Today: de Cataldo and Migliorini's proof of the decomposition theorem (sketch)

Recall the main proponents of this course:

$$\begin{array}{ccc} \text{Poincaré duality,} & \text{weak Lefschetz,} & \text{hard Lefschetz.} \\ \Downarrow & \Downarrow & \Downarrow \\ D^b_c(X), \mathbb{D} & \text{Perverse sheaves} & \text{DT} \end{array}$$

Today we add signs, in the form of the Hodge-Riemann bilinear relations.

Let X be smooth projective with Lefschetz operator $L = (\omega \cdot)$, $\omega \in H^2(X, \mathbb{R})$.

Recall the Hodge decomposition

$$H^{\cdot}(X, \mathbb{C}) = \bigoplus H^{p,q}(X, \mathbb{C}).$$

$H^{p,q}(X, \mathbb{C}) =$ forms which can be written locally in p holomorphic and q anti-holomorphic coordinates. CAN PROB OMIT

$$\overline{H^{p,q}} = H^{q,p}.$$

Because ω is of type $(1,1)$,

$$L(H^{p,q}) \subset H^{p+1,q+1}.$$

i.e. "columns of Hodge diamond are sl_2 subreps".

Eg: X K3:

$$\begin{array}{ll} 4 & \mathbb{C} \\ & 0 \\ 2 & \mathbb{C} = H^{0,2} \oplus \underset{\substack{\| \\ H^{1,1}}}{\mathbb{C}^{20}} \oplus \mathbb{C} = H^{2,0} \\ & 0 \\ 0 & \mathbb{C} \end{array}$$

Hence if we set

$$H^{p,q}_{prim} := \ker L^{n-p-q+1} \subset H^{p,q}$$

then we have

$$H = \bigoplus_{p,q} \underbrace{\bigoplus_{n-p-q \geq m \geq 0} L^m (H^{p,q}_{prim})}_{sl_2 \text{ isotypic component.}} \qquad (*)$$

Consider the form $Q_k(\alpha, \beta) := \int_X \omega^k \wedge \alpha \wedge \beta$ on $H^k(X; \mathbb{C})$.

~~Then Q_k is~~ Q_k non-degenerate $\iff$ hard Lefschetz.

Q_k is alternating for k ~~even~~ odd, symmetric for k even.

and $\overline{Q_k(\alpha, \beta)} = Q_k(\bar{\alpha}, \bar{\beta})$.

Lemma: The decomposition of H induced by (*) is orthogonal with respect to Q_k.

If we set $B_k(\alpha,\beta) := i^k Q(\alpha,\overline{\beta})$ then B_k is Hermitian (i.e. non-degenerate, sesqui-linear and $B_k(\alpha,\beta) = \overline{B_k(\beta,\alpha)}$.)

Hodge-Riemann bilinear relations:

The restriction of B_k to $H^{p,q}_{prim}$ is $(-1)^{\binom{k}{2}} i^{-2q}$ - definite.

Eg: if z is a holomorphic ~~coo~~ coordinate on a complex curve C then a $(1,0)$-form α looks locally like $\gamma(z)dz$ and we compute:

$$B_1(\alpha,\alpha) = i\int_C \gamma(z)\overline{\gamma(z)}\, dz\, d\overline{z} = 2\int_C \|\gamma(z)\|\, dx\wedge dy > 0. \qquad dz\,d\overline{z} = -2i\, dx\wedge dy.$$

Eg: Assume $H^{p,q}=0$ unless $\underbrace{p=q}_{\text{"everything on one column"}}$ "generated by algebraic cycles". then $H^{odd}=0$ and ~~then~~ the factor is $(-1)^{\binom{k}{2}} i^{-k} \Rightarrow Q_k$ is $(-1)^{k/2}$-definite on primitive classes in degree k.

Exercise: 1) Use the Hodge-Riemann relations to show that the signature of a smooth projective variety is determined by its Hodge numbers. Check your formula for a K3 surface (signature $(19,3)$).

