Lecture 12: Hecke algebras and Kazhdan-Lusztig polynomials

Let (W, S) be a Coxeter system. That is

$$W = \langle s \in S \mid s^2 = id,\ \underbrace{st\ldots}_{m_{st} = m_{ts} \text{ terms}} = \underbrace{ts\ldots}\rangle \qquad m_{st} = m_{ts} \in \{2, 3, \ldots, \infty\}.$$

$m_{st} = \infty$ "no relation".

Notation: An expression is a word $\underline{w} = (s, t, \ldots, u)$ in S. $w = st\ldots u$.

We abuse notation and write $\underline{w} = st\ldots u$ but word is important.

A reduced expression or rex is an expression $\underline{w} = s_1 s_2 \ldots s_m$ with m minimal.

We have $\ell(w) = m$ "length" of w.

A subexpression ~~of~~ $\underline{e}$ of $\underline{w} = s_1 \ldots s_m$ is a sequence $\underline{e} = e_1 \ldots e_m$ $e_i \in \{0, 1\}$.

We write $\underline{w}^{\underline{e}} := s_1^{e_1} \ldots s_m^{e_m}$ and say $\underline{e}$ expresses $\underline{w}^{\underline{e}}$.

The Bruhat order is characterised by

$$y \leq x \iff y = \underline{x}^{\underline{e}} \text{ for some rex } \underline{x} \text{ and subexp. } \underline{e}$$

(~~independent~~ $\{\underline{x}^{\underline{e}} \mid \underline{e}$ a subexp. is indep. of rex $\underline{x}\}$)

Given a subexp. $\underline{e}$ of $\underline{w}$ the Bruhat stroll is $x_0 = id, x_1, \ldots, x_m = \underline{w}^{\underline{e}}$

where $x_i = x_{i-1} s_i^{e_i}$. (Same info as subexp. $\underline{e}$)

So we can think of of a subexp. as a path $\{0, \ldots, m\} \ni i \mapsto x_i$ s.t. $x_0 = id$

and s.t.b. $x_i = \begin{cases} x_{i-1} s_i \\ \text{or } x_{i-1} \end{cases}$ for all i.

Given a subexp. $\underline{e}$ of $\underline{w}$ we decorate $\underline{e}$ as follows:

$e_i = 1$	$x_i > x_{i-1}$	U1	we move
$e_i = 1$	$x_i < x_{i-1}$	D1	
$e_0 = 0$	$x_{i-1} s_i > x_{i-1}$	U0	we would have moved up
$e_0 = 0$	$x_{i-1} s_i < x_{i-1}$	D0	" down

Given a subexpression $\underline{e}$ of $\underline{x}$ its (Deodhar) defect is

$$\operatorname{def}(\underline{e}) := \#\text{ of U1's} - \#\text{ of D1's}.$$

Eg: $\underline{w} = sts$ $\qquad \underline{e} = 100$ $\quad \operatorname{def}(e) = 0$ $\qquad \underline{e} = 001$ $\quad \operatorname{def}(\underline{e}) = 2$.

Hecke algebra: assoc. unital algebra over $\mathbb{Z}[v^{\pm 1}]$ generated by $\{H_s \mid s \in S\}$ subject to

$\mathbb{H}$
$$H_s^2 = (v^{-1}-v)H_s + 1 \qquad \forall s \in S$$
$$\underbrace{H_s H_t \cdots}_{m_{st}} = \underbrace{H_t H_s \cdots}_{m_{st}} \qquad \forall \{s,t\} \overset{f}{\subset} S.$$

If we set $v=1$ then $\mathbb{H} \cong \mathbb{Z}W$ the group algebra of W.

Given any rex $\underline{x} = s_1 \ldots s_m$ set $H_x := H_{\underline{x}} := H_{s_1} \ldots H_{s_m}$, does not depend on rex $\underline{x}$.

Thm: $\mathbb{H}$ is a free $\mathbb{Z}[v^{\pm 1}]$-module with basis $\{H_x \mid x \in W\}$.

Check $H_s^{-1} = H_s + (v - v^{-1})H_{id}$. Hence each H_x is invertible. Define

× Kazhdan-Lusztig involution $h \mapsto \overline{h} : H_x \mapsto H_{x^{-1}}^{-1}$, $\mathbb{Z}[v^{\pm 1}] \ni p \mapsto \overline{p} := p(v^{-1})$.

× ω anti-involution: $h \mapsto \omega(h) : H_x \mapsto H_x^{-1}$, $p \mapsto \overline{p}$.

Set $\underline{H}_s := H_s + vH_{id}$ Kazhdan-Lusztig generator, $\overline{\underline{H}_s} = \omega(\underline{H}_s) = \underline{H}_s$.

Thm: There exists a unique basis $\{\underline{H}_x \mid x \in W\}$ s.t.

1) $\overline{\underline{H}_x} = \underline{H}_x$ (self-duality)

2) $\underline{H}_x = H_x + \sum_{y<x} h_{y,x} H_y$ (Bruhat upper triangularity) $h_{y,x} \in v\mathbb{Z}[v]$.

$\{\underline{H}_x\}$ KL basis, $h_{y,x}$ KL polynomials.

Uniqueness: exercise. Existence: post-poned!

Given $h = \sum h_x H_x$ the standard trace is $\varepsilon(h) = h_{id}$. (Check: trace $\Leftrightarrow \varepsilon(ab) = \varepsilon(ba)$).

The standard form on $\mathbb{H}$ is $(a,b) = \varepsilon(\omega(a)b)$. We have

$$(pa, qb) = \overline{p}q(a,b) \qquad \forall p,q \in \mathbb{Z}[v^{\pm 1}] \qquad \text{(semi-linear)}$$

$(\underline{H}_s a, b) = (a, \underline{H}_s b)$ $(a\underline{H}_s, b) = (a, b\underline{H}_s)$ ($\underline{H}_s$ self-biadjoint).

When we categorify, products $\underline{H}_{\underline{x}} := \underline{H}_{s_1} \ldots \underline{H}_{s_m}$ will play an important role.

Deodhar's lemma: Let $\underline{x} = s_1 \ldots s_m$ be an expression.

1) $$\underline{H}_{\underline{x}} = \sum_{\underline{e} \text{ subexp } \underline{x}} \cancel{v^{def(\underline{e})}}\, v^{def(\underline{e})} H_{\underline{x}^{\underline{e}}}.$$

2) $$\varepsilon(\underline{H}_{\underline{x}}) = \sum_{\substack{\underline{e} \text{ subexp } \underline{x} \\ \underline{x}^{\underline{e}} = id}} v^{def(\underline{e})}.$$

Remark: One can also present $\mathbb{H}$ using $\{\underline{H}_s \mid s \in S\}$. See the exercises.