Given any Coxeter group (W, S) we can produce a coloured simplicial complex whose automorphisms are precisely W. This complex is called the *Coxeter complex* and will be denoted $|(W, S)|$.

Let $n = |S|$ denote the rank of W. Its construction is as follows:

- colour the n faces of the $n-1$-simplex Δ by the set S,
- take one such simplex Δ_w for each element $w \in W$,
- glue Δ_w to Δ_{ws} along the wall coloured by s.

For example, consider the symmetric group on three letters:

$$W = \langle s, t \mid s^2 = t^2 = (st)^3 \rangle = \{e, s, t, st, ts, sts\}.$$

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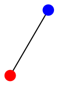

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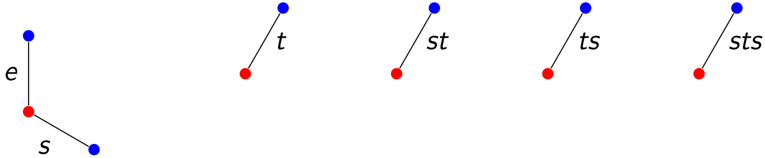

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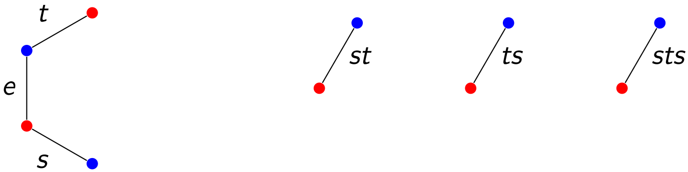

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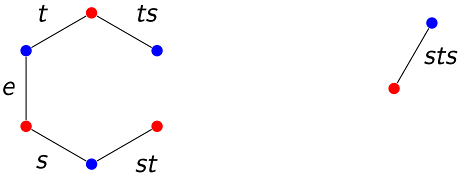

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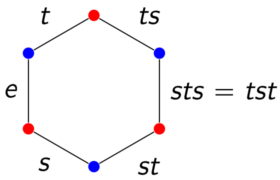
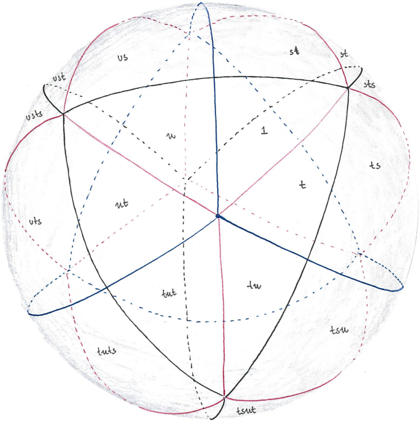
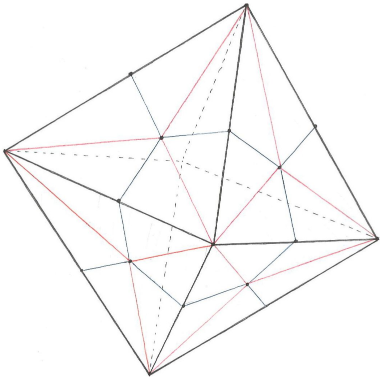


The Coxeter complex of $S_4 =$  :

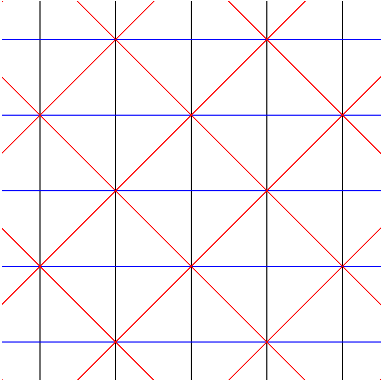


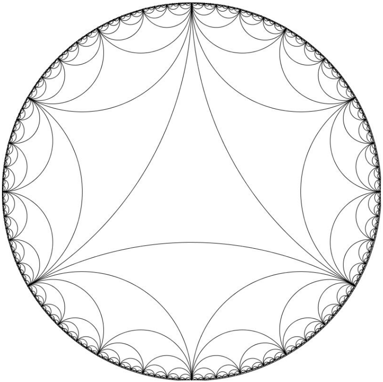
(barycentric subdivision of the tetrahedron).




4

$$s \overset{4}{\text{———}} t \overset{4}{\text{———}} u$$





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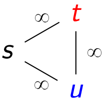
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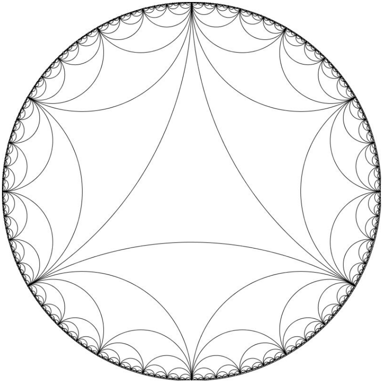

;

Let $\ell : W \to \mathbb{N}$ denote the length function on W. It is easy to describe the length function using the Coxeter complex:

$$\begin{aligned}\ell(w) &= \text{length of a minimal expression for } w \text{ in the generators } s \\ &= \text{number of walls crossed in a minimal path } id \to w \text{ in } |(W,S)|.\end{aligned}$$

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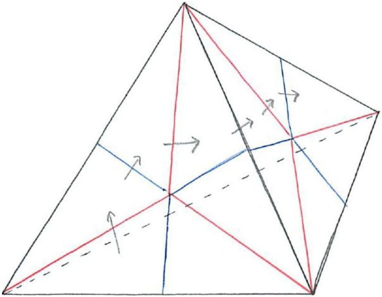

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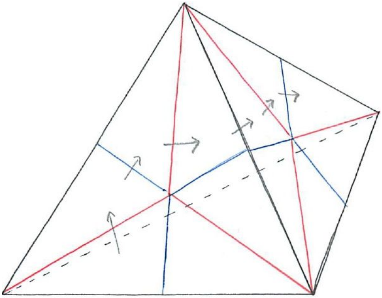

The Bruhat order is trickier...

By construction $|(W,S)|$ has a left action of W.

W also acts on the alcoves of $|(W,S)|$ on the right by

$$\Delta_w \cdot s = \Delta_{ws}.$$

This action is *not* simplicial, but is "local": cross the wall coloured by s.

Using the Coxeter complex makes it easy to visualize elements of the Hecke algebra **H**.

We view an element $f = \sum f_x H_x$ as the assignment of $f_x \in \mathbb{Z}[v^{\pm 1}]$ to the alcove indexed by $x \in W$.

Recall the Kazhdan-Lusztig generator $\underline{H}_s := H_s + vH_{id}$. The formulas for the action of $\underline{H}_s$ on the standard basis can be rewritten

$$H_x \underline{H}_s = \begin{cases} H_{xs} + vH_x & \text{if } \ell(xs) > \ell(x), \\ H_{xs} + v^{-1} H_x & \text{if } \ell(xs) < \ell(x). \end{cases}$$

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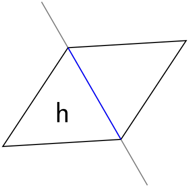
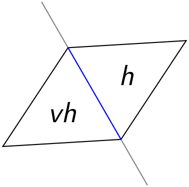
We can visualise this as follows: ("quantized averaging operator")

id



$\cdot \underline{H}_s =$



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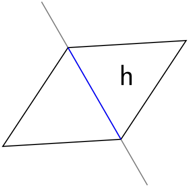

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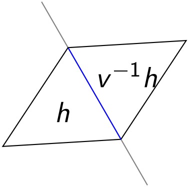
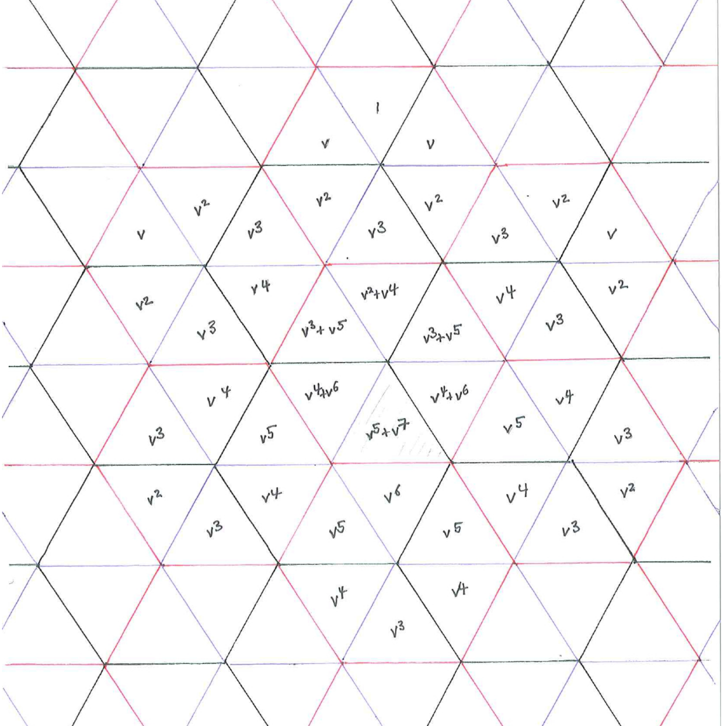


Recall that the Kazhdan and Lusztig basis has the form

$$\underline{H}_x := H_x + \sum_{y<x} h_{y,x} H_y$$

with $h_{y,x} \in v\mathbb{Z}[v]$ and satisfies $\overline{\underline{H}_x} = \underline{H}_x$.

The polynomials $h_{y,x}$ are the *Kazhdan-Lusztig polynomials*.



