⑧ Topology of algebraic varieties and Perverse sheaves III

Recall from last time:

$$\begin{tikzcd} X \subset Y \times \mathbb{P} \arrow[d, "f"'] \arrow[dl] \\ Y \end{tikzcd}$$

$$c_1(\mathcal{O}(1)) \subset H^2(\mathbb{P}) \iff \eta : \underline{\mathbb{Q}}_{\mathbb{P}} \longrightarrow \underline{\mathbb{Q}}_{\mathbb{P}}[2]$$

$\rightsquigarrow$ pull-back

$$\eta : \underline{\mathbb{Q}}_{\mathbb{P}\times Y} \longrightarrow \underline{\mathbb{Q}}_{\mathbb{P}\times Y}[2]$$

$\rightsquigarrow$ restrict

$$\eta : \underline{\mathbb{Q}}_{X} \longrightarrow \underline{\mathbb{Q}}_{X}[2]$$

$\rightsquigarrow$ push-forward.

$$\eta : f_* \underline{\mathbb{Q}}_{X} \longrightarrow f_* \underline{\mathbb{Q}}_{X}[2].$$

Over $\mathcal{U} \subset Y$ contractible: $(f_* \underline{\mathbb{Q}}_X)_U = \underline{\mathbb{Q}}_U \otimes H^\cdot(F)$ $\circlearrowleft$ η Lefschetz operator $F \subset \mathbb{P}$.

hard Lefschetz $\Rightarrow$

__Deligne's theorem:__
f smooth

1) $\eta : \mathcal{H}^{d_F - i}(f_* \underline{\mathbb{Q}}_X) \longrightarrow \mathcal{H}^{d_F + i}(f_* \underline{\mathbb{Q}}_X)$ is an isomorphism.

$$\mathcal{H}^{d_F - i}(f_* \underline{\mathbb{Q}}_X) = R^{d_F - i} f_* \underline{\mathbb{Q}}_X$$

$\boxed{\text{Exercise: } 1) \Rightarrow 2) \text{ do it!}}$

2) $f_* \underline{\mathbb{Q}}_X = \bigoplus \mathcal{H}^i (f_* \underline{\mathbb{Q}}_X)[-i].$ ($\Rightarrow$ E_2-degeneration of Leray-Serre spectral sequence.)

3) (not discussed last time)

each $\mathcal{H}^i$ is semi-simple. (surprising given that $\pi_1(Y)$ can be infinite).

Rmk: (more next time) 3) follows from the fact that each $\mathcal{H}^i$ admits a canonical "primitive" decomposition, and each summand admits a $\pm$ definite form (Hodge-Riemann bilinear relations).

Last time: introduced:

$$\mathrm{Perv}(X) \subset D^b_c(X) \qquad \text{"perverse sheaves"}$$

Facts about $\mathrm{Perv}(X)$: 1) abelian, finite length
(doesn't hold for ordinary sheaves).

Exercise: $\mathcal{U} = \mathbb{C} \setminus \{x_1, \dots, x_n\}$ $\quad j : U \hookrightarrow \mathbb{C}$ inclusion.

1) $j_! \underline{\mathbb{Q}}_U \hookrightarrow \underline{\mathbb{Q}}_{\mathbb{C}}$ is injective in sheaves.

2) $j_! \underline{\mathbb{Q}}_U[1]$, $\underline{\mathbb{Q}}_{\mathbb{C}}[1]$, $\underline{\mathbb{Q}}_{\{x_1, \dots, x_n\}}$ are perverse.

$$\underline{\mathbb{Q}}_{\{x_1, \dots, x_n\}} \longrightarrow j_! \underline{\mathbb{Q}}_U[1] \longrightarrow \underline{\mathbb{Q}}_{\mathbb{C}}[1] \xrightarrow{+1}$$ is a ses in perverse sheaves.

2) simple objects are of the form $IC(\overline{Z}, \mathcal{L})$ where $Z \subset X$ is locally closed, smooth, connected and $\mathcal{L}$ is a local system on $\mathcal{L}$.

If Z is smooth closed then $IC(\overline{Z}, \mathcal{L}) = \mathcal{L}[d_Z]$.

"singular local systems".

3) $\mathrm{Perv}(X)$ is the heart of a t- structure on $D^b_c(X)$.
for "truncation"

Hence we have ${}^p\tau_{\leq 0}$, ${}^p\tau_{>1}$, ${}^p\mathcal{H}^i : D^b_c(X) \to \mathrm{Perv}(X)$.

Decomposition theorem: f ~~arbit~~ projective, X smooth

1) $\eta^i : {}^p\mathcal{H}^{-i}(f_*\underline{\mathbb{Q}}_X[d_X]) \longrightarrow {}^p\mathcal{H}^{i}(f_*\underline{\mathbb{Q}}_X)$ is an isomorphism

(relative hard Lefschetz).

2) (1) $\Rightarrow$) $\quad f_*\underline{\mathbb{Q}}_X[d_X] \cong \bigoplus {}^p\mathcal{H}^{i}(f_*\underline{\mathbb{Q}}_X[d_X])[-i]$.

