C2.1a Lie algebras

Mathematical Institute, University of Oxford
Michaelmas Term 2010

Problem Sheet 6

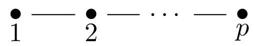
1. Let $V = \oplus_{i=1}^n \mathbb{R}e_i$ and $\widetilde{V} = \{v = \sum \lambda_i e_i \mid \sum \lambda_i = 0\}$. Consider $R = \{e_i - e_j \mid i \neq j\} \subset \widetilde{V}$.

a) Show that $R \subset \widetilde{V}$ is a root system and that the Weyl group W of $R \subset \widetilde{V}$ is isomorphic to the symmetric group S_n.

c) Show that $S = \{e_i - e_{i+1}\}$ for $1 \leq i \leq n-1$ is a basis for R. Hence compute the Coxeter graph of R and conclude that $R \subset \widetilde{V}$ is a root system of type A_{n-1}.

2. In this question we complete the classification of all "simply-laced" Dynkin diagrams.

a) Consider the Coxeter graph



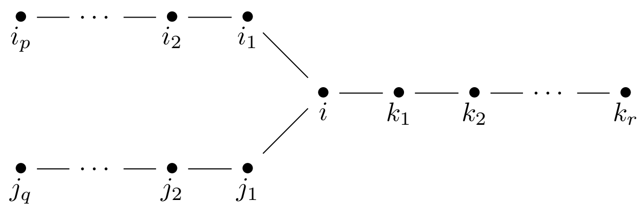
and let V be the corresponding vector space with bilinear form B. That is, V is a real vector space with basis $\{e_i\}_{i=1}^p$ and bilinear form B given by

$$B(e_i, e_j) = \begin{cases} 1 & \text{if } i = j, \\ -\frac{1}{2} & \text{if } j = i \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the element $v = \sum_{i=1}^p i e_i$. Show that

$$B(v, v) = \frac{p(p+1)}{2}.$$

b) Now let Γ be the Coxeter graph



And let (V, B) be the corresponding vector space with symmetric bilinear form B. Define

$$x = e_{i_p} + 2e_{i_{p-1}} + \cdots + pe_{i_1}$$
$$y = e_{j_q} + 2e_{j_{q-1}} + \cdots + qe_{j_1}$$
$$z = e_{j_q} + 2e_{j_{q-1}} + \cdots + qe_{j_1}$$

By considering the distance from e_i to the subspace spanned by x, y and z show that, if B is positive definite, then

$$(p+1)^{-1} + (q+1)^{-1} + (r+1)^{-1} > 1.$$

(*Hint:* Combine the first part of this question with Question 1 of Sheet 5.)

c) Conclude that (p, q, r) is equal (up to permutation) to $(m, 1, 1)$ with $m \geq 0$ or $(1, 2, 2)$, $(1, 2, 3)$ or $(1, 2, 4)$.

d) Conclude that the only possible Coxeter graphs of the form Γ are of type D_m for $m \geq 4$ or E_6, E_7 or E_8.

3. Consider the set $R \subset \bigoplus_{i=1}^n \mathbb{R}e_i$ consisting of vectors $\pm e_i$ and $\pm e_i \pm e_j$ for $i \neq j$.

a) Show that R is a root system. Can you describe the Weyl group of R?

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b) Find a basis for R and conclude that R is a root system of type B_n.

c) Compute the dual root system of R. How does it compare to R? (For the definition of dual root system see Question 5 on Sheet 5.)

4. Consider the set $R \subset \bigoplus_{i=1}^{4} \mathbb{R}e_i$ consisting of the vectors $\pm e_i$, $\pm e_i \pm e_j$ (for $i \neq j$) and $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$.

a) Show that R is a root system. (*Hint:* The formula $s_\alpha s_b s_\alpha = s_{s_\alpha \beta}$ can be used to reduce the number of necessary calculations.)

b) Find a basis for R. Conclude that R is a root system of type F_4.