2) Let $f: X \to Y$ be a smooth projective morphism of algebraic varieties. Use the Hodge-Riemann bilinear relations to deduce that the local systems $R^i f_* \underline{\mathbb{Q}}_X$ admit splittings, with each summand carrying a definite form. Deduce that each $R^i f_* \underline{\mathbb{Q}}_X$ is semi-simple.

An important trick of the trade:

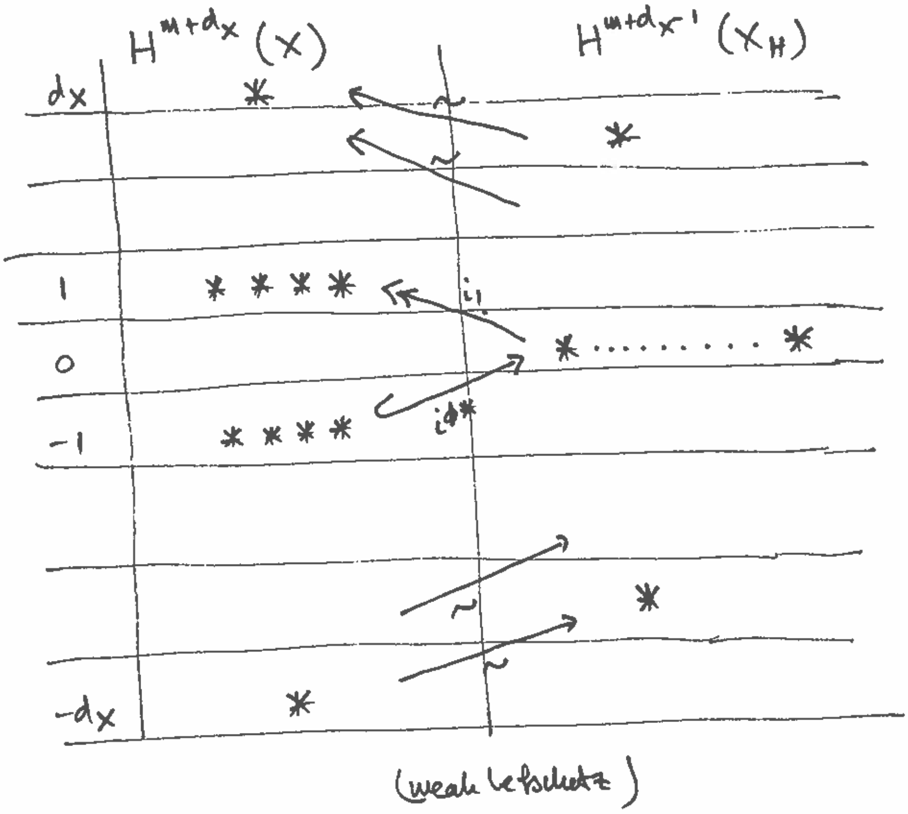
Suppose hard Lefschetz and the Hodge-Riemann bilinear relations hold in dim $n-1$. Then hard Lefschetz holds in dim n.

<u>Idea:</u> Let X be smooth projective and $X_H \subset X$ smooth hyperplane section, $i : X_H \hookrightarrow X$ the inclusion. We have

$i^* : H^m(X) \longrightarrow H^m(X_H)$ restriction

$i_! : H^m(X_H) \longrightarrow H^{m+2}(X)$ adjoint

<u>Fact:</u> $i_! i^* =$ Lefschetz operator.



hard Lefschetz for $H(X_H)$

$\Downarrow$

hard Lefschetz for $H(X)$ except for "most difficult case"

$H^{d_X-1}(X) \longrightarrow H^{d_X+1}(X)$.

hard Lefschetz for $H^{d_X-1}(X) \to H^{d_X+1}(X)$ $\Longleftrightarrow$ form $Q_{d_X-1}(\alpha,\beta) = \langle \alpha, L\beta \rangle = \langle \alpha, i_! i^* \beta \rangle$ non-degenerate.

