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# Topology of varieties & Perverse Sheaves

. $X \in \mathbb{P}^n_{\mathbb{C}}$ smooth projective variety
⇝ compact (complex) manifold
↳ orientable.

. Except said otherwise dim = $\dim_{\mathbb{C}}$

Examples $E = \mathbb{P}^n_{\mathbb{C}}$, $E$ = elliptic curve
$\eta_z = x(x-z)(x-\lambda z)$ $\lambda \neq 0,1$

$E$ = smooth hypersurface.

. Homology - cohomology

1) Singular chain $S_i(X,k)$ ⇝ $H_i(X,k)$
2) ✓ co-chain $S^i(X,k)^*$ ⇝ $H^i(X,k)$
3) locally finite chain $S_i^!(X,k)$ ⇝ Borel-Moore homology $H_i^!(X,k)$
(infinite chain with closed support)
4) locally finite co-chain ⇝ $H^i_!(X,k)$

When $k$ is a field. $H^i_m(X,k)^* = H^m_i(X,k)$
$H^!_m(X,k)^* = H^m_!(X,k)$

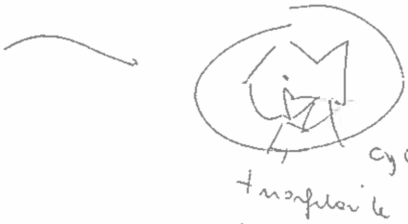
<u>Example</u>

$F: H_1(\mathbb{C}^\times) \to H_1^!(\mathbb{C}^\times)$ | $H_1^!(\mathbb{C}^\times, k) = k$

generator

$\mathbb{C}^\times$

cycle



cycle is homologous to 0

triangulate + compact support.

point + infinity is hol to Chain in domain of $F$.

Ex.

Let $V \subset X$ open, show that $H_x^!(X) \to H_x^!(V)$ restriction map

Why does restriction make sense?

## Main Thms.

Poincaré duality

$k$ is a field.

$M$ compact oriented real manifold $\dim_{\mathbb{R}} = m$, then

$$H_i(M,k) \cong H^{m-i}(M,k)^*$$

Let $X$ be as above, $X_H \subset X$ be a generic ~~by perspective~~ hyperplane section (smooth by Bertini's Thm) $\dim_{\mathbb{C}} X = n$.

Weak Lefschetz Thm

$k$ is an arbitrary ring

The inclusion $X_H \hookrightarrow X$ induces an iso

$$H_\ell(X_H) \to H_\ell(X) \quad \ell < n-1$$

and it is a surjection ~~at~~ $\ell = n-1$.

Moral: everything except for the middle cohomology comes from the hyperplane section

Ex $X$ surface $\dim X = 2$

$X_H \subset X$ a curve

$$i_x : H_0(X_H, \mathbb{Z}) \simeq H_0(X, \mathbb{Z}) \simeq \cancel{H_0(X_H, \mathbb{Z})}$$

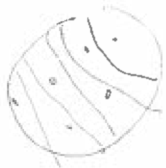
$$\mathbb{Z}^{2g} = H_1(X_H, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \text{ surjective.}$$

Exercise: Show that the Weak Lefschetz Thm is a consequence of

Thm Any affine variety $Y \subset \mathbb{A}$ of dim $n$ has the homotopy type of a CW-complex of real dim $\le n$.

Example

$\mathbb{P}^1\mathbb{C}$ - $l$ points



is homotopy to

Philosophy: Sheaves that make W. Lefschetz Thm work = Perverse Sheaves.

Hard Lefschetz Thm

(original argument by Lefschetz was wrong.
Hodge prove it using Harmonic forms)
Claim some local system was semi simple.