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We want to use the Coxeter complex to understand how to calculate the Kazhdan-Lusztig basis. The first few Kazhdan-Lusztig basis elements are easily defined:

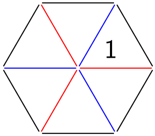
$$\underline{H}_{id} := H_{id}, \quad \underline{H}_s := H_s + vH_{id} \quad \text{for } s \in S.$$

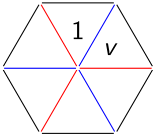
Now the work begins. Suppose that we have calculated $\underline{H}_y$ for all y with $\ell(y) \leqslant \ell(x)$. Choose $s \in S$ with $\ell(xs) > \ell(x)$ and write

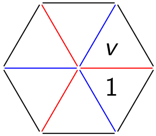
$$\underline{H}_x \underline{H}_s = H_{xs} + \sum_{\ell(y) < \ell(xs)} g_y H_y.$$

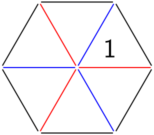
The formula for the action of $\underline{H}_s$ shows that $g_y \in \mathbb{Z}[v]$ for all $y < \ell(xs)$. If all $g_y \in v\mathbb{Z}[v]$ then $\underline{H}_{xs} := \underline{H}_x \underline{H}_s$. Otherwise we set

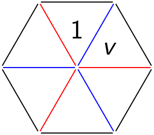
$$\underline{H}_{xs} = \underline{H}_x \underline{H}_s - \sum_{\substack{y \\ \ell(y) < \ell(x)}} g_y(0) \underline{H}_y.$$

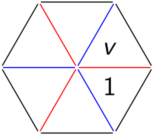
$\underline{H}_{id} =$ 

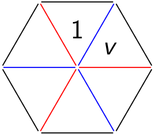
$\underline{H}_{t} =$ 

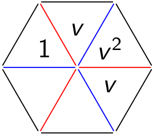
$\underline{H}_{s} =$ 

$\underline{H}_{id} =$ 

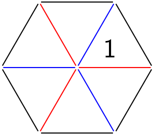
$\underline{H}_{t} =$ 

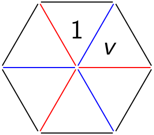
$\underline{H}_{s} =$ 

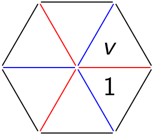
$\underline{H}_{t}\underline{H}_{s} =$ 

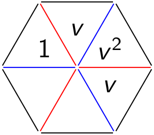
$\cdot\, \underline{H}_{s} =$ 

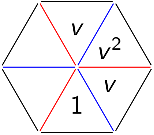
$= \underline{H}_{ts}$

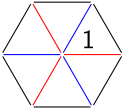
$\underline{H}_{id} =$ 

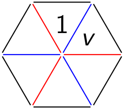
$\underline{H}_{t} =$ 

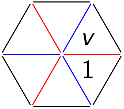
$\underline{H}_{s} =$ 

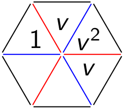
$\underline{H}_{ts} =$ 

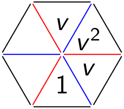
$\underline{H}_{st} =$ 

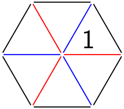
$\underline{H}_{id} =$ 

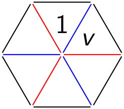
$\underline{H}_{t} =$ 

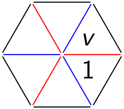
$\underline{H}_{s} =$ 

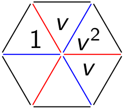
$\underline{H}_{ts} =$ 

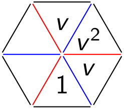
$\underline{H}_{st} =$ 

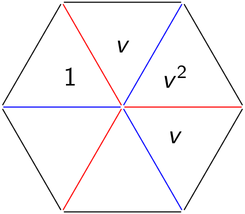
$\underline{H}_{id} =$ 

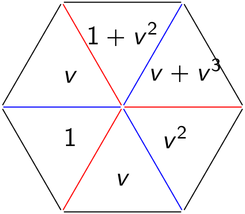
$\underline{H}_{t} =$ 

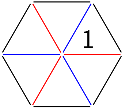
$\underline{H}_{s} =$ 

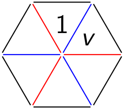
$\underline{H}_{ts} =$ 

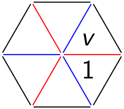
$\underline{H}_{st} =$ 

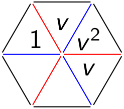
$\underline{H}_{ts}\underline{H}_{t} =$ 

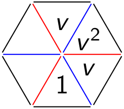
$\cdot\ \underline{H}_{t} =$ 

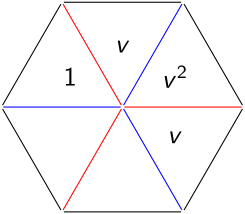
$\underline{H}_{id} =$ 

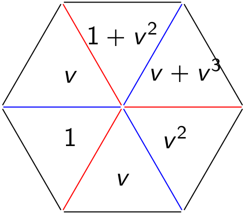
$\underline{H}_{t} =$ 

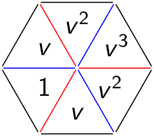
$\underline{H}_{s} =$ 

$\underline{H}_{ts} =$ 

$\underline{H}_{st} =$ 

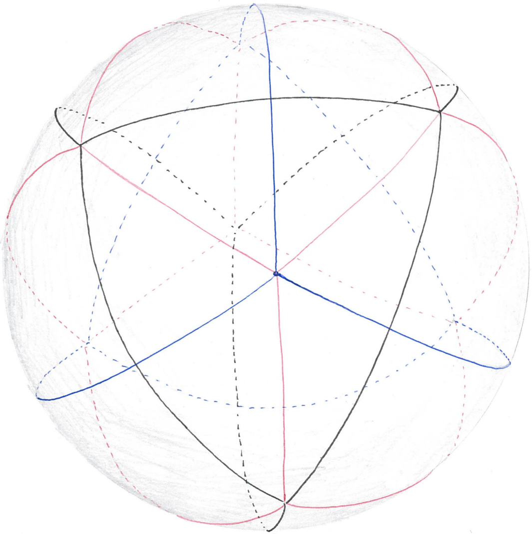
$\underline{H}_{ts}\underline{H}_{t} =$ 

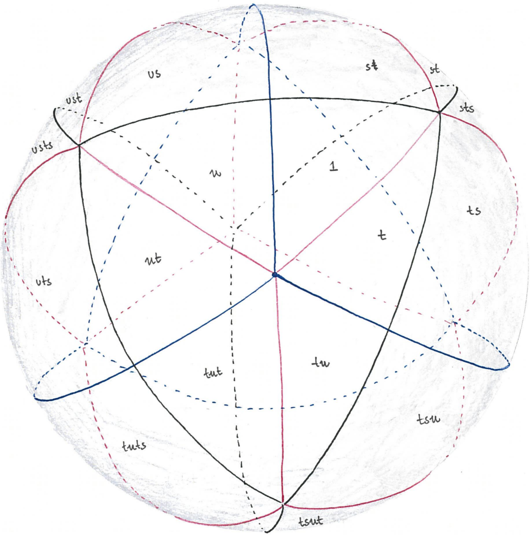
$\cdot \underline{H}_{t} =$ 

Hence: $\underline{H}_{tst} = \underline{H}_{ts}\underline{H}_{t} - \underline{H}_{t} =$ 

For dihedral groups (rank 2) we always have $h_{y,x} = v^{\ell(x)-\ell(y)}$ (Kazhdan-Lusztig basis elements are *smooth*.)

However in higher rank the situation quickly becomes more interesting...

