applied to $G = \mathrm{Perm}(T_{60})$, $d = D_2^{-3}$, and C and E the conjugacy classes consisting of permutations of cycle type 3^{20} and 2^{30}, respectively. The irreducible representations of $\mathrm{Perm}(T_{60})$ are indexed by the 966 467 partitions P of 60 (this count found by Magma's command `NumberOfPartitions`(60)). Using the `SymmetricCharacterValue`(P, π) command for calculating the value $\chi_P(\pi)$ of the character corresponding to P at the element $\pi \in \mathrm{Perm}(T_{60})$, Magma was not able to calculate the sum in (6.6) in a reasonable time. Since we only need $\chi_P(\pi)$ for π having cycle type k^m, a specialized routine was written for efficiently calculating $\chi_P(\pi)$ in this case, and the sum was calculated, and found to be $77\,826\,756 \times 10^5$, as expected. $\square$

We now show that for each $i \in \{1, \ldots, 12\,212\}$ we need only consider 15 conjugates of B_i by elements of $\mathcal{C}$. It is easy to see that the group $\mathcal{S}$ of $s \in \mathrm{Perm}(T_{600})$ which commute with the action of each $k \in K$ are the maps $kt_\alpha \mapsto kk_\alpha t_{\pi(\alpha)}$, where π is a permutation of $\{0,1,2\}$ and $k_0, k_1, k_2 \in K$. Each $s \in \mathcal{S}$ commutes with the action of d_1 and d_2 on T_{600}, and so induces a permutation of T_{60} belonging to $\mathcal{C}$. The subgroup $\mathcal{C}_0$ of $\mathcal{C}$ consisting of the maps (7.5), where $p \in \mathrm{Perm}(T_6)$ permutes the three doubleton sets $\{t_\alpha, wt_\alpha\}$, has index 15. Each $s \in \mathcal{S}$ induces a $C \in \mathcal{C}_0$, and each $C \in \mathcal{C}_0$ is induced by an $s \in \mathcal{S}$. With this notation we have proved the following result:

Lemma 7.3. *Suppose that an action of $\bar{\Gamma}$ on $T_{600} = Kt_0 \cup Kt_1 \cup Kt_2$ is given such that (i) each nontrivial element of finite order acts without fixed points, and (ii) the action of each $k \in K$ is $(k, k't_\alpha) \mapsto kk't_\alpha$. Write $\mathcal{C}$ as a union of cosets $\mathcal{C}_0 C_j$, $j = 1, \ldots, 15$. Then the action of $\bar{\Gamma}$ on T_{600} is conjugate by an element of $\mathcal{S}$ to an action satisfying (i) and (ii) for which the element $b \in \bar{\Gamma}$ induces on T_{60} a permutation $C_j B_i C_j^{-1}$ for some $i \in \{1, \ldots, 12\,212\}$ and $j \in \{1, \ldots, 15\}$.*

Theorem 7.1. *There is no torsion-free subgroup of index 600 in $\bar{\Gamma}_{(\mathcal{C}_1,\emptyset)}$.*

Proof. Suppose that Π is a torsion-free subgroup of $\bar{\Gamma}$ of index 600, and consider the action of $\bar{\Gamma}$ on a transversal T_{600} of Π in $\bar{\Gamma}$. By Lemma 7.3, we may assume that this action satisfies (i) and (ii) of that lemma, and that the action of $b \in \bar{\Gamma}$ on T_{600} has the form

$$b.(d_1^i t) = d_1^{i+f(t)} B(t),$$

where B is one of the 183 180 permutations $C_j B_i C_j^{-1}$ described above, and where $f : T_{60} \to \mathbb{Z}/10\mathbb{Z}$. The conditions that b and $bd_1^{-4}d_2^{-3}$ have order 3 and 2, respectively, can be expressed in terms of f. This gives 50 conditions on the 60 values $f(t)$, which in all cases can be solved with either 11 or 12 free variables. Then the condition that $bd_1d_2^{-3}w = ad_1^5 w$ induces a permutation of T_{600} of cycle type 5^{120} can be tested. In all cases this test eliminated each choice of f. The elimination is speeded up by noticing that not all the free variables have to be chosen before $f(t)$ is known for sufficiently many $t \in T_{60}$ to eliminate the B in question. □

