C2.1a Lie algebras

Mathematical Institute, University of Oxford
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Problem Sheet 2

1. Find the structure constants of $\mathfrak{sl}_2$ with respect to the basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Show that the only ideals of $\mathfrak{sl}_2(\mathbb{C})$ are 0 and itself.

2. If a Lie algebra $\mathfrak{g}$ is a vector space direct sum of two Lie subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$, then we say that $\mathfrak{g}$ is the *direct sum* of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ and write $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$.

a) Show that $\mathfrak{gl}_2(\mathbb{C})$ is the direct sum of $\mathfrak{sl}_2(\mathbb{C})$ and the subalgebra of scalar multiples of the identity matrix.

b) Show that if $\mathfrak{g}$ is the direct sum of Lie subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ then $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are ideals of $\mathfrak{g}$. Show that $Z(\mathfrak{g}) = Z(\mathfrak{g}_1) \oplus Z(\mathfrak{g}_2)$ and $\mathfrak{g}' = \mathfrak{g}_1' \oplus \mathfrak{g}_2'$.

3. Let $\mathfrak{g}$ be a Lie algebra. In lectures it was explained how to pass between a $\mathfrak{g}$-module V and a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Prove that these two procedures are inverse to one another.

4. Classify all Lie algebras $\mathfrak{g}$ with $\dim \mathfrak{g} = 3$ and $Z(\mathfrak{g}) \neq 0$. (Remember that we showed in Sheet 1 that any three dimensional Lie algebras with $\mathfrak{g}' = Z(\mathfrak{g})$ is isomorphic to the Heisenberg Lie algebra.)

5. Let $A \subset \mathfrak{gl}(V)$ denote a subspace consisting of commuting diagonalisable endomorphisms. Show that we may find a basis of V in which each element of A is represented by a diagonal matrix.

6. *(The classical groups)* In this course a fundamental role is played by the classical groups. In this question they will be defined, we will calculate their dimensions and we will look at some small dimensional examples. Assume throughout that k is a field of characteristic $\neq 2$.

a) (*The special linear group* $\mathfrak{sl}_n$) Recall that $\mathfrak{sl}_n \subset \mathfrak{gl}_n$ denotes the subspace of traceless $n \times n$-matrices. On the last sheet we saw that $\mathfrak{sl}_n = \mathfrak{gl}_n'$ and hence $\mathfrak{sl}_n$ is a Lie subalgebra. Give another proof of this fact by proving that $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$ for $n \times n$-matrices $a, b \in \mathfrak{gl}_n$. Calculate the dimension of $\mathfrak{sl}_n$.

b) (*The special orthogonal group* $\mathfrak{so}_n$) Recall the definition of $\mathfrak{gl}_S$ and J_n from Question 3 of Sheet 1. Consider the matrix

$$S = J_n.$$

We define $\mathfrak{so}_n$ to be $\mathfrak{gl}_S$. Find conditions for a matrix to belong to $\mathfrak{so}_n$ and hence calculate its dimension.

c) (*The symplectic group* $\mathfrak{sp}_{2n}$) Consider the matrix

$$S = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}.$$

We define $\mathfrak{sp}_{2n}$ to be $\mathfrak{gl}_S$. We already saw $\mathfrak{sp}_{2n}$ in the first exercise sheet, and (hopefully!) calculated its dimension to be $2n^2 + n$. Give an explicit description of $\mathfrak{sp}_2$ in terms of another Lie algebra occuring on the list above.

d) Show that $\mathfrak{so}_2$ is abelian and that $\mathfrak{sl}_2 \cong \mathfrak{so}_3$

e) (*Harder ... but have a go!*) Show that $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$ and that $\mathfrak{sp}_4 \cong \mathfrak{so}_5$. (*Hint:* Let $\mathfrak{n}$ and $\mathfrak{b}$ denote the strictly upper triangular and upper triangular matrices in $\mathfrak{gl}_4$. First try to understand the structure of $\mathfrak{so}_4 \cap \mathfrak{n}$ and $\mathfrak{sp}_4 \cap \mathfrak{n}$. Then proceed to $\mathfrak{so}_4 \cap \mathfrak{b}$ and $\mathfrak{sp}_4 \cap \mathfrak{b}$. From here it should be clearer how to construct the desired isomorphism.)

One may show that, as well as the list above, there is only one more "exceptional isomorphism" between the classical Lie algebras. If you are feeling courageous you might like to guess what it is (Hint: it involves $\mathfrak{so}_6$). We will return to this question later on in the course.

Geordie Williamson, 15th October 2010