B2b Lie algebras

Mathematical Institute, University of Oxford
Michaelmas Term 2010

Problem Sheet 3

1. Let $\mathfrak{g}$ be a Lie algebra. Suppose that the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is irreducible. What can you say about $\mathfrak{g}$?

2. a) Let k be a field of characteristic 2 and let $\mathfrak{g} = \mathfrak{sl}_2(k)$. Show that $\mathfrak{g}$ is solvable (and even nilpotent) but that the natural representation of $\mathfrak{g}$ is irreducible. Conclude that Lie's theorem is not true in positive characteristic.

b) Let $\mathbb{C}[x]$ denote a polynomial ring in x, and consider the Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{C}[x])$ generated by the endomorphisms given by multiplication by x and $\frac{d}{dx}$. Show that $\mathfrak{g}$ is a three dimensional nilpotent Lie algebra, isomorphic to the Heisenberg algebra. Does $\mathfrak{g}$ fix a line in $\mathbb{C}[x]$? Why doesn't this contradict Lie's theorem?

3. Prove Schur's lemma: if V is a simple finite dimensional $\mathfrak{g}$-module over an algebraically closed field and $\phi \in \mathrm{End}_{\mathfrak{g}}(V)$ is an endomorphism of V commuting with the $\mathfrak{g}$-action, then ϕ is a scalar. (*Hint:* Consider the eigenspaces of ϕ.)

4. Show that $\mathfrak{sl}_n(\mathbb{C})$ is simple. (*Hint:* It might be easier show that $\mathfrak{gl}_n(\mathbb{C})$ has no non-trivial ideals contained in $\mathfrak{sl}_n$.)

5. Let $\mathfrak{g}$ be the set of complex matrices of the form $\begin{pmatrix} \alpha & \beta & \lambda \\ \gamma & \delta & \mu \\ 0 & 0 & 0 \end{pmatrix}$ where $\alpha + \delta = 0$. Show that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_3(\mathbb{C})$. Find the radical of $\mathfrak{g}$ and show that $\mathfrak{g}$ contains a subalgebra isomorphic to $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$. Prove that the only ideal of $\mathfrak{g}$ strictly contained in $\operatorname{rad}\mathfrak{g}$ is $\{0\}$.

6. Let $\mathbb{H}$ denote the associative four dimensional algebra over $\mathbb{R}$ with basis e, i, j and k and multiplication given by $i^2 = j^2 = k^2 = -e$, $ij = k$, $jk = i$, $ki = j$ and requiring that e is the identity. The algebra $\mathbb{H}$ is called the algebra of *quarternions*. Determine $\mathrm{Der}\,\mathbb{H}$, the Lie algebra of derivations of $\mathbb{H}$. Can you identifiy $\mathrm{Der}\,\mathbb{H}$ with a classical Lie algebra?

7. *(Optional question, requires some differential geometry.)* Let A be a finite dimensional algebra over $\mathbb{R}$ or $\mathbb{C}$. Let g_t denote a family of automorphisms of A parametrised by $t \in (\varepsilon, -\varepsilon)$ with $g_0 = \mathrm{id}_A$. Assume furthermore that g_t is differentiable in t.

a) Let D denote the derivative of g_t at $t = 0$. Show that D is a derivation of A. (You may assume that g is analytic in t if you wish.)

b) Conversely, if D is a derivation of A, show that

$$g_t := e^{tD} : \mathbb{R} \to \mathfrak{gl}(A)$$

is a one-parameter family of automorphisms of A.

Geordie Williamson, 22nd October 2010