LECTURE 2.2 DRAWING SBIM IN RANK ≤ 2

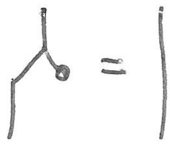
We've defined $\otimes$-cat $\mathbb{BSB}im$, so should draw morphisms. If you're new to this, think of a diagram as a better encoding of a morphism – much easier than symbols. They have a life of their own too – can define a diag. cat... not for now.

WARMUP #1: $S=\{s\}$, $\mathbb{BSB}im$ consists of $B_s^{\otimes n}$ 

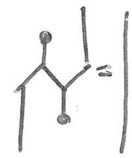
B_s is a frobenius algebra in $\mathbb{BSB}im$ so have 4 structure maps:

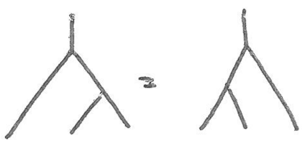
$\overset{R}{\underset{B_s}{\uparrow \mu}}$ $\overset{fg}{\underset{f\otimes g}{\uparrow}}$  $\overset{B_s}{\underset{R}{\uparrow}}$ $\overset{\Delta_s = \frac{\alpha_s}{2}\otimes 1 + 1\otimes \frac{\alpha_s}{2}}{\underset{1}{\uparrow}}$  $\overset{B_s}{\underset{B_s\otimes B_s}{\uparrow}}$ $\overset{\partial_s(f)\otimes 1}{\underset{1\otimes f\otimes 1}{\uparrow}}$  $\overset{B_s\otimes B_s}{\underset{B_s}{\uparrow}}$ $\overset{1\otimes 1\otimes 1}{\underset{1\otimes 1}{\uparrow}}$ 

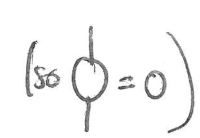
degree +1 +1 −1 −1

<u>Example computation:</u> 

LHS $\partial_s(\frac{\alpha_s}{2})\otimes 1 + \partial_s(1)\otimes\frac{\alpha_s}{2} = 1\otimes 1$ ↥ $1\otimes\frac{\alpha_s}{2}\otimes 1 + 1\otimes 1\otimes\frac{\alpha_s}{2}$ ↥ $1\otimes 1$

RHS $1\otimes 1$ ↑ $1\otimes 1$. 

 since $\partial_s(f)$ symmetric.

<u>Another generator:</u> $\boxed{f}$ $\overset{R}{\underset{R}{\uparrow}}$ $\overset{f}{\underset{1}{\uparrow}}$ <u>Example:</u>  $= \boxed{\alpha_s}$  $= \boxed{\partial f}\,|$ 

<u>Thm:</u>  $\boxed{f}$ generate all R-bimodule morphisms. Let $\cap =$  $\cup =$ 

Relations are: Std Frobenius Relations: Isotopy 

Unit:  Assoc 

⇒ Counit  ⇐ 

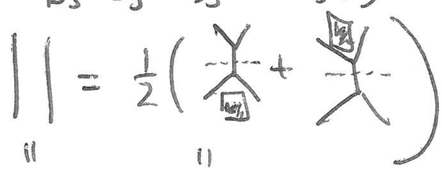
Decomp:  $= \frac{\alpha_s}{2}\,| + |\,\frac{\alpha_s}{2}$ (in general, $\sum a_i | b_i$ for dual bases $\{a_i\}$ $\{b_i\}$ come from $R \cong R^s \otimes R^s(-2)$)

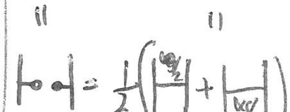
<u>Eval:</u>  $= \mu(\Delta_s) = \boxed{\alpha_s}$  $= |\,\boxed{\partial f}$

<u>Consequences:</u> $f\otimes 1 = \partial_s(f)\left[\frac{\alpha_s}{2}\otimes 1 + 1\otimes\frac{\alpha_s}{2}\right] + 1\otimes s(f)$

$\boxed{f}\,| =$  $\boxed{\partial f} + |\,\boxed{sf}$

$B_s\otimes B_s \cong B_s(1)\oplus B_s(-1)$

 inclusion + projection maps. Check!!



Now $S=\{s,t\}$. Have , but anything else?? How to work with?

