Some simple properties of vector spaces

Theorem Suppose that V is a vector space.

a) The zero vector $0 \in V$ is unique.
   That is, if $x + 0 = x$ and $x + 0' = x$, for all $x \in V$, then $0' = 0$.

b) If $r \in \mathbb{R}$ then $r \cdot 0 = 0$.

c) If $x \in V$ then $0 \cdot x = 0$.

d) The negative of a vector is unique.
   That is, if $x + x' = 0$ and $x + x'' = 0$ then $x' = x''$.

e) If $x \in V$ then $-x$ is the negative of x.

Proof

(a) Suppose that 0 and $0'$ are both zero vectors in V.
Then $x + 0 = x$ and $x + 0' = x$, for all $x \in V$.

Therefore, $0' = 0' + 0$, as 0 is a zero vector,

$= 0 + 0'$, by commutativity,

$= 0$, as $0'$ is a zero vector.

Hence, $0 = 0'$, showing that the zero vector is unique.

Proof that negatives are unique

Suppose that $x + x' = 0$ and $x + x'' = 0$.

Then $x'' = x'' + 0$

$= x'' + (x + x')$, as x' is a negative for x

$= (x'' + x) + x'$, by the distributive law

$= 0 + x'$, as x'' is a negative for x

$= x'$, as x'' is a negative for x

Hence, $x' = x''$. So there is only one negative of a given vector $x \in V$.

As $0 = 0 \cdot x = (1 - 1) \cdot x = x + (-x)$, the vector $-x$ is the (unique) negative of x.

Consequently, we write $-x$ for the negative of x.

Vector subspaces If A is an $n \times m$ matrix then the null space $\mathrm{Null}(A) \subseteq \mathbb{R}^m$ is contained in the bigger vector space $\mathbb{R}^m$.

It often happens that one vector space is contained inside a larger vector space and it is useful to formalize this.

Definition Suppose that V is a vector space. A vector subspace of V is a non-empty subset W of V which is itself a vector space, using the same operations of vector addition and scalar multiplication as V.

We frequently just say that W is a subspace of V.

Examples

- $\mathrm{Null}(A)$ is a vector subspace of $\mathbb{R}^m$
- $\mathbb{P} = \{$all polynomial functions$\}$ is a vector subspace of $\mathbb{F} = \{$all functions $f : \mathbb{R} \longrightarrow \mathbb{R}\}$.
- $\mathbb{P}_n = \{$all polynomial functions of degree at most $n\}$ is a vector subspace of $\mathbb{P}$.
- $\mathrm{Diff}(\mathbb{R}) = \{$all differentiable functions $f : \mathbb{R} \longrightarrow \mathbb{R}\}$ is a vector subspace of $\mathbb{F}$.

Recognizing vector subspaces It turns out that there is a simple test to determine when a subset of a vector space V is a subspace of V.

Theorem

Suppose that V is a vector space and that $W \subseteq V$. Then W is a subspace of V if and only if

- $W \neq \emptyset$;
- (A1) W is closed under vector addition; and,
- (S1) W is closed under scalar multiplication.

Proof If W is a subspace of V then these three conditions are certainly true.

Conversely, suppose that $W \neq \emptyset$ and that W satisfies both $(A1)$ and $(S1)$. We have to show that W is a vector space.

To prove this it is enough to observe that the remaining vector space axioms automatically hold in W because they already hold in V. (exercise!!)

Vector subspace examples

Example 1

$\text{Diff}(\mathbb{R}) = \{ f \in \mathbb{F} : f \text{ is differentiable} \}$ is a vector subspace of $\mathbb{F}$.

In particular, $\text{Diff}(\mathbb{R})$ is a vector space.

We have to check three things:

- $\text{Diff}(\mathbb{R}) \neq 0$: this is clear as the zero function is in $\text{Diff}(\mathbb{R})$.
- $\text{Diff}(\mathbb{R})$ is closed under addition.
  If f and g are differentiable functions then $(f+g)' = f' + g'$,so that $f + g \in \text{Diff}(\mathbb{R})$.
- $\text{Diff}(\mathbb{R})$ is closed under scalar multiplication.
  If $f \in \text{Diff}(\mathbb{R})$ then $(rf)' = rf'$
  so that $rf \in \text{Diff}(\mathbb{R})$.

Example 2

If V is any vector space then $\{0\}$ is a subspace of V.

Example 3 If $V = \mathbb{R}^2$ what are the subspaces of V?

Example 4 If $V = \mathbb{R}^3$ what are the subspaces of V?

Vector subspace examples–null space example

Example 5 Let $A = \begin{bmatrix} 3 & 1 & 3 & 3 \\ 2 & 4 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix}$.

Describe $\mathrm{Null}(A)$ as a subspace of $\mathbb{R}^4$.

As $\mathrm{Null}(A) = \{\, x \in \mathbb{R}^4 : Ax = 0 \,\}$ we have to find all solutions of $Ax = 0$.

We need to find the general solution to $Ax = 0$, so we use Gaussian elimination:

$$\begin{bmatrix} 3 & 1 & 3 & 3 \\ 2 & 4 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1 \longleftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 4 & 1 & 2 \\ 3 & 1 & 3 & 3 \end{bmatrix} \xrightarrow[R_3 := R_3 - 3R_1]{R_2 := R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 4 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \longleftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & -1 & 0 \end{bmatrix} \xrightarrow{R_3 := R_3 - 4R_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 := -R_3} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 := R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence, $\mathrm{Null}(A) = \left\{ \begin{bmatrix} -t \\ 0 \\ 0 \\ t \end{bmatrix} : t \in \mathbb{R} \right\}$

$$= \left\{ t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Example 6 Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -2 & 0 & -6 & -1 & -2 \\ 1 & 8 & 3 & 11 & 13 \\ -1 & 4 & -3 & 3 & 3 \end{bmatrix}$.

Describe $\text{Null}(A)$.

Again, we just apply row operations:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ -2 & 0 & -6 & -1 & -2 \\ 1 & 8 & 3 & 11 & 13 \\ -1 & 4 & -3 & 3 & 3 \end{bmatrix} \xrightarrow[R_4:=R_4+R_1]{\substack{R_2:=R_2+2R_1 \\ R_3:=R_3-R_1}} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 4 & 0 & 7 & 8 \\ 0 & 6 & 0 & 7 & 8 \\ 0 & 6 & 0 & 7 & 8 \end{bmatrix} \xrightarrow[R_4:=R_4-R_3]{R_2:=R_2-R_3} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2:=-\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 6 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_1:=R_1-2R_2]{R_3:=R_3-6R_2} \begin{bmatrix} 1 & 0 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3:=\frac{1}{7}R_3} \begin{bmatrix} 1 & 0 & 3 & 4 & 5 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{8}{7} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1:=R_1-4R_4} \begin{bmatrix} 1 & 0 & 3 & 0 & \frac{3}{7} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{8}{7} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $\text{Null}(A) = \left\{ \begin{bmatrix} -3s - \frac{3}{7}t \\ 0 \\ s \\ -\frac{8}{7}t \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\}$

$$= \left\{ s\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 0 \\ -8 \\ 7 \end{bmatrix} \; : \; s, t \in \mathbb{R} \right\}.$$