C2.1a Lie algebras

Mathematical Institute, University of Oxford
Michaelmas Term 2010

Problem Sheet 5

Please be sure to attempt Question 4!

1. Let V be a Euclidean vector space, and let F be a subspace with orthonormal basis $v_1, \dots, v_k$. Given any vector $v \in V$ with $(v, v) = 1$ show that the square of the distance from v to F is given by

$$1 - \sum_{i=1}^{k} (v_i, v)^2.$$

2. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Show that the Killing form on $\mathfrak{g}$ is identically zero.

3. Let κ denote the Killing form on $\mathfrak{gl}_n(\mathbb{C})$ and let $\mathfrak{h}$, $\mathfrak{n}_+$ and $\mathfrak{n}_-$ denote the subspaces of diagonal, strictly upper triangular and strictly lower triangular matrices respectively.

a) Show that $\mathfrak{h}$ is orthogonal to $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ and that the resriction of κ to $\mathfrak{n}_+ \oplus \mathfrak{n}_-$ is non-degenerate. (*Hint:* It might help to give a formula for κ in terms of matrix units.)

b) Calculate $\mathfrak{n}_+^\perp$.

c) Describe the radical of the restriction of κ to $\mathfrak{h}$, and conclude that the restriction of κ to $\mathfrak{sl}_n(\mathbb{C})$ is non-degenerate.

(We have seen the week before last that $\mathfrak{sl}_n(\mathbb{C})$ is simple. Hence part c) also follows from the Cartan-Killing criterion.)

4. Let V be a finite dimensional real vector space and $R \subset V$ a root system. Fix two roots $\alpha, \beta \in R$. In this exercise we analyse the possible angles between α and β.

Recall from lectures that we may equip V with a positive definite bilinear form $(-, -)$ such that each reflection s_α is orthogonal with respect to this form. We also saw that we have

$$\langle \alpha^\vee, \beta \rangle = 2\frac{(\alpha, \beta)}{(\beta, \beta)}. \qquad (*)$$

a) Let ϕ denote the angle between α and β. Show the relation

$$\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle = 4 \cos^2 \phi$$

Conclude that $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle \in \{0, 1, 2, 3, 4\}$. (*Hint:* Use (*) together with the fact that, in a Euclidean space, we have $(v, w) = |v||w| \cos \tau$, where τ denotes the angle between v and w.)

b) Show that, if α and β are not proportional, we have

$$\phi \in \{\pi/2, \pi/3, 2\pi/3, \pi/4, 3\pi/4, \pi/6, 5\pi/6\}.$$

What can be said about the ratios of the lengths of α and β in each case?

c) Describe $\mathbb{R}\alpha \oplus \mathbb{R}\beta \cap R$ and deduce that any rank 2 root system is isomorphic to $A_1 \times A_1$, A_2, B_2 or G_2 (as claimed in lectures).

d) Show that $\langle \alpha^\vee, \beta \rangle > 0$ then $\beta - \alpha$ is a root.

5. Let $R \subset V$ be a root system. Show that the set $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ is a root system in V^*. (We call $R^\vee$ the *dual root system* to R).

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