Lecture 1.1: Historical introduction and outline

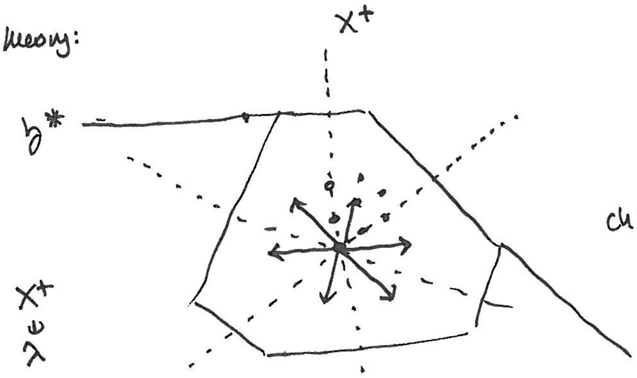
$\mathfrak{g}$ f.d. complex semi-simple Lie algebra (eg. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$)

Rep $\mathfrak{g}$??

| This is a fundamental question in mathematics.
| Has lead to much beautiful mathematics in the past!

Finite dimensional representations: $\mathfrak{h} \subset \mathfrak{g}$ Cartan $R^+ \subset R \subset \mathfrak{h}^*$, W Weyl group
pos. roots, roots
$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$

highest weight theory:



simple f.d. dim $\mathfrak{g}$-modules $\leftrightarrow$ X^+
$L(\lambda) \leftrightarrow \lambda$

Weyl character formula: $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$, $\quad w \cdot \lambda = w(\lambda+\rho) - \rho$.

$$\operatorname{ch} L(\lambda) = \sum_{w \in W} \frac{(-1)\, e^{w\cdot\lambda}}{\prod_{\alpha \in R^+}(1-e^{-\alpha})}$$

$$\Updownarrow$$

$$[L(\lambda)] = \sum_{w\in W} (-1)^{\ell(w)} [\Delta(w\cdot\lambda)].$$

Verma (1966): Uniform algebraic construction of $L(\lambda)$ as a quotient of a Verma module

$$\Delta(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda . \qquad \operatorname{ch} \Delta(\lambda) = \frac{e^\lambda}{\prod_{\alpha\in R^+}(1-e^{-\alpha})}$$

For any $\lambda \in \mathfrak{h}^*$, $\Delta(\lambda)$ always has an irreducible quotient $L(\lambda)$
"simple highest weight module".

Basic question: $\operatorname{ch} L(\lambda)$ for arbitrary $\lambda \in \mathfrak{h}^*$?

$$\Updownarrow$$

Jordan-Hölder multiplicity $[M(\lambda) : L(\mu)] = ?$

Linkage principle $\neq 0 \implies \lambda = w \cdot \mu$ for $w \in W$.

Kazhdan-Lusztig (1979) defined polynomials $h_{x,y} \in \mathbb{Z}[v]$. KL conjecture: $[M(x\cdot 0) ; L(y \cdot 0)] = h_{x,y}(1)$.

$\rightsquigarrow$ all integral weights by Jantzen's translation functors.

Many mysteries:

- KL polynomials are defined for any Coxeter system (W,S).
- KL positivity conjecture $h_{y,x} \in \mathbb{N}[v]$
- Why polynomials?
- ...

$\mathbb{V}(P_x)$ can be described as the largest summand of $S (:= S(\mathfrak{h})$

$$\mathbb{V}(P_x) \overset{\oplus}{\subset} \mathbb{C} \otimes_{S^s} S \otimes \ldots \otimes_{S^u} S \qquad \text{where } \underline{x} = st\ldots u \text{ is a reduced expression.}$$

as a $\mathbb{C}$ right C-module. Key difficulty: show that $\mathbb{V}(P_x)$ is small enough.

Soergel's dream: Establish the "decomposition theorem" algebraically, hence free KL conjecture from geometry, explain positivity, attack modular conjectures etc.