($\Rightarrow$ E_2 degeneration of "perverse Leray-Serre spectral sequence")

3) each ${}^p\mathcal{H}^{i}(f_*\underline{\mathbb{Q}}_X[d_X])$ is semi-simple as an object of $\mathrm{Perv}(Y)$.

$\Rightarrow$ ($\bigoplus$ IC's).

First example: (Weierstraß family)

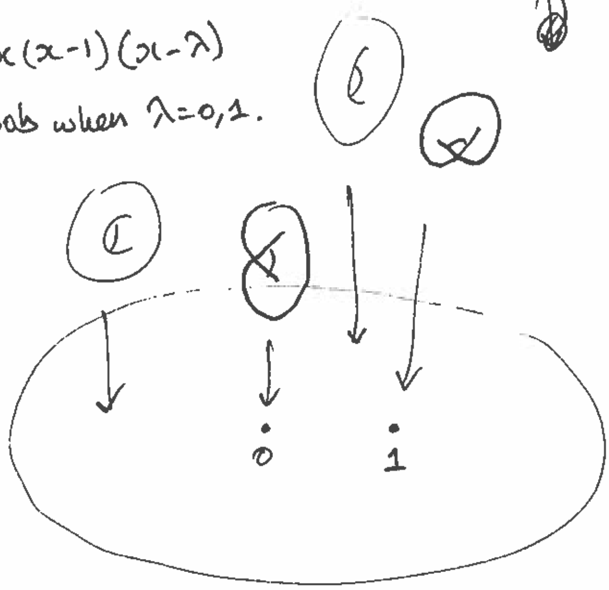
$$X = \{y^2 z = x(x-\cancel{y}z)(x-\lambda z)\} \subset \mathbb{A}^1 \times \mathbb{P}^2_{[x:y:z]}.$$

$$X \xrightarrow{f} \mathbb{A}^1$$

affine:

$y^2 = x(x-1)(x-\lambda)$

repeated roots when $\lambda = 0, 1$.



Step 1: X is not smooth! (But we can ignore this.)

Step 2:

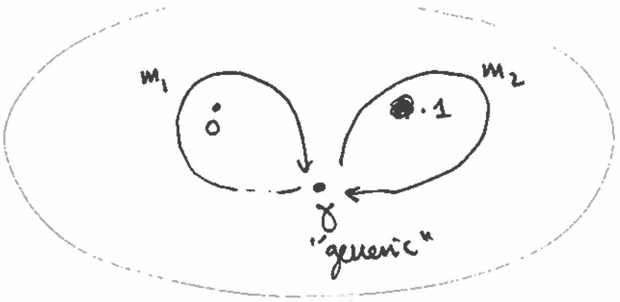
${}^p\mathcal{H}^{-1}(F) = \mathcal{H}^{0}(f_*\underline{\mathbb{Q}}) = \underline{\mathbb{Q}}_{\mathbb{A}^1}$.

${}^p\mathcal{H}^{1}(F) = \mathcal{H}^{2}(f_*\underline{\mathbb{Q}}) = \underline{\mathbb{Q}}_{\mathbb{A}^1}$.

$\eta : {}^p\mathcal{H}^{-1} \xrightarrow{\sim} {}^p\mathcal{H}^{1}$ (enough to check on open locus).

$(\Longrightarrow)$ $\qquad \mathcal{F} \cong {}^p\mathcal{H}^{-1} \oplus {}^p\mathcal{H}^{0} \oplus {}^p\mathcal{H}^{1}$.

Hence it remains to analyze ${}^p\mathcal{H}^0$ = local system of H^1.



$$ {}^p\mathcal{H}^0\big|_{\mathbb{C}\setminus\{0,1\}} = \mathbb{Q}^2 \quad m_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad m_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}. $$

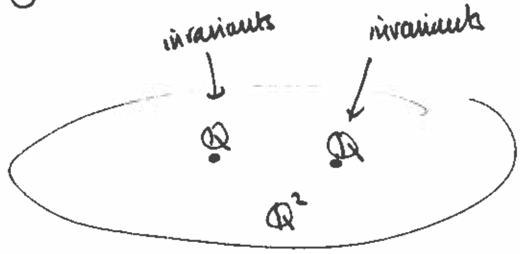
(Dehn twists)^2.

$F_1 = f^{-1}(1)$, $F_0 = f^{-1}(0)$.