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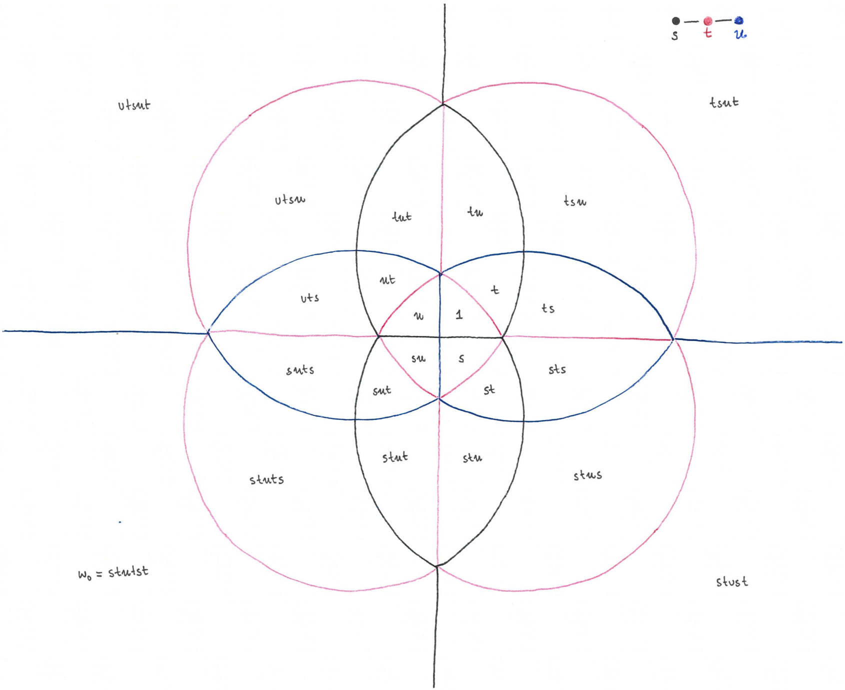
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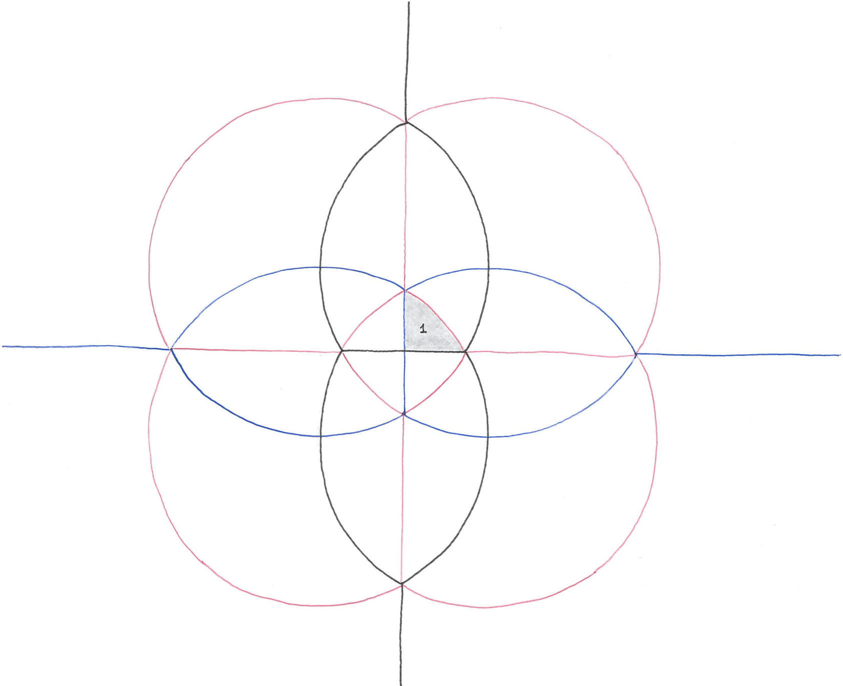
w0 = stutst
stust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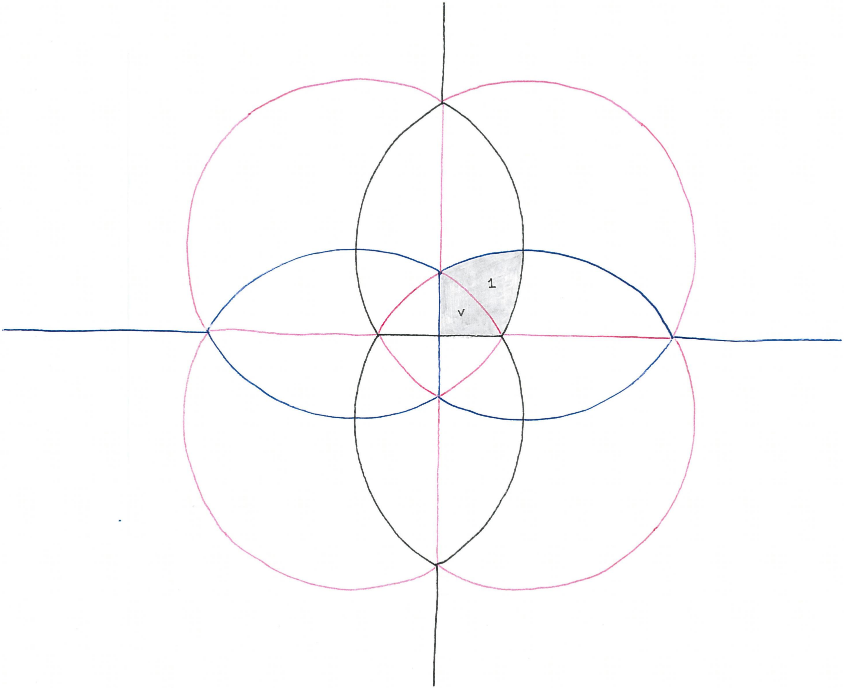
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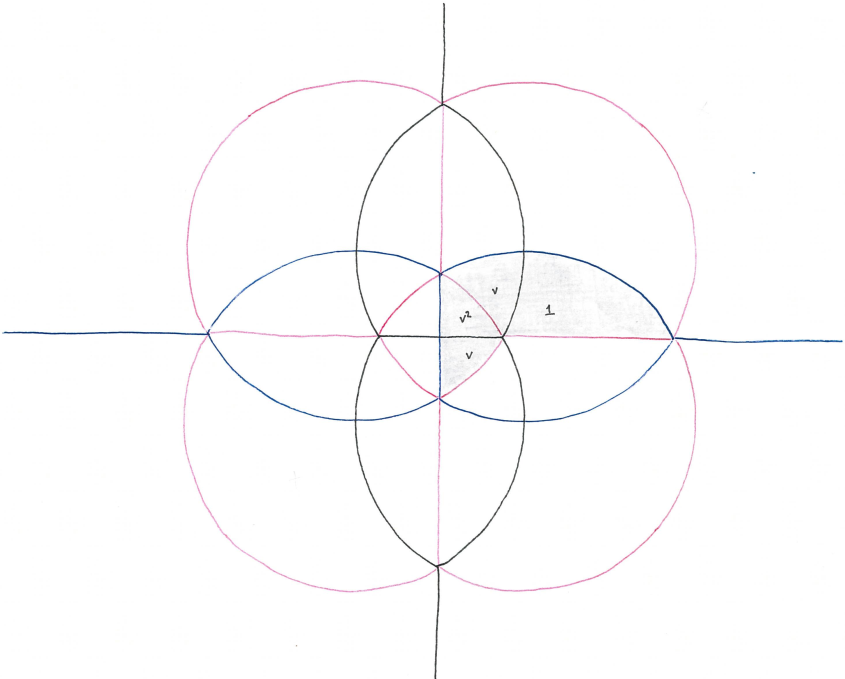
v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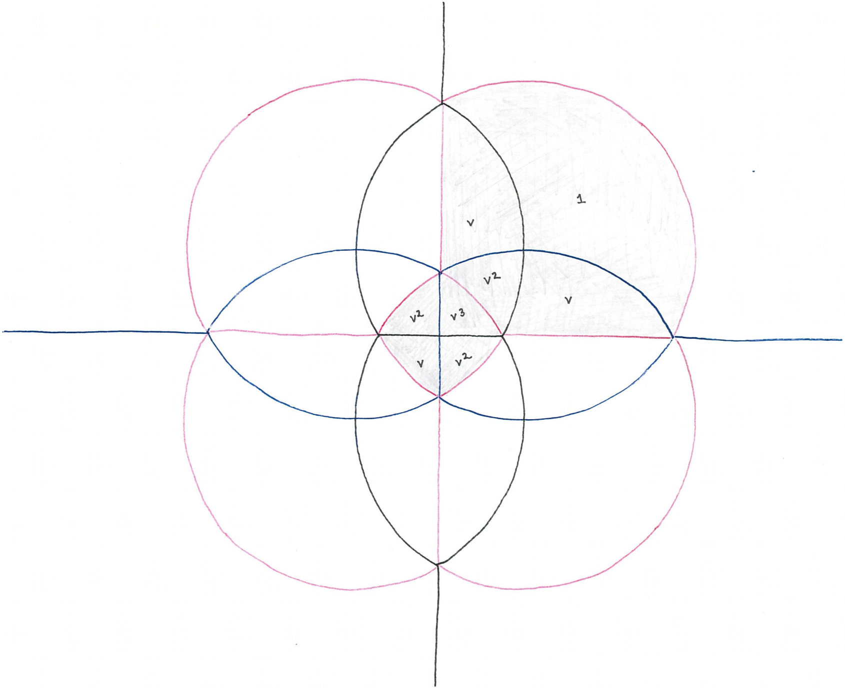
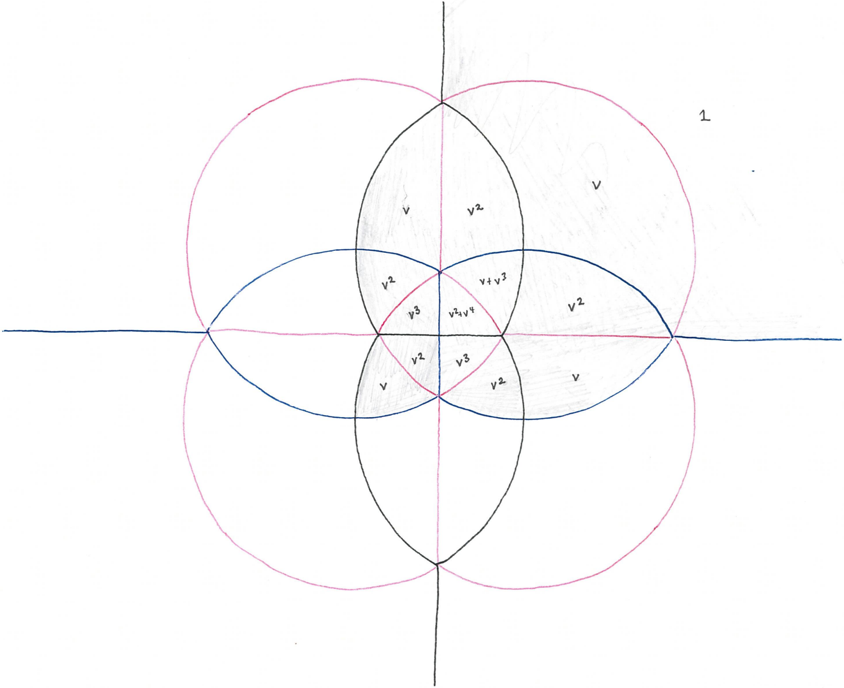
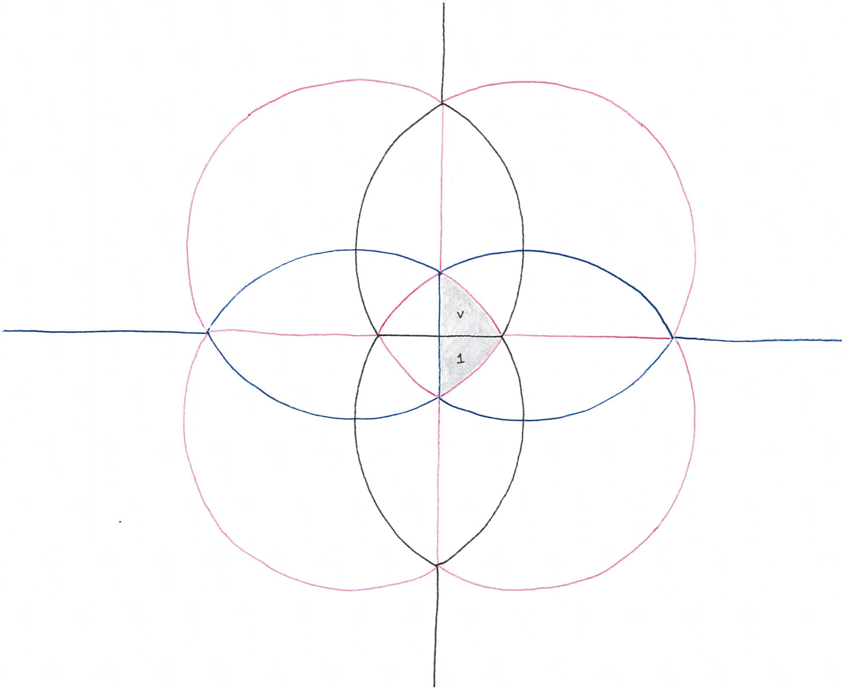
v²
1
v



