C2.1a Lie algebras

Mathematical Institute, University of Oxford
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Problem Sheet 1

1. Let V be a vector space. Let $\mathfrak{gl}(V)$ be the vector space of all linear maps from V to itself. Define a Lie bracket on $\mathfrak{gl}(V)$ by

$$[x, y] := xy - yx \quad \text{for all } x, y \in \mathfrak{gl}(V).$$

Show that $\mathfrak{gl}(V)$ is a Lie algebra.

2. Let A be an algebra over a field k. Recall that a derivation of A is a linear map

$$D : A \to A$$

such that $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. Show that the set of derivations $\operatorname{Der} A \subset \mathfrak{gl}(A)$ form a Lie subalgebra.

3. Let S be an $n \times n$ matrix with entries in a field k. Define

$$\mathfrak{gl}_S := \{x \in \mathfrak{gl}_n \mid x^t S + Sx = 0\}.$$

a) Show that $\mathfrak{gl}_S$ is a Lie subalgebra of $\mathfrak{gl}_n$.

b) Let J be the $n \times n$-matrix:

$$J_n = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}$$

Now let S be the $2n \times 2n$ matrix:

$$\begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}.$$

Find conditions for a matrix to lie in $\mathfrak{gl}_S$ and hence determine the dimension of $\mathfrak{gl}_S$.

4. Let k be a field and $\mathfrak{g} = \mathfrak{gl}_n(k)$. Let $x \in \mathfrak{gl}_n(k)$ be a diagonal matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. By describing a basis of eigenvectors for $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ show that $\operatorname{ad} x$ is diagonalisable, with eigenvalues $\lambda_i - \lambda_j$ for $1 \le i, j \le n$.

5. a) Suppose that $\mathfrak{g}$ is a 3-dimensional Lie algebra over a field k with $\mathfrak{g}'$ of dimension 1 and $\mathfrak{g}' \subset Z(\mathfrak{g})$. Determine the structure constants of $\mathfrak{g}$ with respect to a suitable basis, and show that there is up to isomorphism a unique such algebra. (This Lie algebra is the famous *Heisenberg algebra*.) Find an isomorphism of $\mathfrak{g}$ with a Lie subalgebra of $\mathfrak{gl}_3$.

b) *(Optional harder question)* Classify up to isomorphism all Lie algebras $\mathfrak{g}$ such that $\dim \mathfrak{g}' = 1$ and $\mathfrak{g}' = Z(\mathfrak{g})$.

6. Let V be a vector space with basis $e_1, \dots, e_n$. Let $E_{i,j} : V \to V$ be the endomorphisms defined by

$$E_{i,j}(e_\ell) = \delta_{j,\ell} e_i.$$

Verify the commutation relations:

$$[E_{i,j}, E_{k,\ell}] = \delta_{j,k} E_{i,\ell} - \delta_{\ell,i} E_{k,j}.$$

Recall that $\mathfrak{sl}(V) \subset \mathfrak{gl}(V)$ denotes the subalgebra of traceless endomorphisms. Use the above relations to show that $\mathfrak{sl}(V) = \mathfrak{gl}(V)'$.

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