8. Eliminating the case $(\mathcal{C}_1, \{5\})$

We use the form $F_{(\mathcal{C}_1,\{5\})}$ given the matrix c^*Fc in Subsection 3.1. As we saw in the proof of Lemma 4.4, the stabilizer K of 0 in $\bar{\Gamma}$ is generated by

$$u = \begin{pmatrix} -1 & \zeta^3 & 0 \\ -\zeta^2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 & \zeta^2 & 0 \\ 0 & \zeta^4 & 0 \\ 0 & 0 & \zeta \end{pmatrix},$$

has order 600, and has presentation given by these generators and the relations $u^3 = v^5 = (uv^2)^5 = 1$ and $u(vu^2v) = (vu^2v)u$. Note that we have multiplied the v appearing in the proof of Lemma 4.4 by ζ to arrange that $\det(v) = 1$.

Using the method described in Section 4.4, we find the following element of $\bar{\Gamma}$:

$$a = \begin{pmatrix} -1 & 0 & 0 \\ -2\zeta^2 - 2\zeta - 1 & -\zeta^3 - \zeta^2 + 1 & -(\zeta + 1) \\ -\zeta^2 - 2\zeta - 2 & -\zeta^3 - 3\zeta^2 - \zeta & \zeta^3 + \zeta^2 - 1 \end{pmatrix}.$$

One may verify that $\det(a) = 1$, and that

$$a^2 = (v^3uvua)^3 = (uva)^5 = 1 \text{ and } a(vu^2v^2) = (vu^2v^2)a.$$

We can show that the matrices u, v and a generate $\bar{\Gamma}$, and that with the above relations of K, the above relations involving a give a presentation of $\bar{\Gamma}$. This is however not needed to prove the following result, which excludes there being a fake projective plane arising from the case $(\mathcal{C}_1, \{5\})$.

Theorem 8.1. *There is no torsion-free subgroup of index 600 in $\bar{\Gamma}_{(\mathcal{C}_1,\{5\})}$.*

Proof. Suppose that Π is a torsion-free subgroup of $\bar{\Gamma}$ of index 600. The hypothesis that Π is torsion-free implies that $K \cap \Pi = \{1\}$, and so we can choose K as a set of representatives for the cosets $g\Pi$ of Π in G. So the natural action of $\bar{\Gamma}$ on the coset space $\bar{\Gamma}/\Pi$ induces a homomorphism $\phi : \bar{\Gamma} \to \mathrm{Perm}(K)$ such that

(a) $\phi(k)(k') = kk'$ for all $k, k' \in K$,
(b) if $g \in \bar{\Gamma}$ has finite order, then $\phi(g)$ fixes no point of K.

Let $A \in \mathrm{Perm}(K)$ denote $\phi(a)$. Then $A^2 = (V^3UVUA)^3 = id$ and $A(VU^2V^2) = (VU^2V^2)A$, where $U = \phi(u)$ and $V = \phi(v)$, and neither A nor V^3UVUA has any fixed points. More generally, if $k \in K$ and if ka has finite order, then $\phi(k)A$ can have no fixed points. A back-track search was used to show that there is no permutation A of K with these properties.

The size of the space to be searched was reduced by observing that, when K is viewed as a K-set under left multiplications, its automorphism group consists precisely of the right translations by elements of K. So if $k_1 \in K$, and if $A \in \mathrm{Perm}(K)$ satisfies the above conditions, then the permutation $\tilde{A} : k \mapsto A(kk_1)k_1^{-1}$ does as well. Another way of thinking of this is to consider the change in the action of a when the subgroup Π is replaced by $k_1\Pi k_1^{-1}$.

In particular, if $A(1) = k_0$, then taking $k_1 = (vu^2v^2)^\nu$ for any $\nu \in \{0, \ldots, 4\}$, we have $ak_1 = k_1a$ and so $\tilde{A}(1) = k_1k_0k_1^{-1}$. So we can start our back-track search by assuming that $A(1)$ is one of the 160 representatives of the $\langle vu^2v^2\rangle$-conjugacy classes in K. Note that $\{k \in K : kvu^2v^2 = vu^2v^2k\}$ has order 50, and is generated by vu^2v^2 and uv.

Now $A(1) = k_0$ cannot hold if $k_0^{-1}a$ has finite order, and there are 95 such elements k_0, comprising 31 $\langle vu^2v^2\rangle$-conjugacy classes. So of the 160 conjugacy classes, 31 can be excluded immmediately.