1
v
v2
v
v2
v3
v
v2


1
v
v
v2
v2
v+v3
v3
v2+v4
v2
v2
v3
v
v2
v


v
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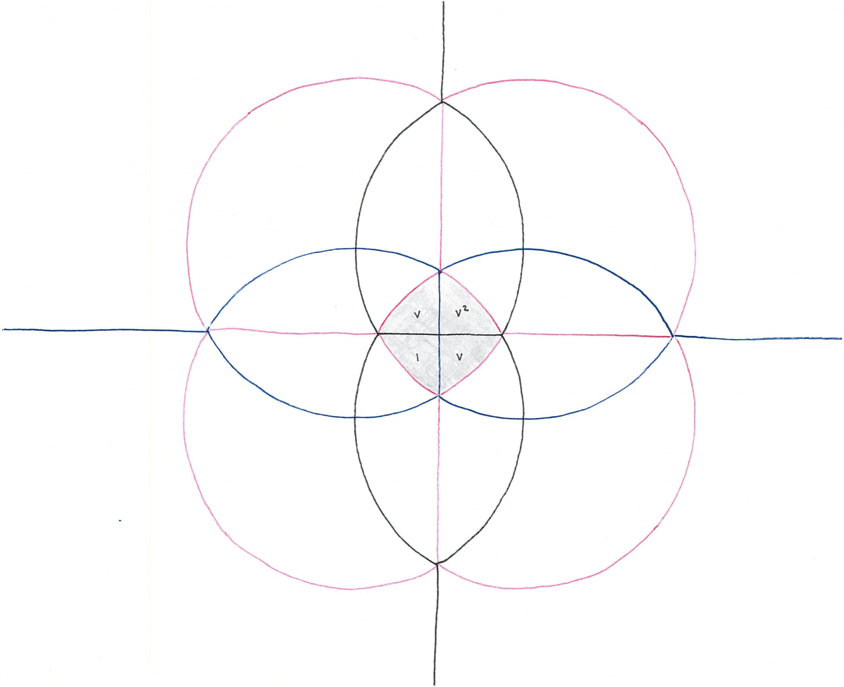
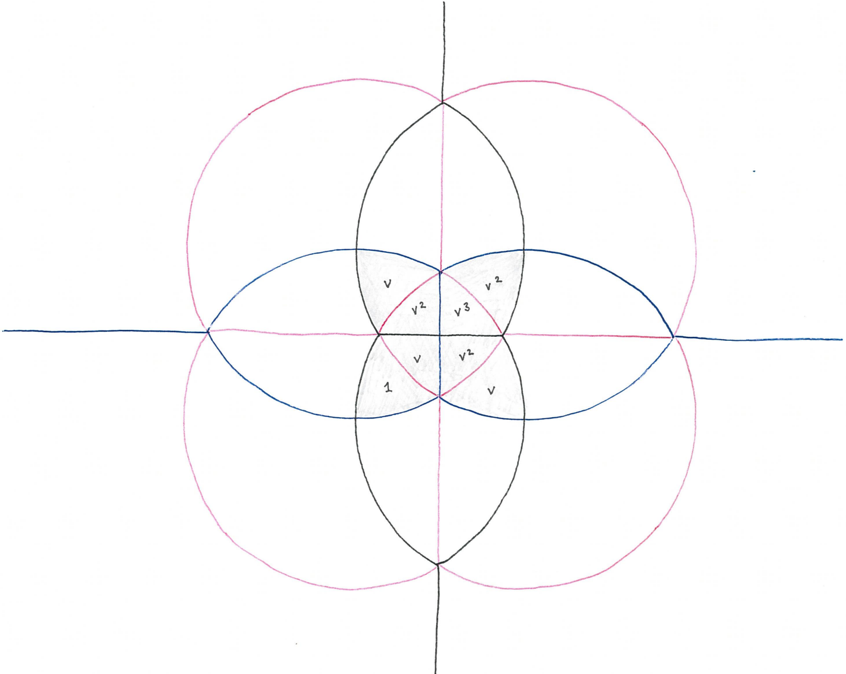
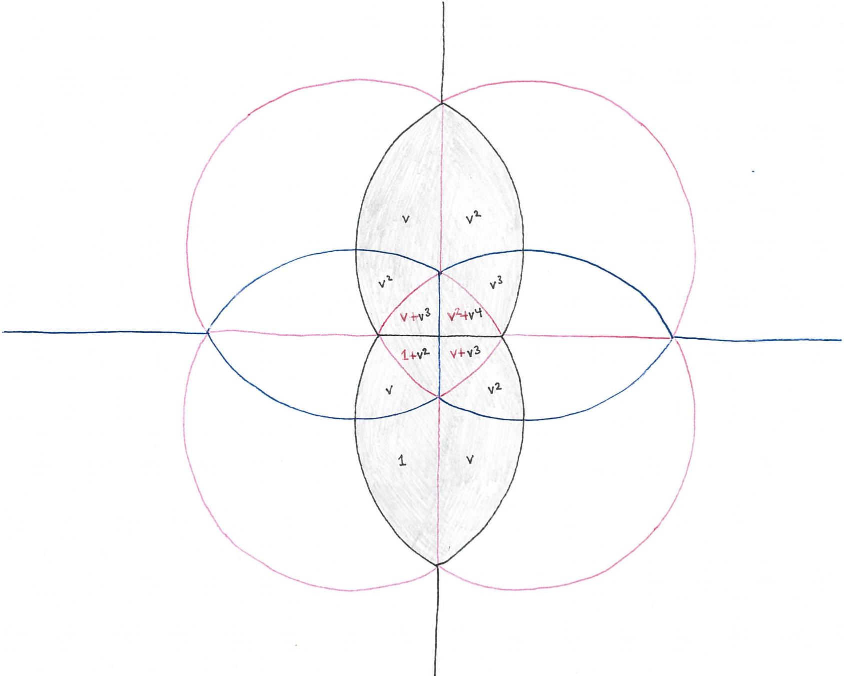
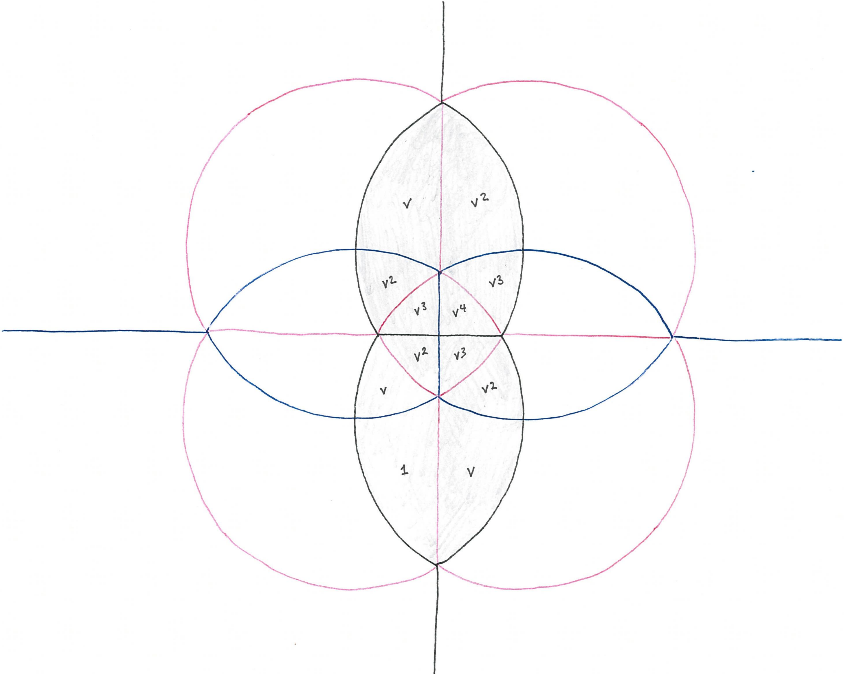
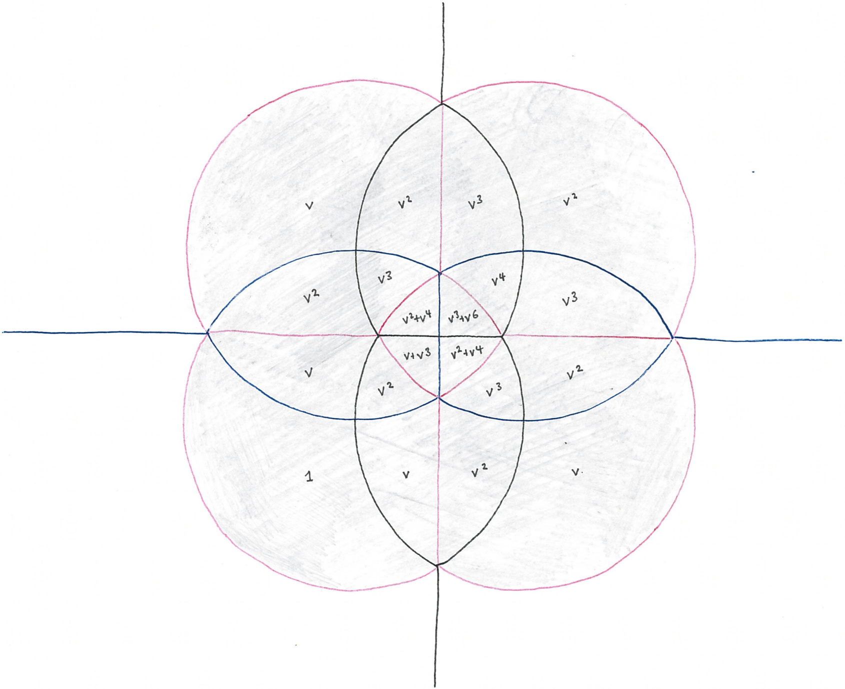

v
v²
1
v



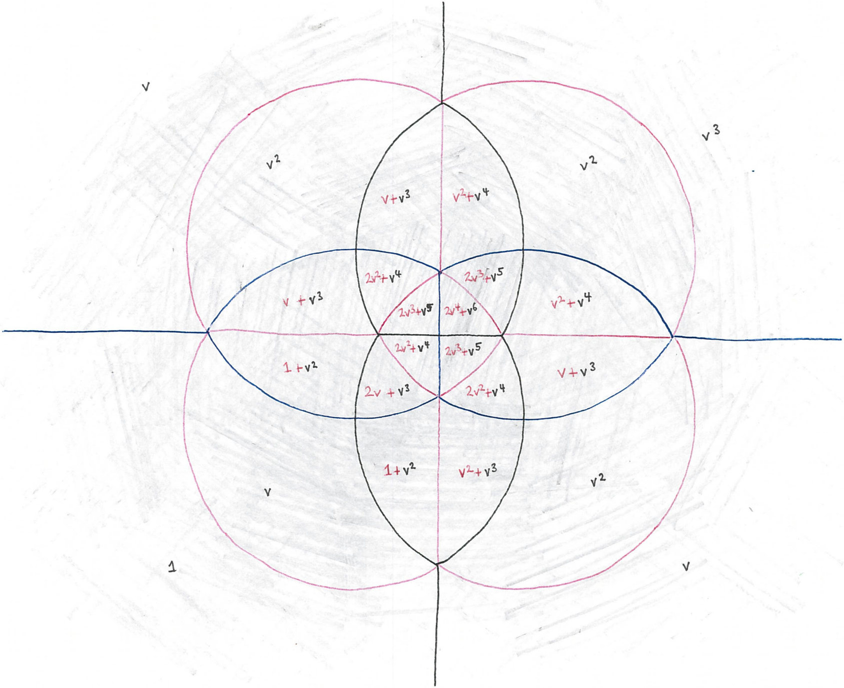
v
v2
v2
v3
v
v2
1
v


v
v²
v²
v³
v+v³
v²+v⁴
1+v²
v+v³
v
v²
1
v


v
v2
v2
v3
v3
v4
v2
v3
v
v2
1
v



v
v²
v3
v²
v3
v4
v²
v3
v²+v4
v³+v6
v+v3
v²+v4
v
v²
v3
v²
1
v
v²
v.


v
v^2
$v+v^3$
v^2+v^4
v^3
v^2
$2v^2+v^4$
$2v^3+v^5$
$v+v^3$
$2v^3+v^5$
$2v^4+v^6$
v^2+v^4
$1+v^2$
$2v^2+v^4$
$2v^3+v^5$
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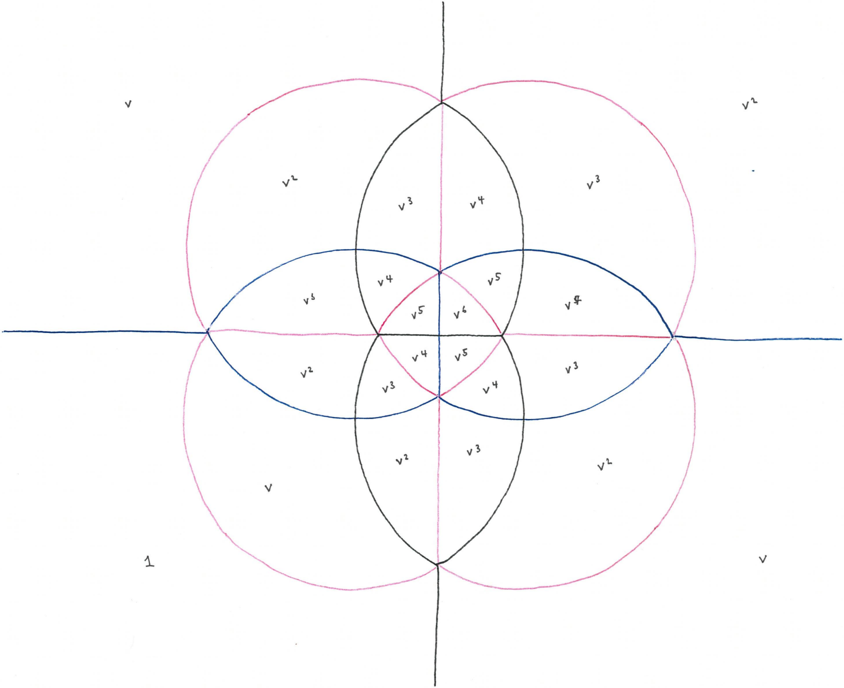
v
v^2
1
v



