Rough plan: last time ... { Poincaré duality ⇝ Verdier duality; weak Lefschetz ⇝ perverse sheaves; hard Lefschetz ⇝ decomposition thm }

TALK AT MPIM MAY 2013

TALK II

for simplicity k is a field.

$D^b_c(X)$ = derived category of constructible $\mathbb{Q}$-sheaves $\underset{\text{full subcat}}{\subset}$ $D(Sh(X; k))$

= full triangulated subcat of $D^b(Sh(X;k))$ generated by $Rg_* \mathcal{L}$ for $g: Z \to X$ and $\mathcal{L}$ a local system on Z

have "constructible".

$$f: X \to Y \qquad \begin{tikzcd} D^b_c(X) \arrow[r, bend left, "Rf_*, Rf_!"] & D^b_c(Y) \arrow[l, bend left, "f^*, f^!"] \end{tikzcd}$$

$(-\otimes-, \quad R\mathcal{H}om)$ "Grothendieck's six functors"

Also: $f_! : Sh(X) \to Sh(Y)$ "direct image with compact support"

$$(f_! \mathcal{F})(U) = \{ s \in \mathcal{F}(f^{-1}U) \mid f|_{\mathrm{supp}\, s} \text{ is proper} \}.$$

$Rf_!$ right derived functor.

(Perhaps: there are interesting constraints on local systems which can occur... "semi-simplicity".)

Exercise: 1) Let $j : U \hookrightarrow X$ is an open embedding. Show that $(j_!, j^*)$ are adjoint.

Mention that $j_!$ is extension by zero. (ie. $(j_! \mathcal{F})_x = \begin{cases} \mathcal{F}_x & \text{if } x \in U \\ 0 & \text{otherwise} \end{cases}$

2) Let $i: Z \hookrightarrow X$ be a closed embedding. Show that $Ri_!$ has a right adjoint $i^!$ "sections with support".

3) Let $f: \mathbb{C} \to pt$ denote the projection. Show that $f_!$ does not have a right adjoint.

Grothendieck-Verdier: $Rf_!$ has a right adjoint $f^!$. It has the following properties:

~~1) If X is smooth and f: X → pt is the projection then f!~~

"Impossible" to compute in general. But 1) and 2) tell us what it is for ~~any~~ ~~open embedded~~ inclusion of a subvariety.

Last manageable case: If $f: X \to Y$ is smooth, Y connected ~~then~~ with fibres of dim d_F then

$$f^! \cong f^*[2d_F].$$

Eg: if X is smooth then $p^! \underline{k}_{pt} \cong \underline{k}_X [2d_X]$ where $p: X \to pt$ is the projection.

For any X set $\omega_X := p^! \underline{k}_{pt}$ $p : X \to pt$ "dualizing sheaf".

Set $\mathbb{D} := R\mathcal{H}om(-, \omega_X)$.   "Verdier duality".

Key basic properties: ~~"open-closed distinguished triangles~~

From now on $f_* := Rf_*$, $f_! := Rf_!$

MAYBE NOT SUCH A GOOD IDEA.

1) Relations with classical cohomology: $f : X \to pt$.

$$H^m(Rf_* \underline{k}_X) = H^m(X; k).$$

$$H^m(Rf_! \underline{k}_X) = H^m_!(X; k).$$

$$H^{-m}(Rf_* \omega_X) = H^!_m(X; k)$$

$$H^{-m}(Rf_! \omega_X) = H^!(X; k).$$

2) Open closed distinguished triangles: $X = U \sqcup Z$ $i : Z \hookrightarrow X \hookleftarrow U : j$.

$$i_! i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F} \xrightarrow{+1}$$

$$j_! j^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \xrightarrow{+1}$$

<u>Exercise</u>: put $\mathcal{F} = \underline{k}_X$ and deduce "all" the long exact sequences of cohomology.

3) <u>Duality</u>: $\mathbb{D}^2 \cong id$, $\mathbb{D} f_* \cong f_! \mathbb{D}$ $\mathbb{D} f^! \cong f^* \mathbb{D}$.

If $\mathcal{L}$ is a local system on X smooth then $\mathbb{D}\mathcal{L} \cong \mathcal{L}^\vee [2d_X]$.

(easy once one knows $\omega_X \cong \underline{k}_X [2d_X]$.)