Here are some other ideas used in the back-track search. The idea was to fill in values for A, that is, set $Ax = y$, one x at a time, considering all possible values for y, and eliminating possibilities as soon as possible. Whenever $Ax = y$, also $Ay = x$ must hold, as $A^2 = id$, and $A(VU^2V^2)^\nu x = (VU^2V^2)^\nu y$ and $A(VU^2V^2)^\nu y = (VU^2V^2)^\nu x$ must hold for $\nu = 0, \ldots, 4$, as $A(VU^2V^2) = (VU^2V^2)A$. The order of VU^2V^2 is 5, and the 10-element subgroup generated by A and VU^2V^2 must act freely. Thus, the 10 points $A(VU^2V^2)^\nu x$ and $A(VU^2V^2)^\nu y$ for $\nu = 0, \ldots, 4$ must all be different. So from the single value $y = Ax$ we can immediately deduce 9 other values for the action of A. We call these *linear* deductions. Thus as we proceed to construct our possible A, we always fill in 10 values at a time.

Taking $A' = V^3UVUA$, the relation $(v^3uvua)^3 = 1$ implies that $A'^3 = id$. If we know $Ax = y$, we also know $A'x = V^3UVUAx = V^3UVUy$, and vice versa — filling in values for A is equivalent to filling in values for A'. If we have filled in the values $A'z = x$ and $A'x = y$, then the further value $A'y = z$ may be deduced. We call these deductions *quadratic*. Choices such that $A'x = x$ can be excluded, as A' must act without fixed points.

In the back-track search, suppose that we arrive at the point where certain values $A'x$ have been determined, either by previous choices or by deductions from those choices, and where various other values $A'x$ remain to be determined. We must choose some x_1 for which $A'x_1$ is still unknown and consider all possibilities for $y_1 = A'x_1$.

Once y_1 is chosen, certain quadratic deductions may be available. For instance, if it is already known that $A'z_1 = x_1$, then any chosen value for y_1 allows us to deduce $A'y_1 = z_1$. The available linear deductions mean that any choice for $y_1 = A'x_1$ will determine 9 other values $A'x$, and 10 values $A'^{-1}x$, and these too may lead to possibilities for quadratic deductions.

Suppose that all possibilities with $A(1) = k_0$ have already been considered, and suppose that while looking at additional possibilities (with different values for $A(1)$) we need to choose $A(k)$, for some k. Then the choice $A(k) = k_0k$ can be excluded, for otherwise, on conjugated by right-translation by k^{-1}, we would obtain $\tilde{A}$ satisfying $\tilde{A}(1) = k_0$. □

9. Eliminating the case $(\mathcal{C}_{11}, \emptyset)$

We use the form $F_{(\mathcal{C}_{11},\emptyset)}$ given in (1.2), where here $x = r + 1$ and $r^2 = 3$. As we saw in the proof of Lemma 4.4, the stabilizer K of 0 in $\bar{\Gamma}$ is generated by the matrices u and v of (4.23), and a presentation for K is given by these generators and the relations $u^3 = 1 = v^4$ and $(uv)^2 = (vu)^2$.

Using the method described in Section 4.4, we find the following element of $\bar{\Gamma}$:

$$b = \begin{pmatrix} 1 & 0 & 0 \\ -2\zeta^3 - \zeta^2 + 2\zeta + 2 & \zeta^3 + \zeta^2 - \zeta - 1 & -\zeta^3 - \zeta^2 \\ \zeta^2 + \zeta & -\zeta^3 - 1 & -\zeta^3 + \zeta + 1 \end{pmatrix}.$$

It satisfies

$$vb = bv, \text{ and } b^3 = (buv)^3 = (buvu)^2 v = 1. \tag{9.1}$$

Moreover, for $g \in \bar{\Gamma}$, the smallest three values of $|g_{33}|^2$ are 1, $r+2$ and $2r+4$, and the method of Section 4.4 shows that these values are attained only by the elements of K, KbK and $Kbu^{-1}bK$, respectively. As shown in Section 11 below (see also [8]), the elements u, v and b, together with the relations given above, form a presentation of $\bar{\Gamma}$.

The following theorem excludes there being a fake projective plane arising from the case $(\mathcal{C}_{11}, \emptyset)$, in view of Hurewicz's Theorem.

