2.3 Generators and relations in general

Write $I \overset{f}{\subset} S$ if I is finitary, i.e. $W_I = \langle I \rangle \subset W$ is finite.

Recall that $\mathbb{H}$ is described by generators and relations as follows:

generators: $\quad H_s \qquad \forall \{s\} \overset{f}{\subset} S.$

relations: $\quad H_s^2 = (v^{-1} - v) H_s + 1 \quad \forall \{s\} \overset{f}{\subset} S$

$$\underbrace{H_s H_t \ldots}_{m_{st}} = \underbrace{H_t H_s \ldots}_{m_{st}} \qquad \forall \{s,t\} \overset{f}{\subset} S \quad \text{i.e. } m_{st} < \infty .$$

We will present $\mathbb{S}\mathrm{Bim}$, and see a generalization of this phenomenon occuring.

Abstract set-up: We can phrase the definition of Soergel bimodules as follows

$$\underset{\substack{\text{(not-additive)} \\ \cap \\ R-\mathrm{Bim}}}{\mathbb{BS}\mathrm{Bim}} = \begin{matrix}\text{objects} \\ BS(\underline{w}) \\ \underline{w} \text{ expression}\end{matrix} \xrightarrow[\text{direct sums}]{\text{add}} \underset{\substack{\cap \\ R-\mathrm{Bim}}}{\langle BS(\underline{w}) \rangle_{\oplus}} \xrightarrow[\text{(Karoubi envelope)}]{\text{take direct summands}} \mathbb{S}\mathrm{Bim}.$$

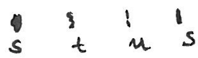
The second two steps are purely formal ("additive Karoubi envelope").

Hence it is enough to describe $\mathbb{BS}\mathrm{Bim}$. Advantage is that objects are completely concrete.

Definition: $\mathcal{D}$ is the monoidal category described as follows:

generators: $\quad s \qquad \forall \{s\} \overset{f}{\subset} S$

We picture an arbitrary object of $\mathcal{D}$ as a sequence of S-coloured dots (finitely many)



on $\mathbb{R}$

generating morphisms: $\quad \boxed{f} \quad \deg = \deg f \qquad f \in R$

deg = -1

deg = 1

$\forall \{s\} \overset{f}{\subset} S$

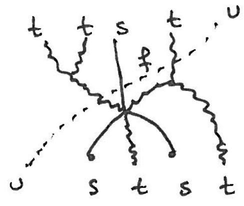
deg 0

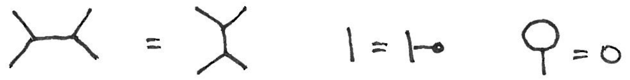
$2m_{st}$-valent vertex $\qquad \forall \{s,t\} \overset{f}{\subset} S.$

An arbitrary morphism is a linear combination of isotopy class of diagrams in $\mathbb{R} \times [0,1]$

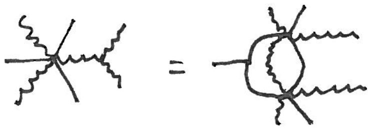
with source = bottom dots and target = top dots.

Ex:



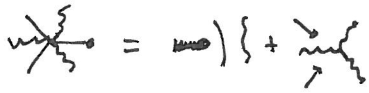
relations:  $\forall \{s\} \overset{f}{\subseteq} S$ "Frobenius relations"

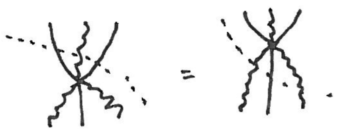
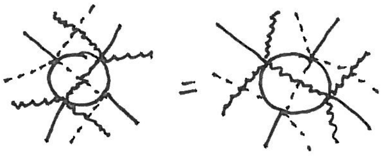
"polynomial relations".

"associativity" 

$\forall \{s,t\} \overset{f}{\subset} S$.

"Jones-Wenzl relation" 

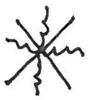
Eg: 

"Zamolodchikov" $A_2 \times A_1$:  A_3:  etc. $\forall \{s,t,u\} \subset S$.

Remark: These relations are homogeneous by inspection, hence Hom spaces in $\mathcal{D}$ are graded.
Remark: It is a MIRACLE that the Zams hold on the nose. This is a subtle (and unsolved) for H_3.
We define a functor $\mathcal{F} : \mathcal{D} \to \mathbb{BSB}im$ as follows:

on objects $\mathcal{F}(\underline{w}) = BS(\underline{w})$

on morphisms:  $\mapsto \begin{matrix} R \\ \uparrow \\ B_s \end{matrix}$ $\begin{matrix} fg \\ \uparrow \\ f\otimes g \end{matrix}$  $\mapsto \begin{matrix} B_s \\ \uparrow \\ R \end{matrix}$ $\begin{matrix} \alpha_s \\ \uparrow \\ 1 \end{matrix}$  $\mapsto \begin{matrix} B_s \\ \uparrow \\ B_s \otimes_R B_s \end{matrix}$ $\begin{matrix} f\otimes(\partial_s g)h \\ \uparrow \\ f\otimes g \otimes h \end{matrix}$  $\mapsto \begin{matrix} B_sB_s \\ \uparrow \\ B_s \end{matrix}$ $\begin{matrix} f\otimes 1 \otimes g. \\ \uparrow \\ f\otimes g \end{matrix}$

image of  difficult to describe explicitly. Unique degree zero map which maps $1\otimes 1 \otimes \dots \mapsto 1 \otimes 1 \otimes \dots$.