$\Longleftrightarrow$ restriction of $\langle -,- \rangle$ to $i^* H^{m-d_X}(X)$ is non-degenerate.

Exercise: This is a consequence of the Hodge-Riemann bilinear relations on $H^\bullet(X_H)$.

Rmk: The induction doesn't close because one can't control the signature on $\ker L \subset H^{d_X}(X)$. This reflects the eternal difficulty in complex algebraic geometry!

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dCM's proof in the semi-small case:

Def: A map $f : X \to Y$ is semi-small if it is surjective and all components of $X \times_Y X$ have dimension bounded by $\dim_{\mathbb{C}} X$.

Exercise: $f : X \to Y$ is semi-small if and only if there exists a stratification $\bigsqcup_{\mu \in \Delta} Y_\mu$ of Y into smooth locally closed subsets such that $f : f^{-1}(Y_\mu) \to Y_\mu$ is a smooth for all $\mu \in \Delta$, and such that, for all $y \in Y_y$

$$\dim F_\mu \leq \frac{1}{2} \operatorname{codim} (F_\mu \subset Y).$$

Ex: • Any semi-small map is generically finite.

• Any resolution of a surface singularity is semi-small.

• Any symplectic resolution is semi-small (e.g. Springer resolution, Hilbert scheme of points on a surface).

Exercise: $f : X \to Y$ surjective ~~semi-small~~, X smooth, f proper.

f semi-small $\iff$ $f_* \underline{\mathbb{Q}}_X(d_X)$ is perverse.

<u>Example</u>: (to convince you that semi-small maps are interesting).

X complex surface.

$X^{[n]}$ its Hilbert scheme of n-points (smooth).

$\pi : X^{[n]} \longrightarrow S^n X$ Hilbert-Chow morphism.

Given $\lambda \vdash n$ a partition consider: $S^\lambda X = \Delta^{\lambda_1} X \times \ldots \times \Delta^{\lambda_m} X \subset X^n$ image of in $S^n X$.

Set $\mathcal{O}_\lambda = S^\lambda X - \bigsqcup_{\mu \trianglerighteq \lambda} S^\mu X$. Then

$$S^n X = \bigsqcup_{\lambda \vdash n} \mathcal{O}_\lambda \quad \text{and} \quad \pi \text{ is smooth over each } \mathcal{O}_\lambda.$$

<u>Exercise</u>: 1) $\dim \mathcal{O}_\lambda = 2\,\ell(\lambda)$ where $\ell(\lambda)$ denotes the number of parts of λ.

2) if $x \in \mathcal{O}_\lambda$ then $\dim \pi^{-1}(x) = \Sigma(\lambda_i - 1)$ and $\pi^{-1}(x)$ is irreducible.

3) $2 \dim \pi^{-1}(x) + \dim \mathcal{O}_\lambda = 2(\Sigma \lambda_i - 1) + 2\,\ell(\lambda) = 2|\lambda| = 2n = \dim S^n X$.

Hence π is semi-small.

4) Each $\overline{\mathcal{O}_\lambda}^{= S^\lambda X}$ has finite quotient singularities and hence

$$IC(\overline{\mathcal{O}_\lambda}) = \underline{\mathbb{Q}}_{\overline{\mathcal{O}_\lambda}}[2\,\ell(\lambda)].$$

3) DT $\Rightarrow$

$$\pi_* \underline{\mathbb{Q}}_{X^{[n]}}(2n) = \bigoplus_{\lambda \vdash n} IC(S^\lambda X)$$

(cf. Göttsche-Soergel).

$$\Rightarrow \quad H^\cdot(X^{[n]}) = \bigoplus_{\lambda \vdash n} \overbrace{H^{\cdot + 2\ell(\lambda)}(S^\lambda X)}^{\text{easily computed}}.$$

For a semi-small map the DT takes the form

$$\underline{\text{DT:}} \qquad f_* \underline{\mathbb{Q}}_X [d_X] = \bigoplus_{\mu \in \Lambda} IC(\overline{X}_\mu, \mathcal{L}_\mu)$$

where $\mathcal{L}_\mu =$

"top" under semi-small assumption.

sheafification of $U \mapsto H^{\frac{1}{2}(\mathrm{codim}(X_\mu \subset Y))}(\pi^{-1}(U))$.

