C2.1a Lie algebras

Mathematical Institute, University of Oxford
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Problem Sheet 7

1. Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and $R \subset \mathfrak{h}^*$ the corresponding root system. Find an expression for the dimension of $\mathfrak{g}$ in terms of R.

2. In this exercise we calculate the root systems of the classical Lie algebras.

i) Show that the diagonal matrices $\mathfrak{h} \subset \mathfrak{sl}_n(\mathbb{C})$ form a Cartan subalgebra. Hence describe the roots $R \subset \mathfrak{h}^*$ and conclude that $\mathfrak{sl}_n(\mathbb{C})$ is a simple Lie algebra of type A_{n-1}.

ii) Find a Cartan subalgebra, and calculate the corrsponding root system for $\mathfrak{g} = \mathfrak{so}_{2n+1}$, $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $\mathfrak{g} = \mathfrak{so}_{2n}$. (*Hint:* Use the definitions of $\mathfrak{so}_n$ and $\mathfrak{sp}_{2n}$ given in Problem Sheet 2, Question 6.)

iii) Calculate the number of roots in the root systems obtained in i) and ii).

iv) In Problem Sheet 2, Question 6 we calculated the dimensions of $\mathfrak{sl}_n$, $\mathfrak{so}_n$, and $\mathfrak{g} = \mathfrak{sp}_{2n}$. Check that your formula obtained in Question 1 gives the same answer in each case.

3. Let V be a real vector space. A *lattice* is a discreet subgroup $Q \subset V$ which spans V over $\mathbb{R}$. Equivalently, a lattice is a subgroup $Q \subset V$ of the form

$$\left\{ \sum \lambda_i \beta_i \;\middle|\; \lambda_i \in \mathbb{Z} \right\}$$

where $\{\beta_i\}_{i=1}^n$ is a basis of V.

Assume that V is equipped with a positive definite inner product $(-,-)$. A lattice $Q \subset V$ is called *integral* if $(\alpha, \beta) \in \mathbb{Z}$ for all $\alpha, \beta \in Q$. A lattice $Q \subset V$ is called *even* if $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in Q$.

i) Prove that an even lattice is integral.

ii) Let $Q \subset V$ be an even lattice. Assume that the set $R_Q = \{\alpha \in V \mid (\alpha, \alpha) = 2\}$ spans V. Show that R_Q is a root system in V.

iii) Let $V = \bigoplus_{i=1}^r \mathbb{R}e_i$ equipped with the standard inner product $(e_i, e_j) = \delta_{ij}$. Consider

$$\Gamma_r = \left\{ \sum a_i e_i \;\middle|\; \begin{array}{c} \text{either all } a_i \in \mathbb{Z} \\ \text{or all } a_i \in \mathbb{Z} + \frac{1}{2} \end{array} \text{ and } \sum a_i \in 2\mathbb{Z} \right\}.$$

Prove that Γ_r is an even lattice if r is divisible by 8.

iv) Consider $\Gamma = \Gamma_8 \subset V = \mathbb{R}^8$. Show that $V \in \Gamma$ is spanned by vectors such that $(v, v) = 2$. Describe the roots in R_Γ.

v) Find a basis for R_Γ and conclude that R_Γ is a root system of type E_8. (*Hint:* Consider the functional $t \in V^*$ given by

$$t = \sum_{i=1}^{7} (i-1)e_i^* + 23e_8^*.$$

where e_i^* is the dual basis of $\{e_i\}_{i=1}^n$ (i.e. $\langle e_i^*, e_j \rangle = \delta_{ij}$ for all i, j). Show that $0 \neq \langle t, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in R_\Gamma$. Calculate the set of roots $\alpha \in R_\Gamma$ with $\langle t, \alpha \rangle = 1$ and use results from lectures.)

v) Let H_7 be the hyperplane orthogonal to $e_7 + e_8$. Show that $R \cap H_7$ is a root system of type E_7.

vi) Let $H_6 \subset R$ be the subspace orthogonal to $e_6 + e_7 + 2e_8$ and $e_7 + e_8$. Show that $R \cap H_6$ is a root system of type E_6.

We have now constructed all the irreducible root systems!

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