Warm up 2: $\mathfrak{sl}_2$ repns. Let (string of n dots) be $V^{\otimes n}$, $V=V_1$ = fund rep basis $\{e_1,e_2\}$

Now $V^{\otimes 2} \underset{\oplus}{\supset} \Lambda^2 V = \mathbb{C}$ so $\exists$ inclusion, proj

Let $\cup$: $1 \mapsto e_1\otimes e_2 - e_2\otimes e_1$ $\quad$ $\cap$: $e_1\otimes e_2 \mapsto -1$, $e_2\otimes e_1 \mapsto +1$, $e_1\otimes e_1,\ e_2\otimes e_2 \mapsto 0$

Check $\bigcirc = -2$

Def: $TL_n = \mathbb{C}\langle$  $\rangle$, get category $\mathcal{TL}$ w/ ob $= \mathbb{N}$, Mor $= \langle \frac{n}{m} \rangle$

Compose = stack, $\bigcirc = -2$.

Rmk: This whole picture has a q-defn, giving Rep $U_q(\mathfrak{sl}_2)$, $V = \mathbb{Q}(q)\langle e_1,e_2\rangle$

$\bigcirc = -(q+q^{-1}) = -[2]$

Thm: $\mathrm{Hom}_{\mathcal{TL}}(m,n) \cong \mathrm{Hom}_{U_q(\mathfrak{sl}_2)}(V^{\otimes m}, V^{\otimes n})$

Now $V_n \underset{\oplus}{\subset} V^{\otimes n}$ so $\exists$ idemp $V^{\otimes n} \to V_n \to V^{\otimes n}$ $\boxed{JW_n} \in TL_n$ Jones-Wenzl Projector

Ex: $\boxed{JW_2} = || + \frac{1}{[2]}$ $\qquad$ $\boxed{JW_3} = ||| + \frac{[2]}{[3]}(\ldots) + \frac{1}{[3]}(\ldots)$

$[3] = [2]^2 - 1$.

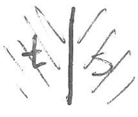
Properties:
1. JW_n Can be defined w/ coeffs $\frac{[k]}{[n]}$ for $k \leq n$
2. Nice recursion formulae
3. $!$ morphism s.t. $\boxed{JW_n} = 0$ and coeff of $||||$ is 1.

exercises

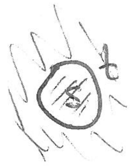
But V_n not the only summand of $V^{\otimes n}$. $V^{\otimes n} \cong V_k^{\oplus c_{n,k}}$, can deduce from

$$V\otimes V_n = V_{n+1} \oplus V_{n-1} \text{ for } n \geq 1, \quad V\otimes V_0 = V_1$$

so TL_n has all these idempotents too (+ isoms within isotypic component)

Now for the real deal: $S=\{s,t\}$. Before I start drawing $\mathbb{BS}$Bim, gonna do something else:  is ${}_{R^t}R_{R^s}(1)$ (rank 2 for either side)  is ${}_{R^s}R_{R^t}$. Mixing two Frob exts!

have  $R^t \leftarrow {}_{R^t}R \otimes_{R^s} R_{R^t}$, $\partial_t(fg) \leftarrow f\otimes g$ $\qquad$  $R \otimes_{R^s} R \leftarrow R^t$, $\frac{\alpha_s}{2}\otimes 1 + 1 \otimes \frac{\alpha_s}{2} \leftarrow 1$

They satisfy  and  $= \partial_t(\alpha_s) = a_{st} = -2\cos\left(\frac{\pi}{m}\right)$ (3)

$= -(q+q^{-1})$ for $q = \zeta_{2m}$

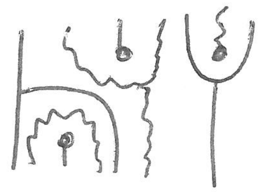
$= -[2]$

So there is a map from $_V{}^{0}T\mathcal{L}$ into $\otimes$s of these guys.