v
v2
v2
v3
v4
v3
v4
v5
v5
v5
v6
v4
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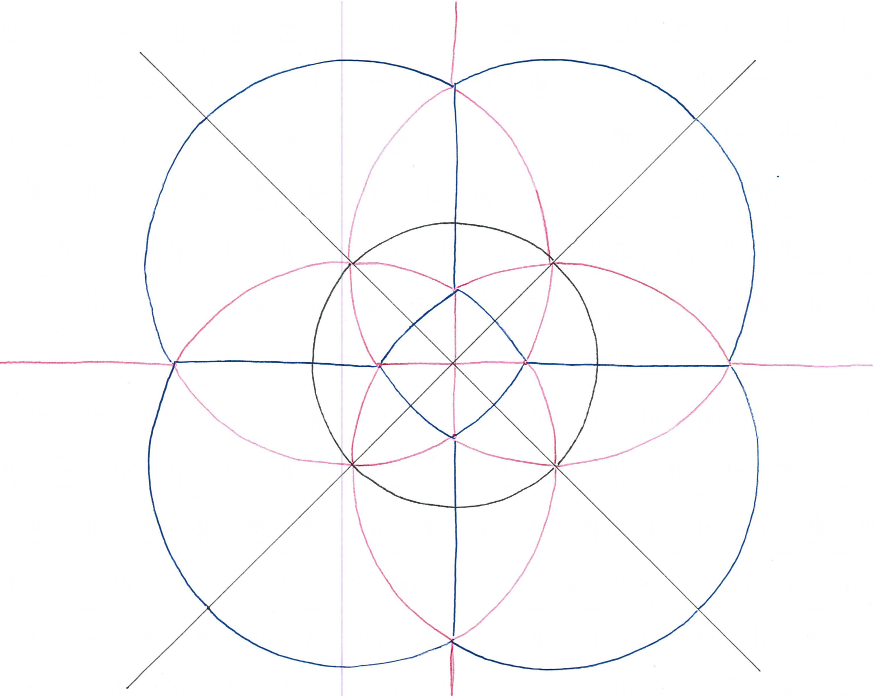
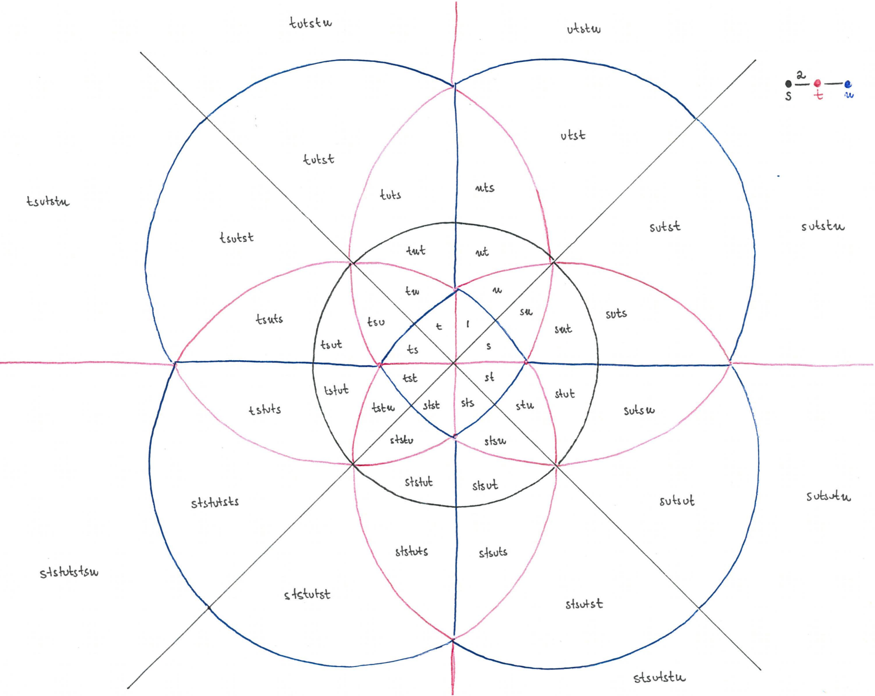
v3
v2
v
1
v





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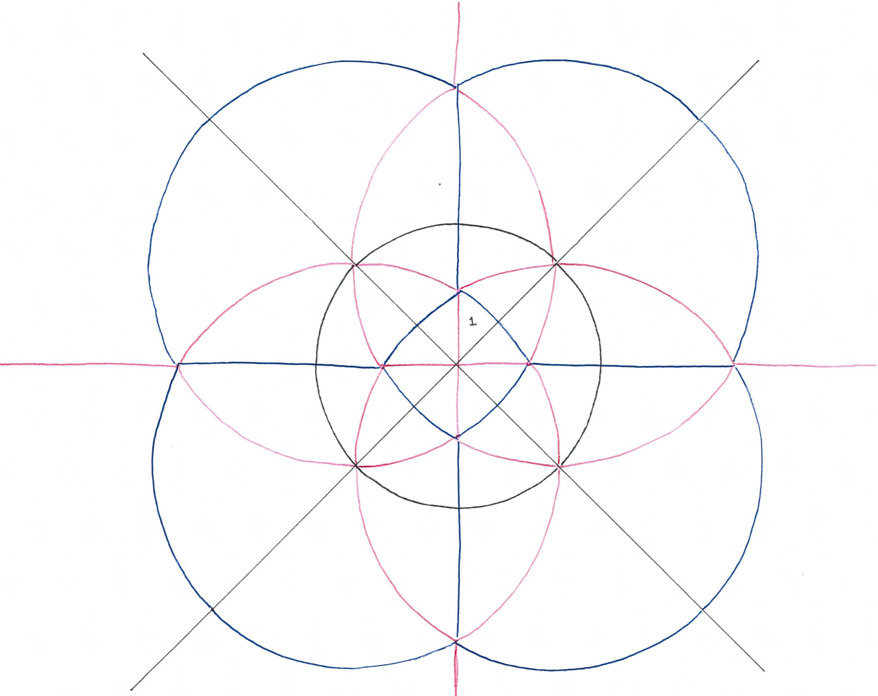
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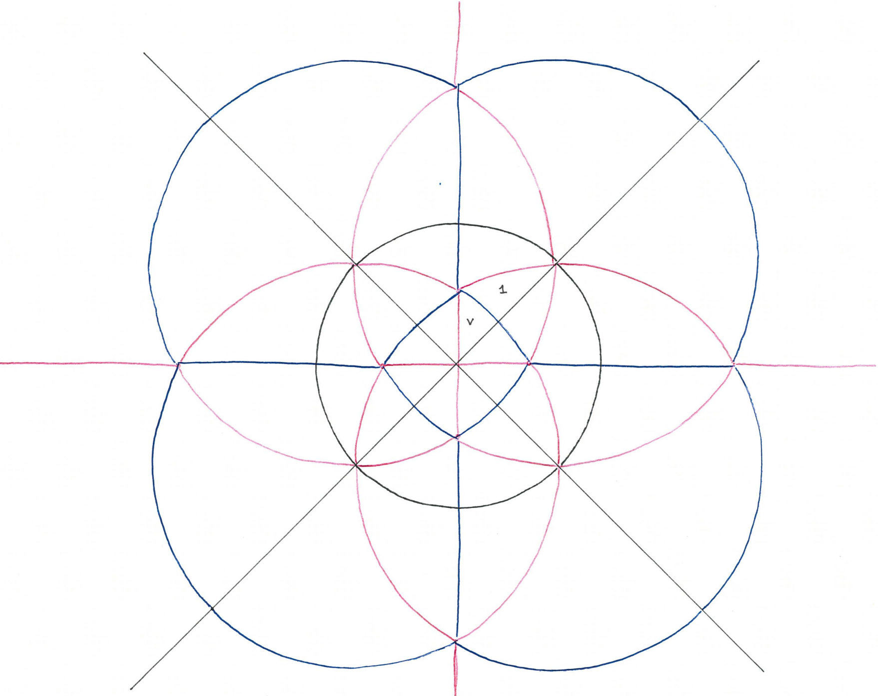
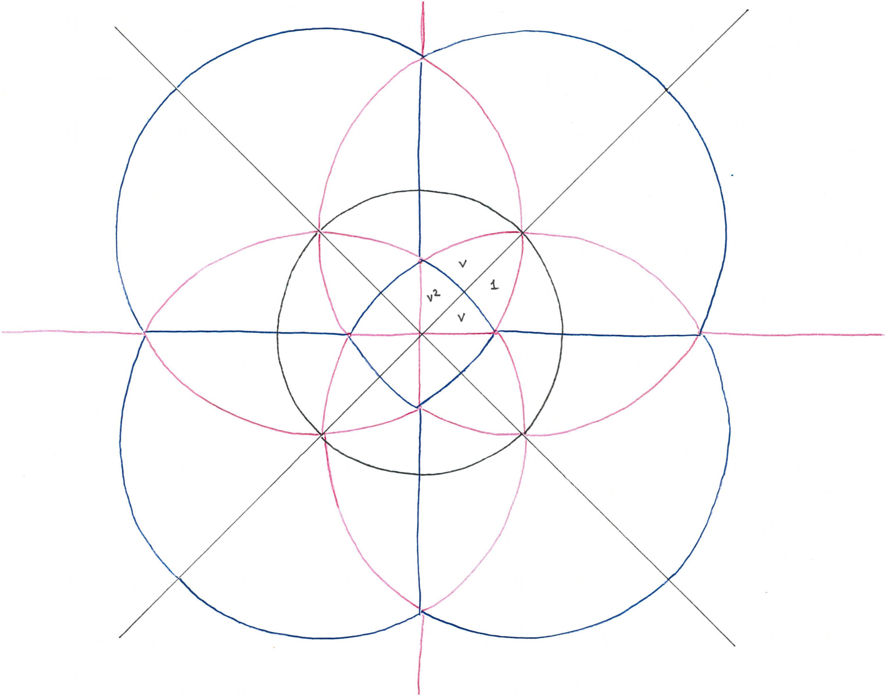
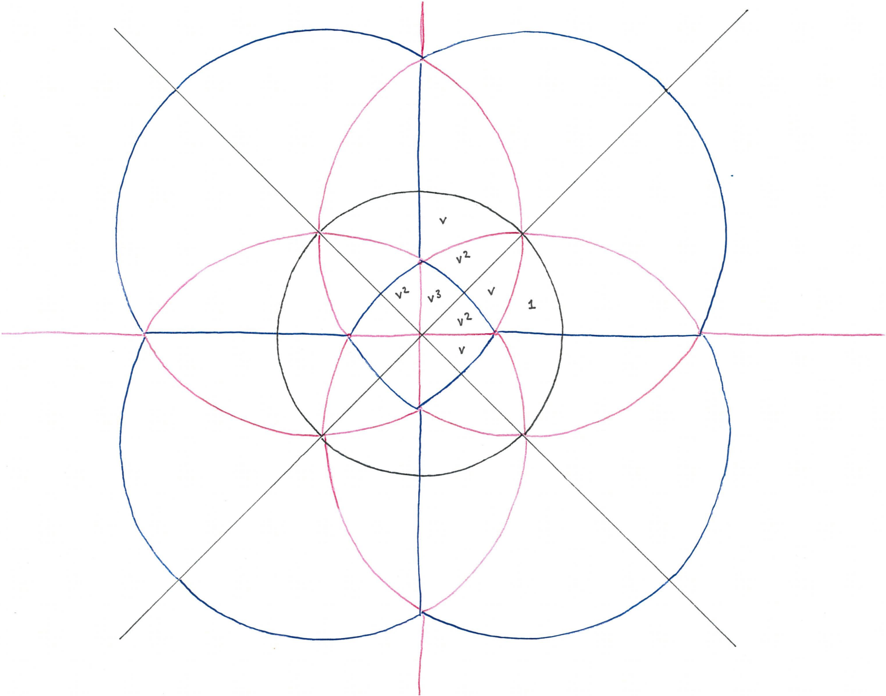
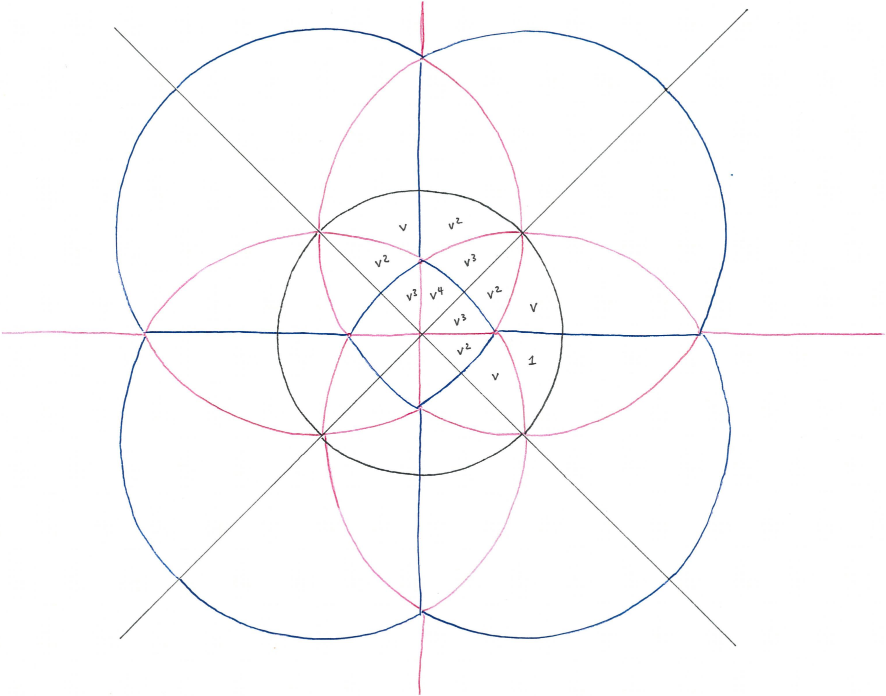
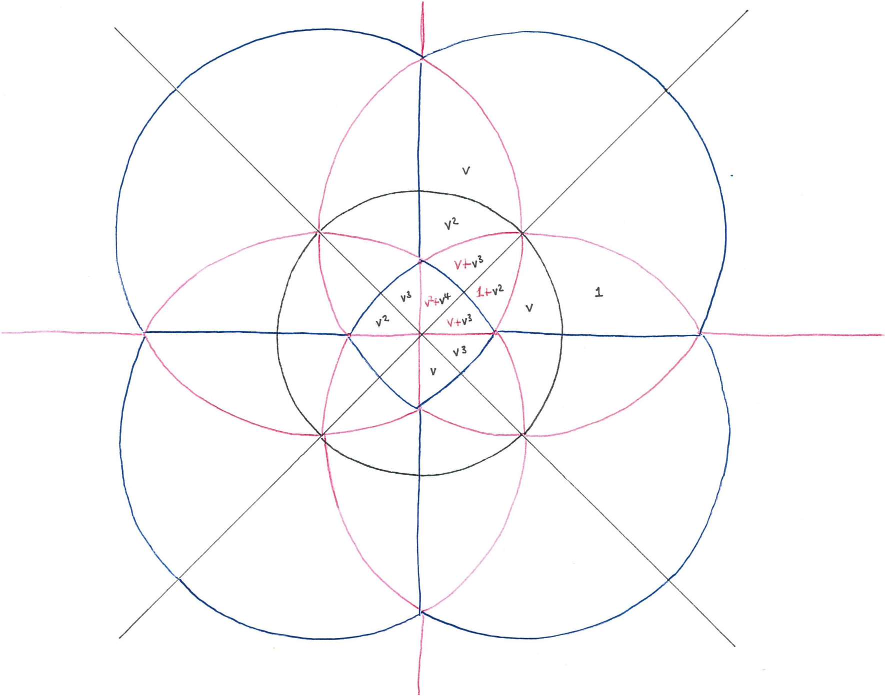