$\Rightarrow$ Poincaré duality:

$$p_* \mathbb{D}\mathcal{L} \cong p_* \mathcal{L}^\vee [2d_X]$$

$$\mathbb{D} p_! \mathcal{L} \cong p_* \mathbb{D}\mathcal{L}$$

$$\Rightarrow \quad H^{-m}_!(X, \mathcal{L})^* \cong H^{2d_X+m}(X, \mathcal{L}^\vee).$$

Now $\Rightarrow \mathcal{L} \cong \mathbb{D}(\mathcal{L}^\vee [2d_X])$

$$p_* \mathcal{L} \cong p_* \mathbb{D}(\mathcal{L}^\vee [2d_X]) \cong \mathbb{D}(p_! \mathcal{L}^\vee [2d_X]).$$

Apply H^m and use $H^m(\mathbb{D} C^\cdot) = H^{-m}(C^\cdot)^*$.

$$\Rightarrow H^m(X, \mathcal{L}) \cong H^{2d_X - m}(X, \mathcal{L}^\vee).$$

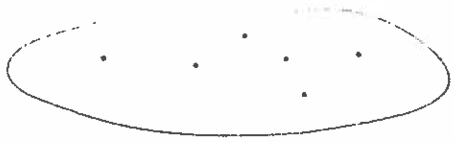
Perverse sheaves: ~~$\dim X = d_X$~~. Assume X is smooth.

Exercise: If $Z \overset{i}{\subset} X$ is a smooth subvariety of dimension d_Z and $\mathcal{L}$ is a ~~smooth subvariety~~ local system then $\mathbb{D}(i_*\mathcal{L}[d_Z]) \cong i_*\mathcal{L}^\vee[d_Z]$.

Eg: skyscrapers on points are self-dual, $\underline{k}_X(d_X)$ is ~~[illegible]~~ self-dual.

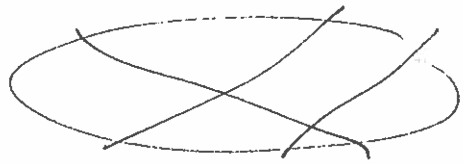
$\mathcal{F}$ is perverse if $\dim \operatorname{supp} \mathcal{H}^{-m}(\mathcal{F}) \leq m$ and $\dim \operatorname{supp} \mathcal{H}^{-m}(\mathbb{D}\mathcal{F}) \leq m$.

Eg: $\mathcal{H}^0$ ~~$(\mathcal{F})$~~

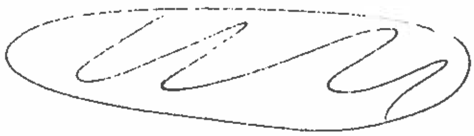


$\vdots$

$\mathcal{H}^{-d_X+1}(\mathcal{F})$



$\mathcal{H}^{d_X}(\mathcal{F})$



Recall ~~[illegible]~~ from last time:

U smooth affine, homotopy type:

$H^m(U) = 0$ for $m > d_U$

$\Rightarrow H^m_!(U) = 0$ for $m < d_U$. $\Longrightarrow$ weak Lefschetz.

Now if $\mathcal{F} \in D^b_c(U)$ is constructible then

Artin: $H^m(U, \mathcal{F}) = 0$ for $m > d_U$.

Exercise: show that $H^m_!(U, \mathcal{F}) = 0$ for $m < d_U$ fails in general!

Seeking a good class of sheaves for which both conditions hold

lead to perverse sheaves.

Exercise: Show that ~~[illegible]~~
for any perverse sheaf $\mathcal{F}$ on X and $U \subset X$ affine open

$$H^m(U, \mathcal{F}) = 0 = H^m_!(U, \mathbb{D}\mathcal{F}) \quad \text{for } m > 0.$$

$\Longrightarrow$ "weak Lefschetz theorem holds universally".

Basic facts about perverse sheaves:

* $\mathrm{Perv}(X) \subset D^b_c(X)$ is ~~[illegible]~~ a finite length abelian category

(This is a miracle!)

The simple objects of Perv

$$\mathrm{Irr}\,\mathrm{Perv}(X) \longleftrightarrow \left\{ \text{pairs } (Z, \mathcal{L}_Z) \right\} \Big/ \sim$$

Z: locally closed subvariety; $\mathcal{L}_Z$: irreducible local system

$(Z, \mathcal{L}_Z) \sim (Z', \mathcal{L}_{Z'})$ if $\dim Z =$

$(Z, \mathcal{L}_Z) \sim (Z', \mathcal{L}_{Z}|_{Z'})$

$Z' \subset Z$ open dense

$\updownarrow$

$IC(\overline{Z}, \mathcal{L}_Z)$

"intersection cohomology complex"

* $\mathrm{Perv}(X) \overset{\text{Riemann-Hilbert}}{\cong}$ regular holonomic $\mathcal{D}$-modules. (This explains abelianness).