Theorem 9.1. *There is, up to conjugacy, exactly one torsion-free subgroup* Π *of index 864 in* $\bar{\Gamma}_{(\mathcal{C}_{11},\emptyset)}$. *Its abelianization is isomorphic to* $\mathbb{Z}^2$, *so that the Betti number* b_1 *of the surface* $B(\mathbb{C}^2)/\Pi$ *is 2.*

Proof. The proof is very similar to that of Theorem 7.1. If Π is a torsion-free subgroup of $\bar{\Gamma}$ of index 864, then (see Lemma 7.1) there is a homomorphism $\varphi : \bar{\Gamma} \to \mathrm{Perm}(T)$, where T is a disjoint union $Kt_0 \cup Kt_1 \cup Kt_2$, such that

(a) $(g,t) \mapsto \varphi(g)(t)$ is a transitive action of $\bar{\Gamma}$ on T,
(b) if $g \in \bar{\Gamma} \setminus \{1\}$ is of finite order, the permutation $\varphi(g)$ fixes no points of T,
(c) $\varphi(k)(k't_\alpha) = kk't_\alpha$ for $k, k \in K$ and $\alpha = 0, 1, 2$.

If φ is such a homomorphism, let $B = \varphi(b)$, $U = \varphi(u)$ and $V = \varphi(v)$. Then by (c), U and V are known. By (9.1), we must have $BV = VB$ and $B^3 = (BUV)^3 = (BUVU)^2 V = id$. A back-track search was done to find all permutations $B \in \mathrm{Perm}(T)$ satisfying these conditions. It incorporated the ideas described in the proof of Theorem 8.1 above. A permutation B was quickly found, and then (using the second part of Lemma 7.1, the corresponding subgroup Π formed. See [8, Proposition 3.5] for details about how we verified that Π is torsion-free. Magma's routine `AbelianQuotientInvariants` verified that the abelianization of Π is $\mathbb{Z}^2$. After a lengthy search, all other possibilities for B were also found. Magma's `IsConjugate` command verified that the corresponding subgroups Π are all conjugate to each other. □

Further properties of the surface $B(\mathbb{C}^2)/\Pi$ are studied in [4].

10. Eliminating the case $(\mathcal{C}_{11}, \{2\})$

We use the form $F_{(\mathcal{C}_{11},\{2\})}$ given the matrix c^*Fc in Subsection 3.4. As we saw in the proof of Lemma 4.4, the stabilizer K of 0 in $\bar{\Gamma}$ is generated by

$$d_1 = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad d_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and } w = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

These satisfy

$$d_1^{12} = d_2^{12} = w^2 = I \text{ and } wd_1w^{-1} = d_2, \tag{10.1}$$

and these generators and relations form a presentation of K.

Using the method described in Section 4.4, we find the following element of $\bar{\Gamma}$:

$$a = \begin{pmatrix} \zeta^2 - 1 & 0 & 0 \\ 0 & -\zeta^3 - \zeta^2 & \zeta^3 \\ 0 & -\zeta^3 - 2\zeta^2 + 1 & \zeta^3 + \zeta^2 - 1 \end{pmatrix}.$$

One may verify that

$$a^3 = 1,\ ad_1 = d_1a,\ (ad_2)^2 = d_1,\ (wa)^3 = (aw)^3,\ \text{and } (a^{-1}waw)^4 = 1. \quad (10.2)$$

We can show that the elements d_1, d_2, w and a, together with these relations and those of K, give a presentation of $\bar{\Gamma}$. This is however not needed to prove the following theorem, which excludes there being a fake projective plane arising from the case $(\mathcal{C}_{11}, \{2\})$.

Theorem 10.1. *There is no torsion-free subgroup of index 288 in* $\bar{\Gamma}_{(\mathcal{C}_{11},\{2\})}$.

Proof. The proof is similar to the proof of Theorem 7.1, though easier because the index 288 is the same as the order of K (and smaller than 600). Again this index is too large for Magma's `LowIndexSubgroups` routine to complete in a reasonable time, and so we proved this theorem in the following way.