1
v

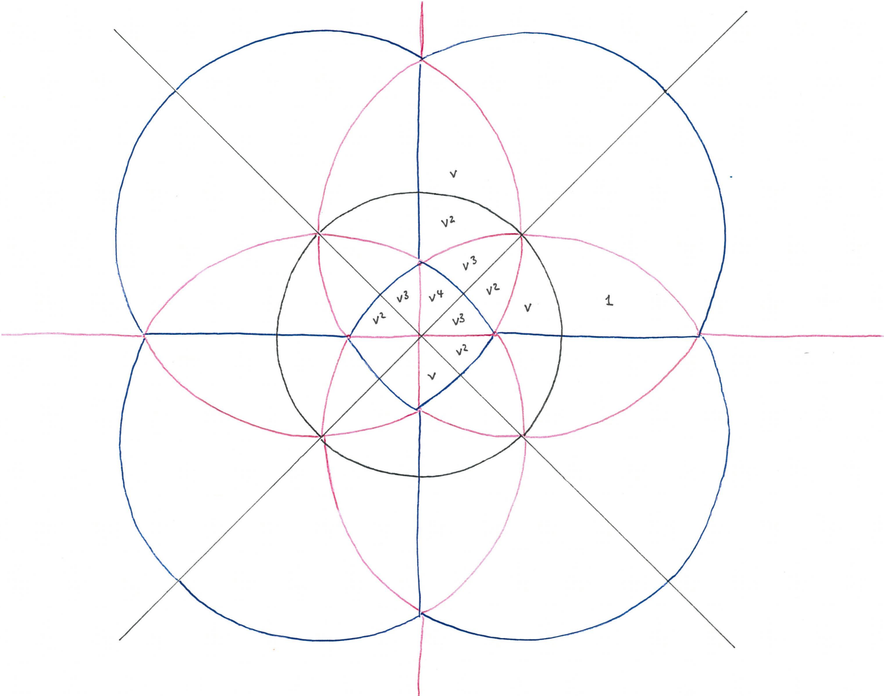

v
1
v2
v

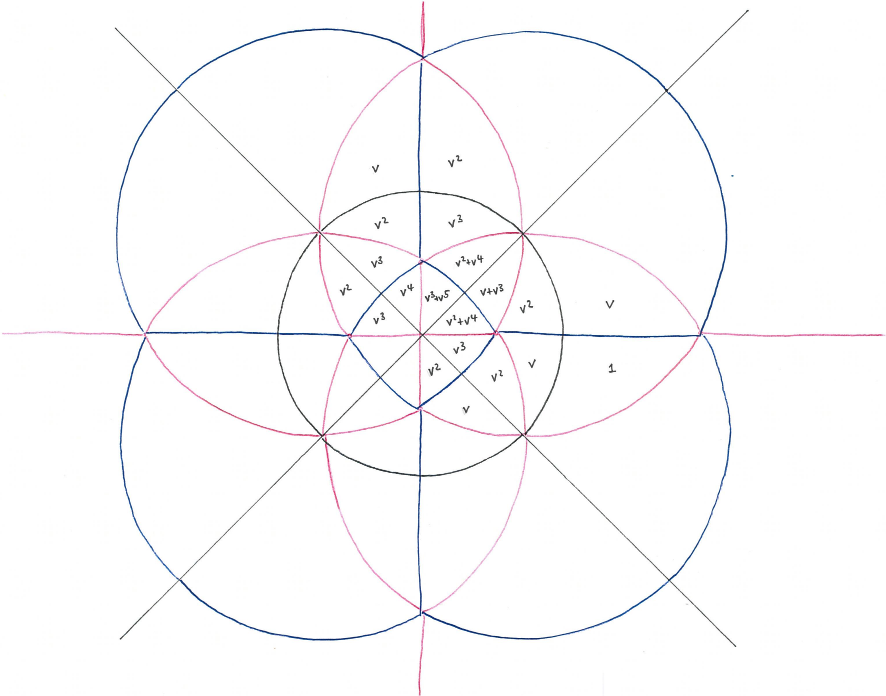


v
v2
v2
v3
v
1
v2
v


v
v2
v2
v3
v3
v4
v2
v
v3
v2
1
v


v
v2
v+v3
1+v2
1
v
v3
v2+v4
v2
v+v3
v3
v


v
v2
v3
v3
v4
v2
1
v
v2
v3
v2
v


v
v2
v2
v3
v3
v2+v4
v2
v4
v3+v5
v+v3
v2
v
v3
v2+v4
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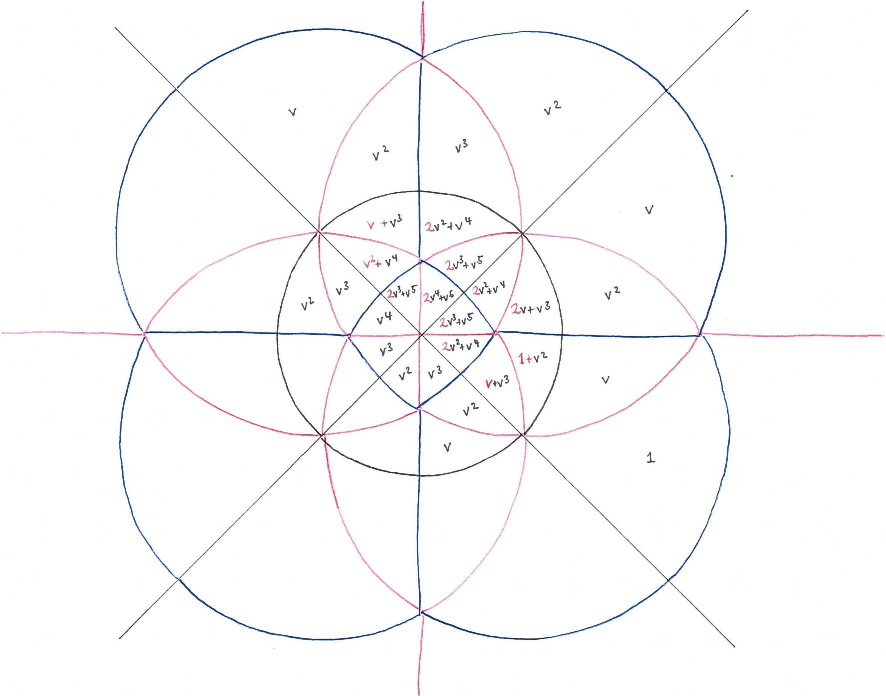
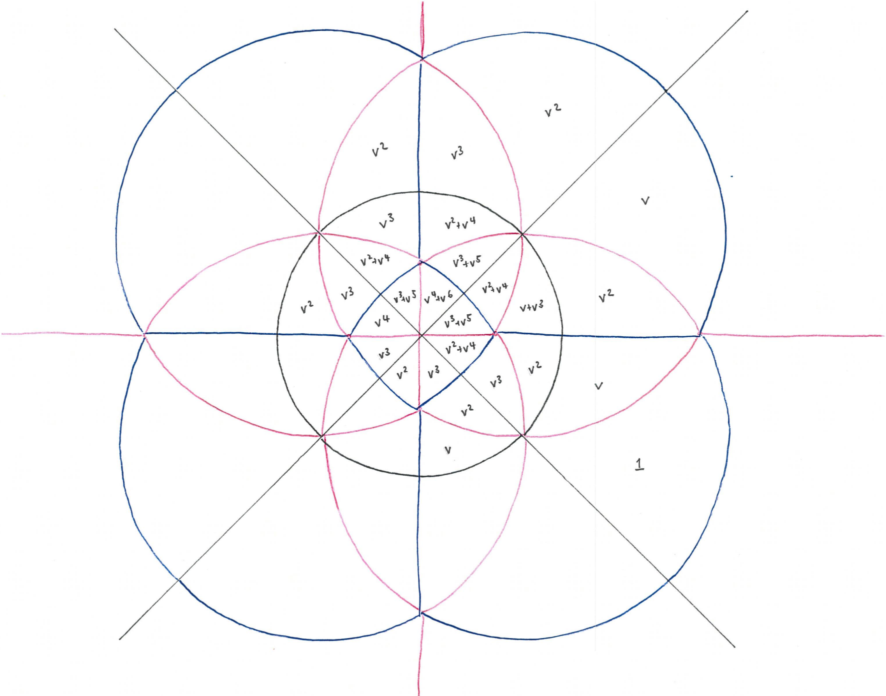
v
1
v2
v2
v