+ resolution of singularities

The perverse sheaf formalism + "normal slices" reduces the question

to the following:

suppose that $f : X \longrightarrow Y$ proper, X projective, smooth,

$\{y_1, \dots, y_m\}$ the set of points with $\dim \pi^{-1}(y_i) = \frac{1}{2} \dim X$.

maximal allowed under semi-small hypothesis.
"most singular points"

Then the intersection form on $H^{BM}_{top}(\pi^{-1}(y_i))$ induced by

$\pi^{-1}(y_i) \subset X$ is non-degenerate.

Recall from last week that for resolutions of surface singularities

this was Grauert's contractibility criterion.

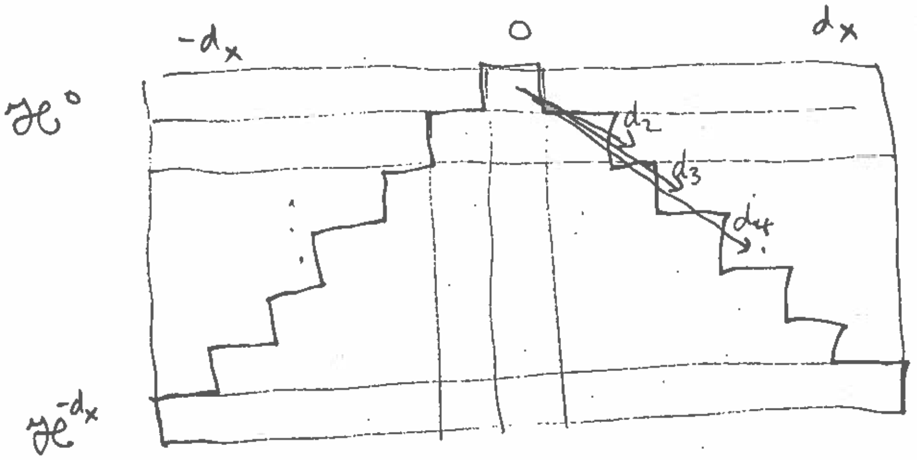
For simplicity assume $m = 1$. "unique most singular point".

Step 1: $\mathrm{cl} : H^{BM}_{d_X}(\pi^{-1}(y_1)) \longrightarrow H^n(X)$ is injective.

"classes of irreducible components are linearly independent".

Poincaré duality: enough to show $H^n(X) \xrightarrow{i^*} H^n(\pi^{-1}(y_1))$ surjective.

Leray-Serre spectral sequence for the filtration $\tau_{\le i}\pi_*\underline{\mathbb{Q}}_X[d_X]$:



$$\Longrightarrow \quad H^{d_X}(X) \longrightarrow H^{d_X}(\tau_{\ge 0}\pi_*\underline{\mathbb{Q}}_X[d_X]) \quad \text{is surjective.}$$

$$\| $$

$$H^{top}(\pi^{-1}(y)). \qquad \square$$

Step 2: cl is an isometry. (Obvious from geometry).

Step 3: cl has image contained in $H^{\frac{1}{2}d_X, \frac{1}{2}d_X}(X)_{prim}$ with respect to $\pi^*\beta$ where β is an ample class on the base.

(Because any generic hyperplane section of β avoids y_1).

Hence we would be done if we knew that β satisfies hard Lefschetz and the Hodge-Riemann bilinear relations.

This is true!

1) use weak Lefschetz trick to get hard Lefschetz for β.

2) realize β as the limit of a family of classes to get Hodge-Riemann relations for β (using that signatures can't change in families).