Definition

$X \subset \mathbb{P}^n\mathbb{C}$

$i^*: H^*(\mathbb{P}^n\mathbb{C}) \to H^*(X)$ ring

$$H^*(\mathbb{P}^n\mathbb{C}) \cong \mathbb{Q}[\beta]/\beta^{n+1}, \quad \deg \beta = 2$$

Set $\eta = i^*\beta$

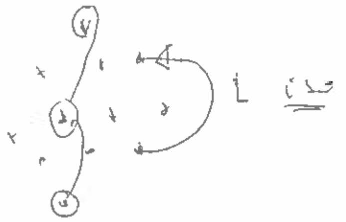
$L$ = operator on $H^*(X)$ given by multiplication by $\eta$

## Hard Lefschetz Thm

$$L^i : H^{n-i}(X) \longrightarrow H^{n+i} \quad \text{is an iso}$$

Example

dim $X = 2$



Exercise : Show that the Hard Lefschetz Thm is equivalent to the existence of a hom

$$\rho : sl(2) \longrightarrow gl\,(H^{\bullet}(X))$$

$$\mathfrak{g} = \mathbb{Q}f \oplus \mathbb{Q}e \oplus \mathbb{Q}h$$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Such that

$$\rho(h)\gamma = j\gamma$$
$$\rho(e)\gamma = L\gamma$$

Set

$$P_L^{n-i} = \ker L^{i+1} \subset H^{n-i}(X)$$

Hard Lefschetz Th $\sim$ $H^i(X) = \bigoplus_{i \geq j \geq 0} L^j(P_L^{n-i}) = \bigoplus_{i \geq 0} \left( \bigoplus_{0 \leq j \leq i} L^j P_L^{n-i} \right)$

Example

$$P^2 = \ker L \subset H^2 \quad 4$$
$$P^1 = \ker L^2 \subset H^1 \quad \boxed{2}$$
$$P^0 = \ker L^3 \subset H^0$$

| indices seems wrong ~~$P^i = \ker L^{n+1-i} \subset H^i$~~

~~Then~~

~~$P^0 = \ker L^3 \cap H$~~

Goal of the course

(1) What happen if $X$ is singular?

(2) What about proper maps $f: X \to Y$?

Answers to (1) & (2) are related via "Decomposition Thm" of Beilinson - Bernstein (Deligne - Gabber) ~ 1981

To make sense of this we need the technology of constructible sheaves, their derived category & Perverse sheaves.

Recall $f: X \to Y$ a map of spaces

$Sh(X,k)$ = abelian category of sheaves of $k$-vector spaces on $X$

$D(Sh(X,k))$ the derived category

$$\begin{tikzcd} Sh(X,k) \arrow[r, bend left, "f_*"] & Sh(Y,k) \arrow[l, bend left, "f^*"] \end{tikzcd}$$

left exact

adjoint to each other ( ... in the compact)

$Rf_*$ $$\begin{tikzcd} D(Sh(X,k)) \arrow[r, bend left, "Rf_*"] & D\,Sh(Y,k) \arrow[l, bend left, "Rf^*"] \end{tikzcd}$$

Given a $k$-module $V$, Constant sheaf with values on $k$ is a sheaf

$V_X(U)$ = continuous function: $U \to V$ (Discrete topology)

Key property (motivation for the definition of constructible)

$f: X \to \{pt\}$

$$H^i\big(\underbrace{Rf_*(\underbrace{k_X}_{\text{constant sheaf}})}_{\in D(Sh)(pt,k)}\big) = H^i(X,k)$$

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle f_2} & \\ \{pt\} & & \end{array}$$

use $Rf_* = Rf_{2*} \circ Rf_{1*}$ . In Deligne's proof of Weil's conjecture

it is important

$$\begin{array}{ccc} X & \longrightarrow & P \\ \downarrow & \swarrow & \\ \{pt\} & & \end{array}$$

hyperplane section

decompose into 2 steps
understand things by
induction reducing it to $\mathbb{CP}^1$.