* $\mathrm{Perv}(X)$ is the heart of a t-structure on $D^b_c(X)$.

$\rightsquigarrow$ truncation functors: ${}^p\mathcal{H}^m : D^b_c(X) \longrightarrow \mathrm{Perv}(X)$.

$IC(\overline{Z}, \mathcal{L})$ is the unique complex satisfying
- $\to$ $IC(\overline{Z}, \mathcal{L})|_Z \cong \mathcal{L}[d_Z]$
- $\to$ $\mathrm{supp}\, IC(\overline{Z}, \mathcal{L}) = \overline{Z}$.
- $\dim \mathrm{supp}\, \mathcal{H}^{-m}(\overline{Z}, \mathcal{L}) < m \quad \forall m < d_Z$.

Tell story about Goresky-MacPherson and Deligne...

• Local systems version:

$$\eta : f_* \underline{\mathbb{Q}}_X \to f_* \underline{\mathbb{Q}}_X [2].$$

Over any simply connected open subset $\mathcal{U}$ we have

$$f_* \underline{\mathbb{Q}}_{\mathcal{U}} \cong \underline{\mathbb{Q}}_{\mathcal{U}} \otimes H^{\cdot}(F)$$

and η is ~~simply multiplication by~~ the Lefschetz operator on $H^{\cdot}(F)$.

$\rightsquigarrow$ $\eta^i : \mathcal{H}^{d_F - m}(f_* \underline{\mathbb{Q}}_X) \longrightarrow \mathcal{H}^{d_F + m}(f_* \underline{\mathbb{Q}}_X)$

is an isomorphism.

<u>Exercise:</u> Suppose that C is an object in a derived category and $C \xrightarrow{\eta} C[2]$ is a morphism s.t. the induced map

$$\eta^m : \cancel{C \to C[2m]}$$

$$C[-m] \to C[m].$$

satisfies $H^0(\eta^m) = H^{-m}(C) \to H^m(C)$ is an iso.

Then $C \cong \bigoplus H^m(C)[-m]$.

Deligne's theorem: (decomp. thm. take 1)

$f : X \to Y$ smooth projective map of algebraic varieties ($\Rightarrow C^\infty$-fibration)

Y connected, fibre F.

The Leray-Serre spectral sequence $H^p(Y, \underline{H^q(F)}) \Rightarrow H^{p+q}(X)$

degenerates at E_2.

First proof: assume Y is simply connected.

$$\begin{tikzcd} F \arrow[r, hook] & X \arrow[r, hook] \arrow[d] & \mathbb{P}_Y \arrow[dl] \\ & Y & \end{tikzcd}$$

$\rightsquigarrow$ $F \subset \mathbb{P}$ $\rightsquigarrow$ Lefschetz operator

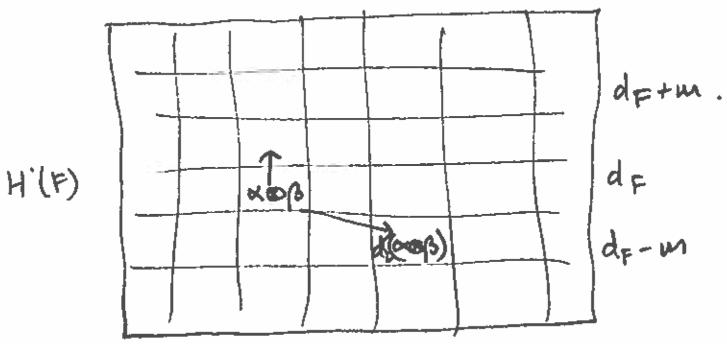
$\eta \in H^2(F)$.

$\quad \shortparallel$

$c_1(\mathcal{O}(1))$.

hard Lefschetz for $H^\cdot(F)$: $\quad H^\cdot(F) = \bigoplus_{i \geq m \geq 0} \eta^m P^{-d_F - i}$ where

$P^{d_F - i} = \ker \eta^{i+1} \subset H^{d_F - i}(F)$.



Now $d_2 : E^{pq} \to E^{p+2, q-1}$ is a map of $H^\cdot(F)$-modules $\Rightarrow$ compatible with Lefschetz operator.

if $\alpha \in P^{-d_F - i}$ and $\beta \in H^m(Y)$ then

$$d_2(\alpha \otimes \beta) \in H^{-d_F - i - 1}(F) \otimes H^{m+2}(Y)$$

$$\eta^{i+1} d_2(\alpha \otimes \beta) = d_2(\eta^{i+1} \alpha \otimes \beta) = 0.$$

hard Lefschetz $\Rightarrow$ $d_2(\alpha \otimes \beta) = 0$. $\square$