<u>Claim:</u>
$$ \left.\begin{aligned} H^1(F_1) &= H^1(F_\gamma)^{(\text{invariants under } m_1)} \\ H^1(F_0) &= H^1(F_\gamma)^{(\text{invariants under } m_2)}. \end{aligned}\right\} \Leftarrow \text{"local invariant cycle theorem!"} $$

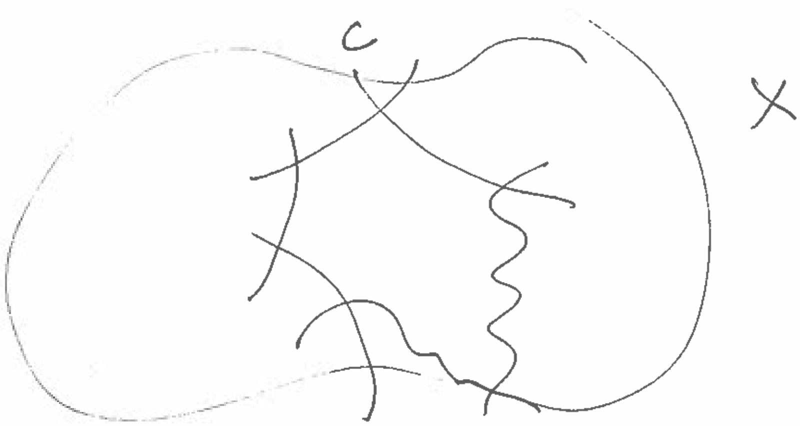
$\Longrightarrow$ $\mathcal{L} = {}^p\mathcal{H}^0|_U$ is simple, but for global reasons.

$IC(\mathbb{C}, \mathcal{L}) =$



<u>Example 2:</u> Resolution of a surface singularity.

Let X be a smooth surface and $C \subset X$ a connected collection of curves.



draw a picture! of the contraction.

Grauert's criterion:

$\exists\ f : X \to Y$ algebraic "contracting C" (i.e. $f(C) = pt$ and $f : X \setminus C \to Y \setminus f(C)$ is an isomorphism.) $\iff$ intersection form on $H_2^{BM}(C)$ induced by $C \subset S$ is negative definite.

Ex: We can contract $\mathbb{P}^1 \cong C \subset X \iff \underset{\substack{\uparrow \\ \text{normal bundle}}}{N_{\mathbb{P}^1/X}} \cong \mathcal{O}(n)$ for $n < 0$.

Now: assume we can contract C:

stalks of $f_* \underline{\mathbb{Q}}_X[2]$: set $\{y_0\} = f(C)$

	-2	-1	0
$\cancel{X}\, Y - y_0$	$\underline{\mathbb{Q}}_{Y-y_0}$	0	0
y_0	$H^0(C)$	$H^1(C)$	$H^2(C)$

$$\mathbb{D}(f_* \underline{\mathbb{Q}}_X[2]) \cong f_! \mathbb{D} \underline{\mathbb{Q}}_X[2] \cong f_* \underline{\mathbb{Q}}_X[2].$$

$i^* f_* \underline{\mathbb{Q}}_X[2] \cong f_! i^* \underline{\mathbb{Q}}$ $\implies$ $f_* \underline{\mathbb{Q}}_X[2]$ is perverse.

Hence here, parts 1) and 2) of the decomposition theorem are devoid of content. Despite its simple appearance, understanding maps for which $f_* \underline{\mathbb{Q}}_X(d_X)$ is perverse (so-called "semi-small" maps) ~~p~~provides the key to the decomposition theorem. in the approach of de C-M.

(Deligne's theorem is ~~furthest~~ far away from the base case!)

By considering supports only $\rightsquigarrow$ $IC(Y)$ occurs.

Only $IC(y_0) = \underline{\mathbb{Q}}_{y_0}$ can occur otherwise.

How do we decide how often it occurs:

<u>Lemma:</u> Let I be an indecomposable object in an additive category $/k$ satisfying the Krull-Schmidt theorem. Assume $\mathrm{End}(I)=k$. Then the multiplicity of I as a direct summand of X is equal to the rank of

$$\mathrm{Hom}(\cancel{X\otimes} I, \overset{X}{\cancel{\mathbb{Q}}}) \times \mathrm{Hom}(X, I) \longrightarrow \mathrm{End}(I) = k.$$

Hence:

$$B: \quad \mathrm{Hom}(IC(y_0), f_* \underline{\mathbb{Q}}_X[2]) \times \mathrm{Hom}(f_* \underline{\mathbb{Q}}_X[2], IC(y_0)) \longrightarrow \mathbb{Q}.$$

<u>Exercise:</u> Use adjunctions to identify $\quad \mathbin{/\!/} \quad \mathbin{/\!/} \quad H_2^{BM}(C)$.

<u>Fact:</u> B is the intersection form induced by the inclusion $C \subset X$.

$\rightsquigarrow \quad IC(y_0)$ occurs $H_2^{BM}(C)$ times in $f_* \underline{\mathbb{Q}}_X[2]$.

$$\rightsquigarrow \quad f_* \underline{\mathbb{Q}}_X[2] \cong IC(Y) \oplus H_2^{BM}(C) \otimes IC(y_0).$$

<u>Third example:</u>

$$\begin{tikzcd} \text{Hitchin system for } G \arrow[dr, "\pi"] & & \text{Hitchin system for } G^\vee \arrow[dl, "\pi^\vee"'] \\ & \text{Hitchin Base} & \end{tikzcd}$$

NGO: understand $\pi_* \mathbb{Q}$ $\pi^\vee_* \mathbb{Q}$ over a nice locus

$\Downarrow$

fundamental lemma.