Ben and I claim that Soergel's dream can be realized.

Two key ideas:

Monday / Wed

1) categorification: KL conjecture is a statement about objects, but to understand these objects one needs to understand morphisms. Using Soergel bimodules we will explain how $\mathcal{O}_0$ can be described by (very complicated) "generators and relations" (problem going back to Gelfand ~1979). Here diagrammatics / higher algebra plays a key role.

Wed / Thurs

2) work of de Cataldo and Migliorini: dCM give a new proof of the decomposition theorem using real Hodge theory. We consider Soergel's theory over $\mathbb{R}$ and show algebraically that $\mathbb{V}(P_x)$ carries $\mathbb{R}$-Hodge structures.

Perspectives:

× at the moment there is only limited interaction between 1) and 2). These should become more closely knit in the future (explicit idempotents etc.)

× theory works for any Coxeter system!

"$\mathcal{O}_0$" for any Coxeter system.

Many interesting questions here.

Proved 1981 by Beilinson-Bernstein, Brylinski-Kashiwara. The proof is a miracle of 20th cent maths.

$\mathfrak{g}$-mod $\overset{\text{localization}}{\rightsquigarrow}$ $\mathcal{D}_{G/B}$-mod (algebraic differential equations on G/B) $\overset{\text{Kashiwara, Mebkhout 1980}}{\rightsquigarrow}$ R.H. correspondence (solve) Perv(G/B) $\mathbb{C}$-coefficients (monodromy) $\overset{\text{"de } \mathbb{F}_q \text{ à } \mathbb{C}\text{"}}{\rightsquigarrow}$ BBD-G (1981) Perv$((G/B)_{\mathbb{F}_q}, \overline{\mathbb{Q}}_\ell)$ ℓ-adic perverse sheaves $\overset{\text{KL 1980}}{\longleftrightarrow}$ Hecke algebras and KL polynomials

↑ uses Deligne's proof of Weil conjectures (1974)

Bernstein: "keep moving sideways until you run into Deligne's theory of weights".

This proof + Springer and DL theory gave rise to geometric rep theory.

Soergel 1990: New proof using "modules over coinvariants".

Let $\mathcal{O}_0 = \langle \Delta(w\cdot 0)\rangle_{w\in W} \subset$ weight modules "principal block of category $\mathcal{O}$"

↑ finite length abelian $\mathbb{C}$-category

simples: $L_x := L(x\cdot 0)$

Vermas: $\Delta_x := \Delta(x\cdot 0)$

projectives: $P_x \twoheadrightarrow L_x$

Let $C = S(\mathfrak{h})/(S(\mathfrak{h})^W_+) = H^*(G^\vee/B^\vee;\mathbb{C})$

$G^\vee$ semi-simple group with $\mathrm{Lie}\, G^\vee = \mathfrak{g}^\vee$.

Soergel: $\mathrm{End}(P_{w_0}) = C$ and $\mathbb{V} := \mathrm{Hom}(P_{w_0}, -)$ gives a functor

$$\mathbb{V}: \mathcal{O}_0 \to \text{mod-}C$$

Soergel's thm: $\mathbb{V}(P_x) \overset{(*)}{=} IH^*(\overline{BxB^\vee/B^\vee})$ intersection cohomology of a Schubert variety

* $\mathbb{V}(P_x)$ has an "elementary" definition in terms of modules over C.
* $(*) \Leftarrow$ KL conjecture.
* $(*)$ explains why KL polys are polys.

$$A = \mathrm{End}\Big(\bigoplus_{x\in W} P_x\Big) \quad \text{"algebra of category } \mathcal{O}\text{"}$$

$$= \mathrm{End}_C\Big(\bigoplus_{x\in W} \mathbb{V}(P_x)\Big)$$

(graded.) $\rightsquigarrow$ A gets a grading (hence $\mathcal{O}$)

$\rightsquigarrow$ Koszul duality.