Suppose that Π is a torsion-free subgroup of $\bar{\Gamma}$ of index 288. The hypothesis that Π is torsion-free implies that $K \cap \Pi = \{1\}$, and so we can choose K as a set of coset representatives for Π in G. So the natural action of $\bar{\Gamma}$ on the coset space $\bar{\Gamma}/\Pi$ induces a homomorphism $\phi : \bar{\Gamma} \to \mathrm{Perm}(K)$ such that

(i) $\phi(k)(k') = kk'$ for all $k, k' \in K$,
(ii) if $g \in \bar{\Gamma}$ has finite order, then $\phi(g)$ fixes no point of K.

Our aim is to show that there is no such homomorphism.

We start by dividing K into 24 $\langle d_1\rangle$-orbits:

$$K = \bigcup_{t \in T_{24}} \{t, d_1t, \ldots, d_1^{11}t\},$$

where $T_{24} = \{d_2^j w^\epsilon \ : \ j \in \{0, \ldots, 11\} \text{ and } \epsilon \in \{0, 1\}\}$. Since both a and d_2 commute with d_1, $\phi(a)$ and $\phi(d_2)$ induce permutations A and D_2 of T_{24} of order 3 and 12, respectively. Since $(ad_2)^2 = d_1$ (modulo scalars), AD_2 is a permutation of order 2. The hypothesis that Π is torsion-free implies that A, D_2 and AD_2 have no fixed points, and so their cycle types are 3^8, 12^2 and 2^{12}, respectively.

Lemma 10.1. *There are exactly* 3204 *permutations* $A \in \mathrm{Perm}(T_{24})$ *such that* A *has cycle type* 3^8 *and* AD_2 *has cycle type* 2^{12}. *Let* $\mathcal{A}$ *denote the set of these* A*'s, and let* $\mathcal{C}$ *denote the commutator in* $\mathrm{Perm}(T_{24})$ *of* D_2. *Then* $\mathcal{C}$ *acts by conjugation on* $\mathcal{A}$, *and* $\mathcal{A}$ *is the union of exactly 25 orbits under this action.*

Proof. The value of $|\mathcal{A}|$ was found by Lemma 6.1, applied to $G = \mathrm{Perm}(T_{24})$, $d = D_2$, and C and E the conjugacy classes consisting of permutations of cycle type 3^8 and 2^{12}, respectively. The irreducible representations of $\mathrm{Perm}(T_{24})$ are indexed by the 1575 partitions P of 24 (this count found by Magma's command `NumberOfPartitions`(24)). Using the command `SymmetricCharacterValue`(P, π) for calculating the value $\chi_P(\pi)$ of the character corresponding to P at the element $\pi \in \mathrm{Perm}(T_{24})$, Magma was able to quickly calculate the sum in (6.6).

The 3204 elements of $\mathcal{A}$ was then found using a back-track computer search. Now $\mathcal{C}$ consists of the permutations $d_2^j w^\epsilon \mapsto d_2^{j+\tau_\epsilon} w^{\pi(\epsilon)}$, where π is a permutation of $\{0, 1\}$, and where $\tau_0, \tau_1 \in \{0, \ldots, 11\}$. So $|\mathcal{C}| = 2! \times 12^2 = 288$. We calculated the orbits in $\mathcal{A}$ under the action of $\mathcal{C}$, and found there were 25 of them, and chose orbit representatives $A_1, \ldots, A_{25}$. These representatives are listed in the file ".../gpc11_2_alist.txt". As a check that this is a complete list of representatives, for each i, the centralizer $\mathcal{C}_i$ in $\mathcal{C}$ of A_i was found. It turned out that the centralizer sizes $|\mathcal{C}_i|$ were as in the following table:

Centralizer size	1	2	4	6	8
Number of i's	2	15	4	3	1

Thus the the union of the orbits of these 25 A_i's has cardinality

$$2! \times 12^2 \times \big(2 + 15/2 + 4/4 + 3/6 + 1/8\big) = 3204,$$

and so is all of $\mathcal{A}$. □

The permutations of K which commute with all the left multiplications $k' \mapsto kk'$ ($k \in K$) are just the right multiplications $\rho(k_0) : k' \mapsto k'k_0$ ($k_0 \in K$). So if $\phi : \bar{\Gamma} \to \mathrm{Perm}(K)$ satisfies (i) and (ii) above, then for each $k_0 \in K$, $\phi' : \gamma \mapsto \rho(k_0)\circ\phi(\gamma)\circ\rho(k_0^{-1})$ is a group homomorphism $\bar{\Gamma} \to \mathrm{Perm}(K)$ which also satisfies (i) and (ii). Since $\rho(k_0)$ commutes with both $\phi(d_1)$ and $\phi(d_2)$, it induces a permutation C of T_{24} which commutes with D_2. So if $\phi(a)$ induces the permutation A of T_{24}, then $\phi'(a)$ induces the permutation CAC^{-1}