v
v^2
v^2
v^3
v
$v+v^3$
$2v^2+v^4$
v^2+v^4
$2v^3+v^5$
v^2
v^3
$2v^3+v^5$
$2v^4+v^6$
$2v^2+v^4$
$2v+v^3$
v^2
v^4
$2v^3+v^5$
v^3
$2v^2+v^4$
$1+v^2$
v^2
v^3
$v+v^3$
v
v^2
v
1



v2
v2
v3
v
v3
v2+v4
v2+v4
v3+v5
v2
v3
v3+v5
v4+v6
v2+v4
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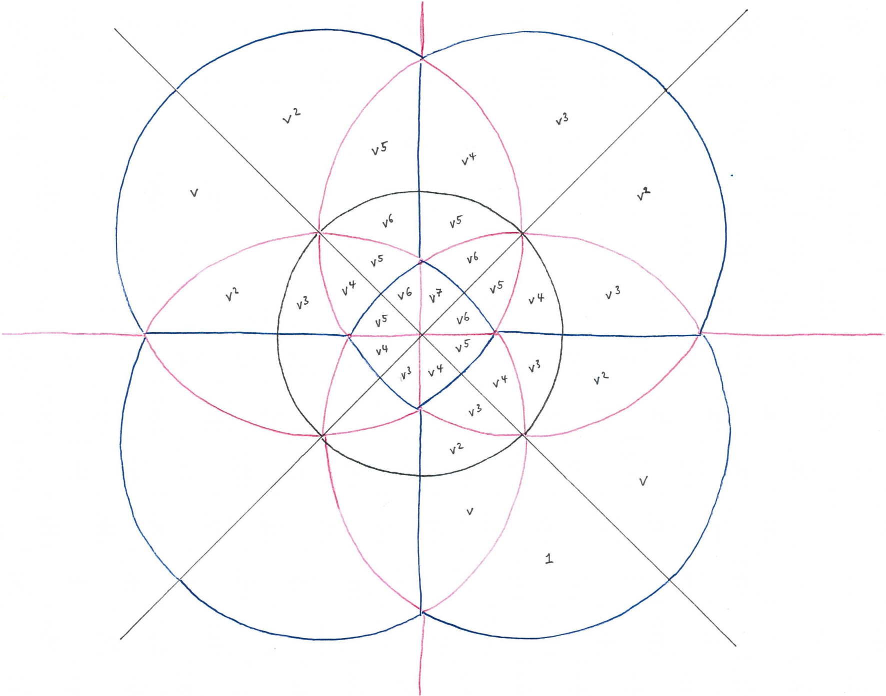
v2
v
v2
v
1


v2
v3
v5
v4
v
v2
v6
v5
v5
v6
v2
v4
v3
v6
v7
v5
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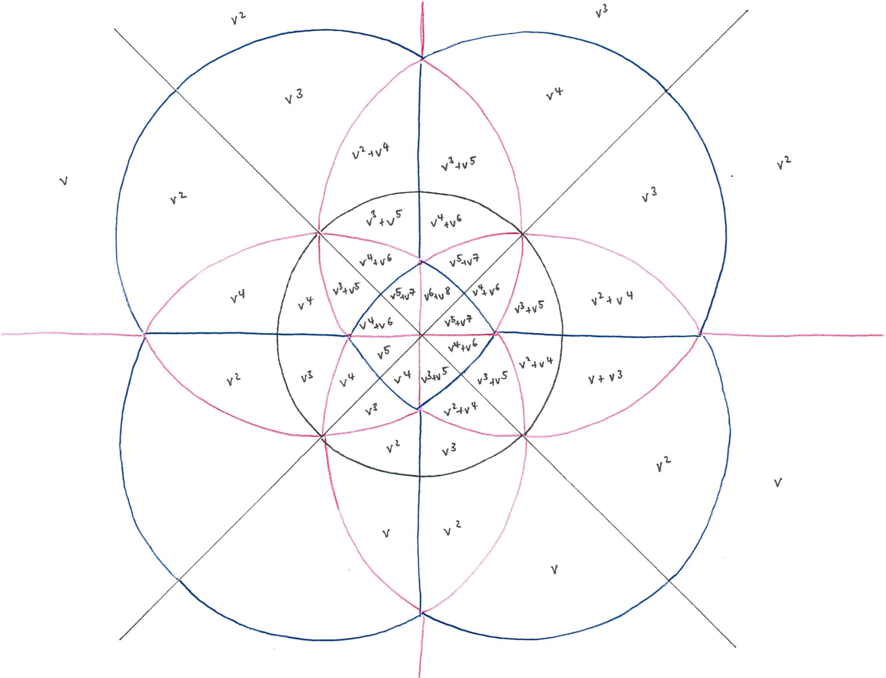
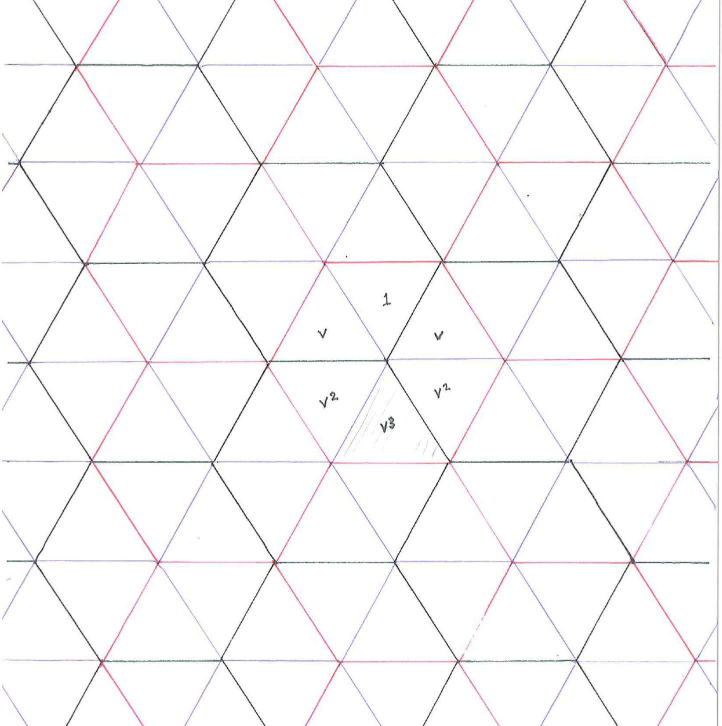
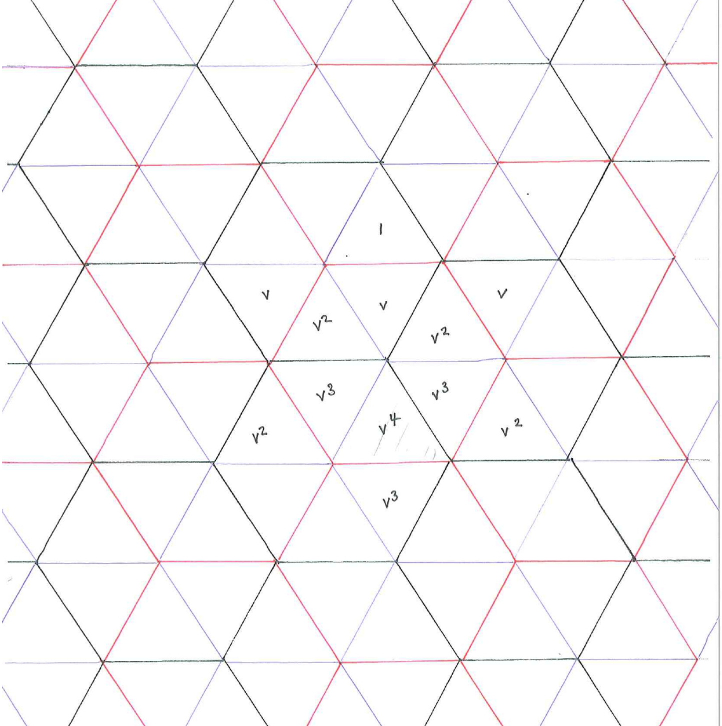
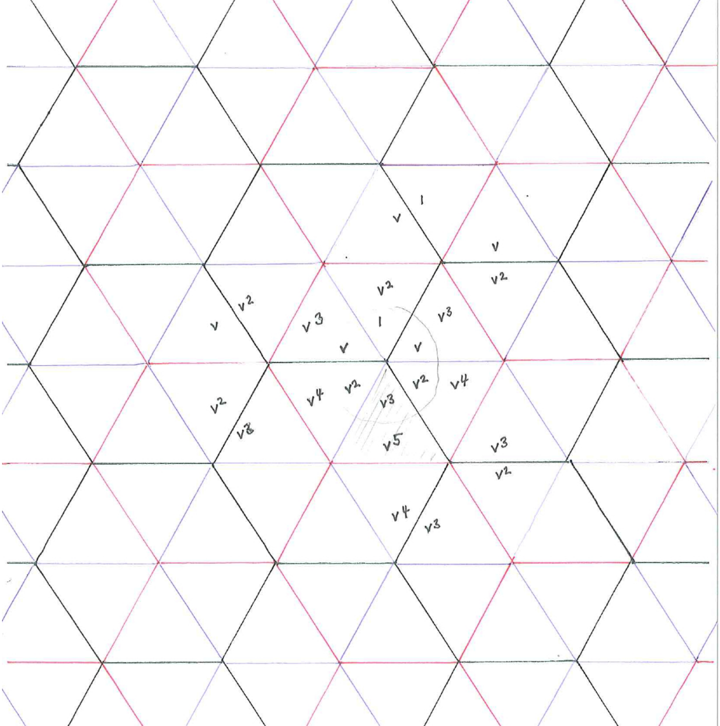
v3
v2
v
v
1


v^2
v^3
$v\,3$
$v\,4$
v
v^2+v^4
v^3+v^5
v^2
$v\,2$
$v\,3$
v^3+v^5
v^4+v^6
v^4+v^6
v^5+v^7
$v4$
$v\,4$
v^3+v^5
v^5+v^7
v^6+v^8
v^4+v^6
v^3+v^5
$v^2+v\,4$
$v^4+v\,6$
v^5+v^7
$v5$
v^4+v^6
$v^2+v\,4$
v^2
$v3$
$v\,4$
$v\,4$
v^3+v^5
v^3+v5
$v+v\,3$
$v3$
v^2+v^4
v^2
$v\,3$
v^2
v
v
$v\,^2$
v