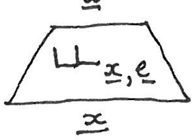
Thm: $\mathcal{F} : \mathcal{D} \to \mathbb{BSB}im$ is an equivalence of monoidal categories. Hence if we let $Kar(\mathcal{D})$ denote the graded additive Karoubian envelope of $\mathcal{D}$ we have an equivalence

$$\mathcal{F} : Kar(\mathcal{D}) \xrightarrow{\sim} \mathbb{SB}im.$$

Libedinsky's light leaves: To understand $\mathcal{H}$ one has to work to get a basis $\{H_x\}$ (cf. Humphreys Bourbaki)

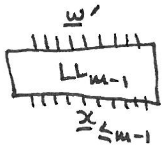
To understand $\mathcal{D}$ one has to really work to get a basis for morphisms.

Fix $\underline{x}$ an expression.

Input: a subexpression $\underline{e} = e_1 e_2 \dots e_m$ of $\underline{x}$ $\rightsquigarrow$ Output: a morphism  in $\mathcal{D}$ where $\underline{w}$ is a rex for $\underline{x}^{\underline{e}}$.

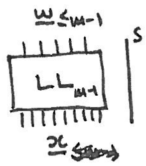
Non-canonical! $\deg(LL_{\underline{x},\underline{e}}) = \operatorname{def}(\underline{e})$.

Construction is inductive. Suppose we know $LL_{m-1} := LL_{\underline{x}_{\leq m-1}, \underline{e}_{\leq m-1}}$.



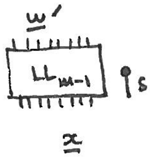
Four possibilities for e_k (assume $x_m = s$).

<u>U1:</u> $LL_m :=$



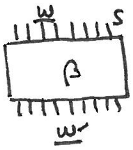
$\deg LL_m = \deg LL_{m-1}$

<u>U0:</u> $LL_m :=$



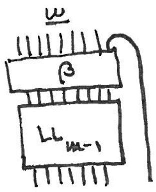
$\deg LL_m := LL_{m-1} + 1$

<u>D:</u> In this case $w_{m-1}s < w$ and hence there exists a sequence of rex moves



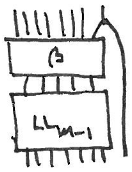
$\underline{w}' \xrightarrow{\beta} \underline{w}'' = \ldots s$

<u>D1:</u> $LL_m :=$



$\deg LL_m = \deg LL_{m-1}$

<u>D0:</u> $LL_m :=$



$\deg LL_m = \deg LL_{m-1} - 1$.

<u>Remark:</u> More generally we allow any rex moves after the construction of LL_m as a "light leaf".

Non-canonical because the choice of β and $\underline{w}''$ is not canonical.

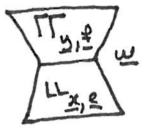
<u>VERBAL:</u> There are interesting computational questions here.

<u>Light leaves theorem:</u> $\mathrm{Hom}(\underline{x}, \emptyset)$ is free as a left or right R-module with basis $\{LL_{\underline{x},\underline{e}} \mid \underline{e} \text{ subexp. s.t. } \underline{x}^{\underline{e}} = id\}$.

<u>Double leaves:</u> Let $M(\underline{x}, y) = \{\underline{e} \text{ subexp of } \underline{x} \text{ with } \underline{x}^{\underline{e}} = y\}$.

For any $\underline{x}, \underline{y}, w \in W$, $\underline{e} \in M(\underline{x}, w)$, $\underline{f} \in M(\underline{y}, w)$ set

$\mathbb{LL}_{\underline{e}, w, \underline{f}} :=$



← vertical flip.

<u>Thm:</u> $\mathrm{Hom}(\underline{x}, \underline{y})$ is a free left or right R-module with basis

$\{\mathbb{LL}_{\underline{e}, w, \underline{f}}\}$.

It is essential to the structure of Soergel bimodules that the double leaves basis gives ① the structure of an "object adapted cellular category". For us the most important point will be that, for any ideal $I \subset W$ (i.e. $x \le y \in I \Rightarrow x \in I$) we have an ideal

D_I := all morphisms factoring through objects $\underline{x}$ where $x \in I$.

<u>If time permits:</u> examples of light leaves.