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A local system is a sheaf $\mathcal{F}$ on $X$ : $X$ has an open covering $X = \bigcup_i U_i$

$\mathcal{F}|_{U_i} = \underline{V}_{U_i}$ constant sheaf of some f.g. k-module


stalk
vector space
with discrete top

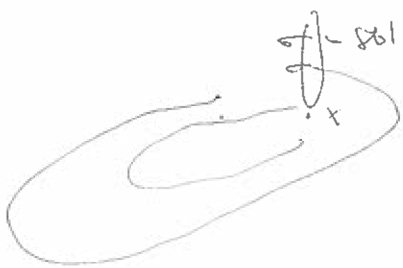
can only move in $\rightarrow$ direction

( Not the case for constructible --- ) ?

<u>Ex</u> $Loc(X,k)$ is an abelian subcategory $Sh(X,k)$

<u>Thm</u> $X$ is connected and has a universal space, then one has an equivalence.

$Loc(X,k) \simeq$ finite representations $(\pi_1(X,k))$




stalk

moving along cycle
return to stalk of $x$ ---

Given a local system $\mathcal{L}$, let $\mathcal{L}^\vee$ the local system of dual rep.

<u>Canonical example of a Local</u> System

Let $f : X \to Y$ be a smooth proper algebraic map of algebraic varieties.

(smooth ~ surjective assumption
Diff Geom'l $df$ is is onto i.e. surjective)

Then by Ehresmann's Lemma $f$ is a $C^\infty$ - fibration, i.e.

we can find a covering of $Y$ by open $V_i$

$$f^{-1}(V_i) = V_i \times S_i \quad \text{smooth proper.}$$

Then the "sheafification" of

$$u \mapsto H^i(f^{-1}(u)) \quad \underline{\text{is a local System}}$$

$$\simeq (R^i f_* \underline{k}_X)$$

$$\| $$

$$H^i(Rf_* \underline{k}_X)$$

"It measures how cohomology changes".

<u>Exercise</u> Consider

$$y^2 z = x(x-z)(x-\lambda z) \quad \subset \mathbb{P}^2 \times \mathbb{A}^1$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad x\,y\,z \qquad \lambda$$

(see his homepage for help)

If $\lambda \neq 0, 1$, polynomial has distinct roots.

$\subset \mathbb{C}$
delete $\{0\} \cup \{1\}$

What is the monodromy of this family
$H^1$(fibers)

<u>Moral</u>: Local systems ~ when you consider smooth maps.
Constructible sheaves ~ arbitrary algebraic maps
$$R^i f_* = H^i(Rf_*(-))$$

Let $\Lambda$ be a partition of $X$ into finitely many locally closed smooth connected subvarieties . $X = \bigcup_{\lambda \in \Lambda} X_\lambda$

A sheaf $\mathcal{F}$ is $\Lambda$ constructible if

$\mathcal{F}|_{X_\lambda}$ is a local system for each $\lambda \in \Lambda$

A sheaf $\mathcal{F}$ is constructible if $\exists$ a partition:

$\mathcal{F}$ is $\Lambda$ constructible.

A complex $\mathcal{F} \in D(Sh(X,k))$ is constructible if only finitely many $H^i(\mathcal{F})$ are non-zero and all $H^i(\mathcal{F})$ are constructible.

(1.8)

Key definition : Constructible derived category

$D_c^b(X,k) =$ full subcat $D(Sh(X,k))$ consisting of constructible complexes

(Consider all sheaves and for all consider those with constructible cohomology $\neq$ take constructible sheaves and take the derived category.)

Very non-trivial

For any $f: X \to Y$ algebraic, we get induced

$$D_c^b(X,k) \underset{Rf^*}{\overset{Rf_*}{\rightleftarrows}} D_c^b(Y,k)$$

For example $\underset{=}{\overset{f}{\exp}} : \mathbb{R} \to S^1$

$Rf_*$ (constant sheaf on $\mathbb{R}$) is not constructible