1
v
v
v²
v²
v³



1
v
v
v
v2
v2
v3
v3
v2
v4
v2
v3



v 1
v
v2
v2
v2
v
v3
v3
1
v
v
v2
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v2
v3
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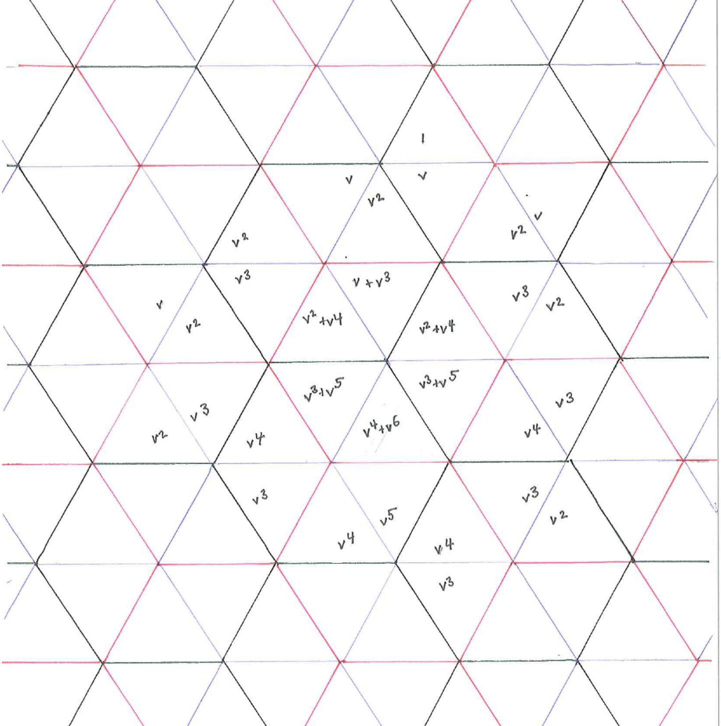
v3
v2
v4
v3



1
v
v
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v
v^2
v^2
v^3
$v+v^3$
v
v^2
v^3
v^2
v^2+v^4
v^2+v^4
v^3+v^5
v^3+v^5
v^3
v^2
v^3
v^4
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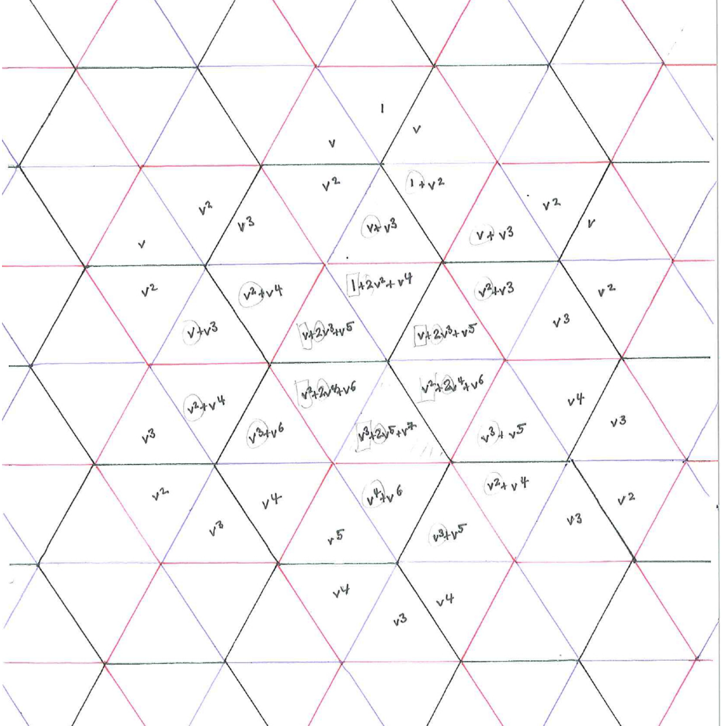
v^3
v^2
v^5
v^4
v^4
v^3


1
v
v
v2
1+v2
v2
v2
v3
v+v3
v+v3
v
v
v2
v2+v4
1+2v2+v4
v2+v3
v2
v+v3
v+2v3+v5
v+2v3+v5
v3
v2+2v4+v6
v2+2v4+v6
v4
v2+v4
v3+v6
v3+2v5+v7
v3+v5
v3
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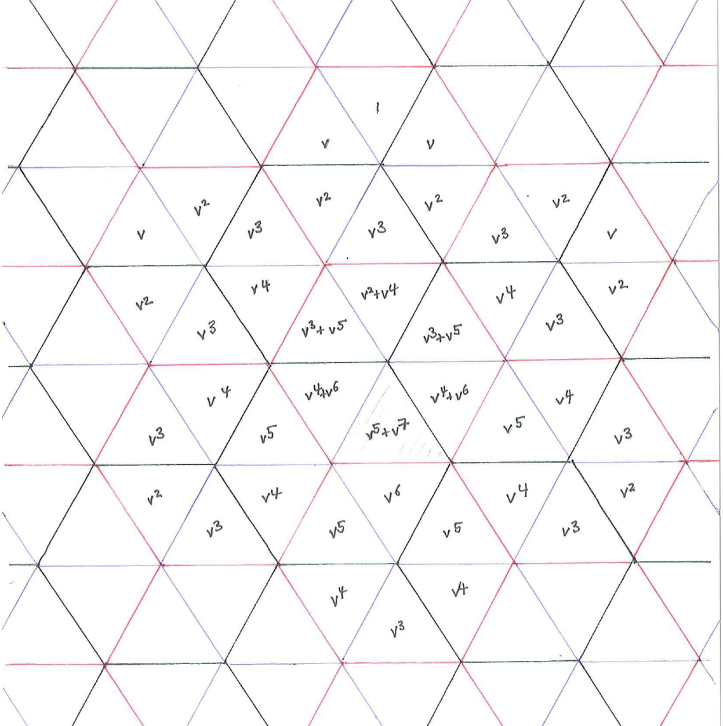
v3
v4
v4
v3


1
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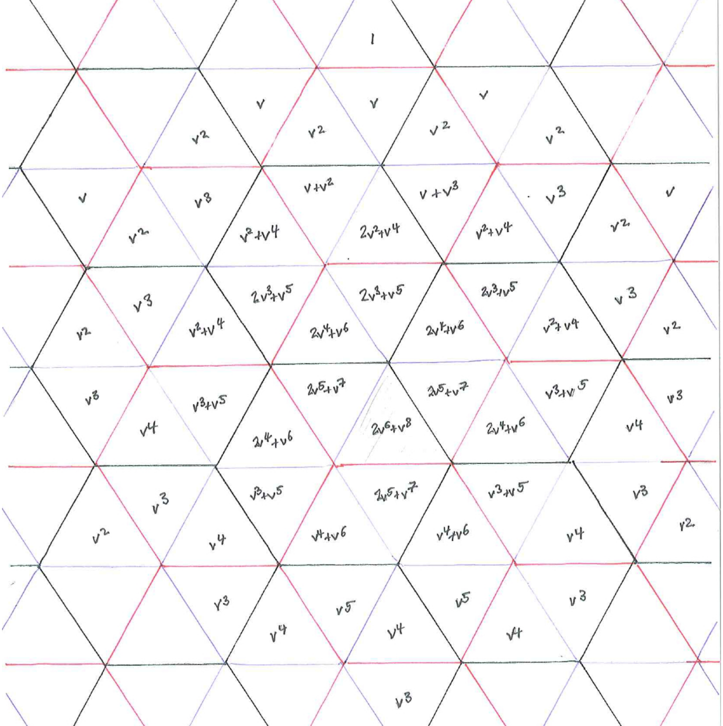
v^4
v^4
v^3


1
v
v
v
v^2
v^2
v^2
v^2
v
v^3
$v+v^2$
$v+v^3$
v^3
v
v^2
v^2+v^4
$2v^2+v^4$
v^2+v^4
v^2
v^3
$2v^3+v^5$
$2v^3+v^5$
$2v^3+v^5$
v^3
v^2
v^2+v^4
$2v^4+v^6$
$2v^4+v^6$
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v^3
v^3+v^5
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$2v^4+v^6$
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v^3

Kazhdan-Lusztig positivity conjecture (1979):

$$h_{x,y} \in \mathbb{Z}_{\geqslant 0}[v]$$

Kazhdan-Lusztig positivity conjecture (1979):

$$h_{x,y} \in \mathbb{Z}_{\geqslant 0}[v]$$

Established for crystallographic W by Kazhdan and Lusztig in 1980, using Deligne's proof of the Weil conjectures.

Crystallographic: $m_{st} \in \{2, 3, 4, 6, \infty\}$.

Why are Kazhdan-Lusztig polynomials hard?

Why are Kazhdan-Lusztig polynomials hard?

Polo's Theorem (1999)

For any $P \in 1 + q\mathbb{Z}_{\geqslant 0}[q]$ there exists an m such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.

Why are Kazhdan-Lusztig polynomials hard?

Polo's Theorem (1999)

For any $P \in 1 + q\mathbb{Z}_{\geqslant 0}[q]$ there exists an m such that $v^m P(v^{-2})$ occurs as a Kazhdan-Lusztig polynomial in some symmetric group.

Roughly: all positive polynomials are Kazhdan-Lusztig polynomials!

The most complicated Kazhdan-Lusztig-Vogan polynomial computed by the *Atlas of Lie groups and Representations* project:

$$
\begin{aligned}
152q^{22} + 3\,472q^{21} + 38\,791q^{20} + 293\,021q^{19} + 1\,370\,892q^{18} + \\
+4\,067\,059q^{17} + 7\,964\,012q^{16} + 11\,159\,003q^{15} + \\
+11\,808\,808q^{14} + 9\,859\,915q^{13} + 6\,778\,956q^{12} + \\
+3\,964\,369q^{11} + 2\,015\,441q^{10} + 906\,567q^{9} + \\
+363\,611q^{8} + 129\,820q^{7} + 41\,239q^{6} + \\
+11\,426q^{5} + 2\,677q^{4} + 492q^{3} + 61q^{2} + 3q
\end{aligned}
$$

(This polynomial is associated to the reflection group of type E_8. See `www.liegroups.org`.)