Lemma 10.2. *Any C belonging to the centralizer $\mathcal{C}$ of D_2 in* $\mathrm{Perm}(T_{24})$ *can be induced from some $\rho(k_0)$, $k_0 \in K$. So if there is a group homomorphism $\phi : \bar{\Gamma} \to$* $\mathrm{Perm}(K)$ *satisfying (i) and (ii) and so that $\phi(a)$ induces the permutation A of T_{24}, then for any $C \in \mathcal{C}$ there is a group homomorphism $\phi' : \bar{\Gamma} \to$* $\mathrm{Perm}(K)$ *satisfying (i) and (ii) and so that $\phi'(a)$ induces the permutation CAC^{-1} of T_{24}.*

Proof. Right multiplication by w induces the involution $d_2^j w^\epsilon \mapsto d_2^j w^{1-\epsilon}$ of T_{24}, right multiplication by d_1 fixes each $d_2^j w^0$ and induces the cycle of length 12 in T_{24} mapping each $d_2^j w^1$ to $d_2^{j+1 \,(\mathrm{mod}\, 12)} w^1$. These two maps generate $\mathcal{C}$. □

We now complete the proof of Theorem 10.1. In view of Lemma 10.2, we can assume that the permutation $A = \phi(a)$ of T_{24} is one of the 25 orbit representatives A_i of Lemma 10.1. Thus, for one of these A's, the action of a on K has the form

$$a.(d_1^i t) = d_1^{i+f(t)} A(t),$$

where $f : T_{24} \to \mathbb{Z}/12\mathbb{Z}$. The conditions that $a^3 = 1$ and $(ad_2)^2 = d_1$ can be expressed in terms of f. This gives 20 linear conditions on the 24 values $f(t)$, which in all 25 cases can be solved with 5 free variables. Then the condition that $(wa)^3 = (aw)^3$ can be tested. In all cases this test eliminated each choice of f. □

11. Finding presentations of the groups $\bar{\Gamma}$

Lemma 4.2 is useful for seeing explicitly the discreteness of the set of distances $d(0, g.0)$, $g \in \bar{\Gamma}$. For example, when $\mathfrak{o}_k = \mathbb{Z}[r]$, then by (4.3), $\cosh^2(d(0, g0)) = |g_{33}|^2 = p_{33} + rq_{33}$ for integers p_{33}, q_{33}, and the proof of Lemma 4.2 shows that $rq_{33} \le p_{33} \le rq_{33} + 1$, so that $2p_{33} - 1 \le |g_{33}|^2 \le 2p_{33}$. If also $g' \in \bar{\Gamma}$, with $|g'_{33}|^2 = p'_{33} + rq'_{33} < |g_{33}|^2$, then either $p'_{33} < p_{33}$ or $q'_{33} < q_{33}$ or both. If $p'_{33} < p_{33}$, then

$$\cosh^2(d(0, g'.0)) = |g'_{33}|^2 \le 2p'_{33} \le 2p_{33} - 2 \le |g_{33}|^2 - 1 = \cosh^2(d(0, g.0)) - 1.$$

Note that $p_{33} < p'_{33}$ cannot happen, as otherwise $q_{33} \le \frac{1}{r} p_{33} \le \frac{1}{r}(p'_{33} - 1) \le q'_{33}$, and so $|g_{33}|^2 < |g'_{33}|^2$. Finally, if $p'_{33} = p_{33}$ (and $|g'_{33}| < |g_{33}|^2$ still), then $q'_{33} < q_{33}$, and so again $|g'_{33}|^2 = p'_{33} + rq'_{33} \le p_{33} + r(q_{33} - 1) = |g_{33}|^2 - r < |g_{33}|^2 - 1$.