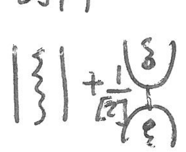
Some 2-colored version of

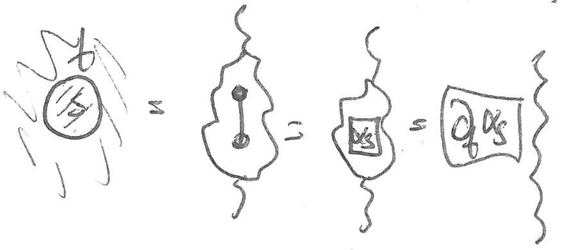
A morphism $_{R^t}R \otimes_{R^s} R \otimes_{R^t} R_{R^s}$ gives a morphism $R \otimes_{R^t} R \otimes_{R^s} R \otimes_{R^t} R \otimes_{R^s} R = B_t \otimes B_s \otimes B_t \otimes B_s$

So every elt of ${}^{0}T\mathcal{L}$ gives a morphism of $\mathbb{BS}$Bim! Def Retract.

 becomes 

$JW_2 = || + \frac{1}{[2]}$  becomes





Still an idempotent (when $[2]\neq 0$ i.e. $m_{st} \geq 2$.)

in fact, $[m_{st}]=0$ gives $B_{sts} \overset{\oplus}{\subset} B_sB_tB_s$.

Just as $V\otimes V_n \cong V_{n+1}\oplus V_{n-1}$ $\quad V\otimes V_0 \cong V_1$

so have $B_s \otimes B_{\underbrace{tst\ldots}_{n+1}} \cong B_{\underbrace{sts\ldots}_{n+2}} \oplus B_{\underbrace{st\ldots}_{n}}$ $\quad m-2 \geq n \geq 1$ $\quad B_s B_t = B_{st}$ (see exercises)

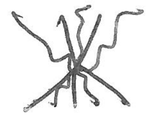
Can now explicitly decompose all alternating $\mathbb{BS}$Bim for $m=\infty$, when all $[m]\neq 0$.

Rmk: This connection $\mathbb{S}\mathrm{Bim}_{A_1} \overset{\sim}{\longleftrightarrow} sl_2$-rep is (quantum) geometric Satake!!!

When $m<\infty$, JW_m undefined, no good way to break down $\underbrace{B_sB_t\ldots}_{k}$ for $k\geq m+1$ using ${}^{0}T\mathcal{L}$

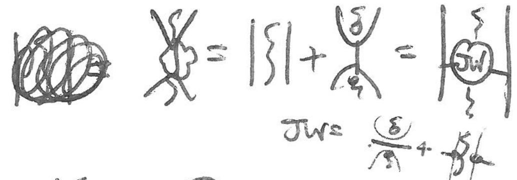
But JW_{m-1} defined, $[m-1]=1$, its even rotation-invariant in TL!!

Gives $B_{\underbrace{sts\ldots}_{m}} \overset{\oplus}{\subset} \underbrace{B_sB_t\ldots}_{m}$ and $B_{tst\ldots} \overset{\oplus}{\subset} B_tB_s\ldots$ a common summand.

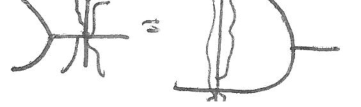
So $\exists$ deg 0 map $B_sB_t\ldots \rightleftarrows B_tB_s\ldots$ draw as  new generator.