Let

$$d_0 = 0 < d_1 < d_2 < \cdots$$

be the distinct values taken by $d(0, g.0)$, $g \in \bar{\Gamma}$. When $\mathfrak{o}_k = \mathbb{Z}[r]$, respectively $\mathbb{Z}[(r+1)/2]$, we have $\cosh^2(d_n) = p_n + q_n r$, respectively $(p_n + rq_n)/2$, for certain integers p_n and q_n. For example, in the case $(\mathcal{C}_{11}, \emptyset)$, the first few $p_n + q_n r$ are:

$$1,\ 2 + r,\ 4 + 2r,\ 6 + 3r,\ 7 + 4r,\ 11 + 6r,\ \ldots$$

We find all possible $g_{11}, g_{12}, g_{13}, g_{23}, g_{33} \in \mathfrak{o}_\ell$ satisfying the column 3 and row 1 conditions and $|g_{33}|^2 = \cosh^2(d_n)$, and then for each $\theta \in \mathfrak{o}_\ell$ such that $|\theta| = 1$, we use the method of proof of Lemma 4.1 to form the unique $g \in M_{3\times3}(\ell)$ with the five specified entries such that $g^* F g = F$ and $\det(g) = \theta$, then test whether the g_{ij}'s are in $\mathfrak{o}_\ell$. In this way, we can form

$$S_n = \{g \in \bar{\Gamma} : d(0, g.0) \le d_n\}.$$

Now

$$K = S_0 \subset S_1 \subset S_1 \subset S_2 \subset \cdots, \quad \text{and} \quad \bigcup_n S_n = \bar{\Gamma}.$$

For any $S \subset \bar{\Gamma}$, we can form

$$\mathcal{F}_S = \{z \in B(\mathbb{C}^2) : d(0,z) \leq d(g.0,z) \text{ for all } g \in S\},$$

and

$$r_S = \sup\{d(0,z) : z \in \mathcal{F}_S\}.$$

Write $\mathcal{F}_n = \mathcal{F}_S$ and $r_n = r_S$ for $S = S_n$. Then

$$B(\mathbb{C}^2) = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \mathcal{F}_1 \supset \cdots \quad \text{and} \quad \bigcap_n \mathcal{F}_n = \mathcal{F}_{\bar{\Gamma}},$$

and $\mathcal{F}_{\bar{\Gamma}}$ is the Dirichlet fundamental domain for $\bar{\Gamma}$. Also,

$$\infty = r_0 \geq r_1 \geq r_2 \geq \cdots$$

Lemma 11.1. *If $d_n \geq r_n$, then S_n generates $\bar{\Gamma}$.*

Proof. Suppose that $\langle S_n \rangle \subsetneqq \bar{\Gamma}$. Choose $h \in \bar{\Gamma} \setminus \langle S_n \rangle$ with $d(0,h.0)$ minimal. If $g \in S_n$, then $g^{-1}h \notin \langle S_n \rangle$, and so

$$d(0,h.0) \leq d(0,(g^{-1}h).0) = d(g.0,h.0) \quad \text{for all } g \in S_n.$$

Hence $h.0 \in \mathcal{F}_n$. But then $d(0,h.0) \leq r_n$, and by hypothesis $r_n \leq d_n$. Hence $h \in S_n$, a contradiction. □

Lemma 11.2. *If $d_n \geq 2r_n$, then*

(a) *$\mathcal{F}_n = \mathcal{F}_{\bar{\Gamma}}$ and $r_n = r_{\bar{\Gamma}}$.*
(b) *S_n, together with the relations $g_1g_2g_3 = 1$ which hold for $g_1, g_2, g_3 \in S_n$, form a presentation for $\bar{\Gamma}$.*

Proof. (a) Suppose that $z \in \mathcal{F}_n \setminus \mathcal{F}_{\bar{\Gamma}}$. As $z \notin \mathcal{F}_{\bar{\Gamma}}$, there must exist a $g \in \bar{\Gamma}$ such that $d(g.0,z) < d(0,z)$. But using $d(0,z) \leq r_n$, we have

$$d(0,g.0) \leq d(0,z) + d(z,g.0) < 2d(0,z) \leq 2r_n \leq d_n,$$

so that $g \in S_n$. But then $d(g.0,z) < d(0,z)$ implies that $z \notin \mathcal{F}_n$, a contradiction.

(b) follows from a general result (Theorem I.8.10 in Bridson & Häfliger's book) about group actions on topological spaces. □

Using Proposition 2.1 in [8], we can replace S_n by S_{n-1} in Lemmas 11.1 and 11.2(b).

The following is useful for giving lower bounds on r_n.