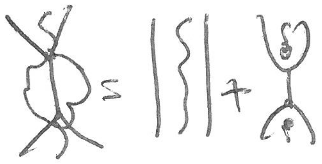
B_{w_0}

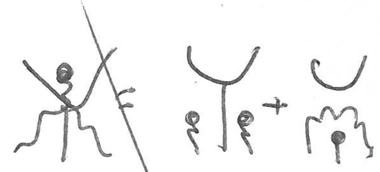
and 

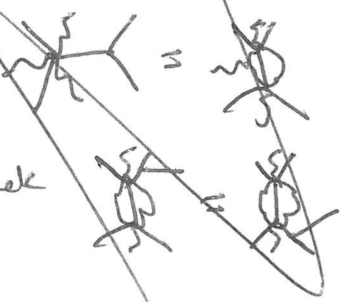
Ex: $m=3$ 

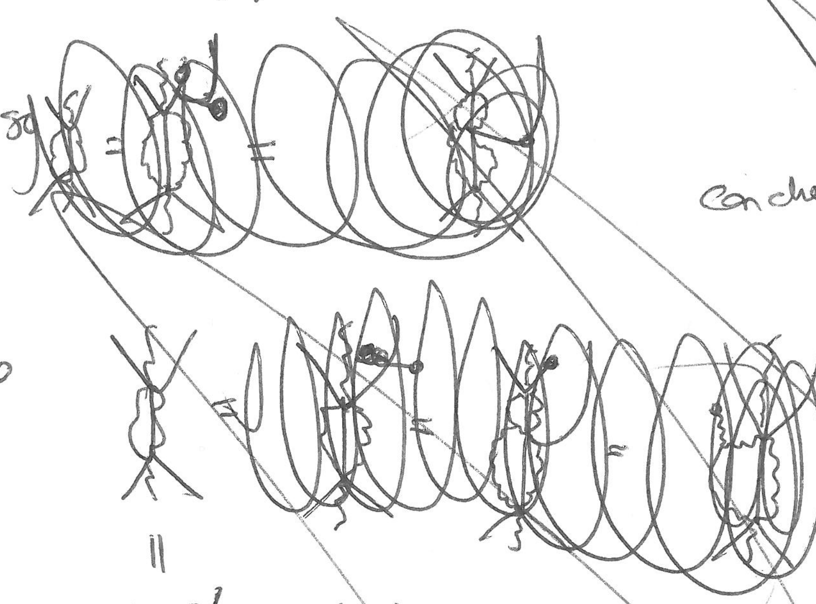
$JW=$ 

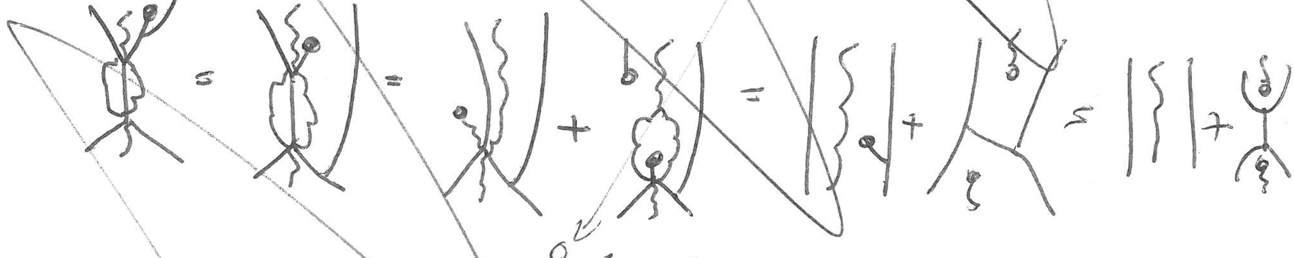
Rels: (1) Isotopy  (2)  (3)  $m=3$ example

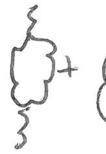
Ex: Why 

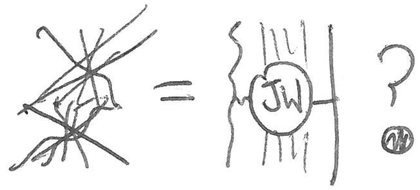
we have 

can check 

So 

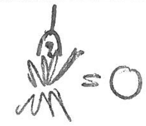


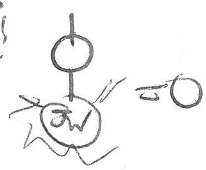
has  both zero

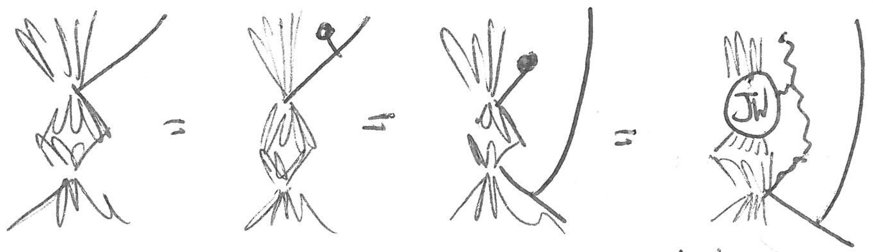
Ex: Why  ?

Claim: 

using ③ twice

Claim: 

Pf: 



all terms except  have 

which kills