Lemma 11.3. *Suppose that $\eta \in B(\mathbb{C}^2)$ is nonzero. Let m be the midpoint of the hyperbolic segment $[0,\eta]$. Then $m \in \mathcal{F}_S$ if and only if*

$$0 \leq |g.0|^2 - 2\mathrm{Re}\langle g.0,\eta\rangle t + |\langle g.0,\eta\rangle|^2 t^2 \quad \text{for all } g \in S, \tag{11.1}$$

where $t = \left(1 - \sqrt{1-|\eta|^2}\right)/|\eta|^2$.

Proof. From (4.2) we have

$$\cosh^2(d(g.0,m)) = \frac{|1 - \langle g.0,m\rangle|^2}{(1-|g.0|^2)(1-|m|^2)},$$

Hence $d(0,m) \leq d(g.0,m)$ if and only if

$$1 - |g.0|^2 \leq |1 - \langle g.0,m\rangle|^2,$$

or equivalently,

$$0 \leq |g.0|^2 - 2\mathrm{Re}\langle g.0,m\rangle + |\langle g.0,m\rangle|^2.$$

Now $m = t\eta$ for $t = \left(1 - \sqrt{1-|\eta|^2}\right)/|\eta|^2$, and so the result is proved. □

In particular, if the condition in Lemma 11.3 is satisfied by $\eta = h.0$, then $r_S \geq \frac{1}{2}d(0,h.0)$.

Lemma 11.4. *For the* $(\mathcal{C}_{11}, \emptyset)$ *example,*

$$r_1 = r_2 = \cdots = \frac{1}{2}d_2 = \frac{1}{2}\cosh^{-1}(1+\sqrt{3}),$$

so that we take $n = 2$ *in Lemmas 11.1 and 11.2.*

Proof. As mentioned before Theorem 9.1, we have $\cosh^2(d_1) = r+2$ and $\cosh^2(d_2) = 2r+4$ $(= (r+1)^2)$, with $S_1 = K \cup KbK$ and $S_2 = K \cup KbK \cup Kbu^{-1}bK$. We saw in [8, Lemma 4.3] that the midpoint of $[0, h.0]$ is in S_2 for $h = bu^{-1}b$, and so $r_2 \geq \frac{1}{2}d_2$. In [8, Proposition 4.1], we saw that for $z = (z_1, z_2) \in \mathcal{F}_1$, we have $|z_1|^2 + |z_2|^2 \leq 2r - 3$, and as $d(0,z) = \frac{1}{2}\log\big((1+|z|)/(1-|z|)\big)$, this means that $\cosh(2d(0,z)) \leq r+1$. Thus $2r_2 \leq 2r_1 \leq d_2$. □

For the other groups $\bar{\Gamma}$ under consideration, we calculated r_n only numerically, though with some effort, other exact calculations may be possible. For example, for the $(\mathcal{C}_{11}, \{2\})$ case, the first nine values of $\cosh^2(d(0, g.0)) = |g_{33}|^2$ were found to be

$$1,\ 2+r,\ 6+3r,\ 7+4r,\ 11+6r,\ 16+9r,\ 20+11r,\ 21+12r \text{ and } 25+14r.$$

Denoting these $\cosh^2(d_0), \dots, \cosh^2(d_8)$, the method of Section 4.4 found that S_7 is the union of 18 distinct double cosets KgK. Using Lemma 11.3, we find that for an $h \in \bar{\Gamma}$ satisfying $d(0, h.0) = d_7$, the midpoint of $[0, h.0]$ is in $\mathcal{F}_7$. Hence $r_7 \geq \frac{1}{2}d_7$. Numerical calculations indicate that equality holds here. Assuming only that r_7 has been calculated with sufficient accuracy to be sure that $r_7 < \frac{1}{2}d_8$, we have $r_8 \leq r_7 \leq \frac{1}{2}d_8$, and can apply Lemma 11.2 with $n = 8$ to get presentation of $\bar{\Gamma}$. One may verify that all 20 double cosets of elements g satisfying $d(0, g.0) \leq d_8$ lie in $\langle d_1, d_2, w, a\rangle$, so that Lemma 11.1 shows that d_1, d_2, w and a generate $\bar{\Gamma}$. Lemma 11.2 with $n = 8$ may give relations which turn out to be unnecessary, but these may be eliminated with special arguments, if a presentation of $\bar{\Gamma}$ is needed which is as simple as possible.

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