# LECTURE 24

# HOW TO DRAW BS BIMODULES

Before: How to draw/encode morphisms. Now: how to draw/encode elements.

Choose RHS, find basis of $BS(\underline{w})$ as right $R$-mod. Size is $2^{\ell(\underline{w})}$ and 2 descriptions.

~~Description 1: Zero-One Sequences~~ First, silly observation: any elt of $BS(\underline{w})$ is $\varphi(1\otimes 1\otimes \cdots \otimes 1)$ for some $\varphi \in End_{R\text{-}Bim}(BS(\underline{w}))$ i.e.

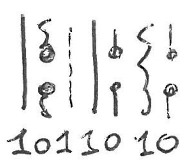
**Description 1:** 01-sequences. Consider $B_s$. We know $\alpha_s\otimes 1$, $1\otimes 1$ are right $R$-basis.

Better basis $c_1 = 1\otimes 1$ canonical

$c_s = \frac{1}{2}(\alpha_s\otimes 1 + 1\otimes\alpha_s)$ also canonical $fc_s = c_s f$ $(\ \cdot\ )(1\otimes 1) = c_s$.

Remember $f| = \partial_s f + | sf$ so $f\otimes g = c_s \partial_s(f)g + c_1 s(f)g$.

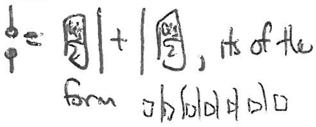
Consider $B_sB_t \cdots B_u$. $\underline{\varepsilon} \in \{0,1\}^{\ell(\underline{w})}$ $\varphi_{\underline{\varepsilon}} =$



$\varphi_{\underline{\varepsilon}}(1\otimes\cdots\otimes 1) \equiv c_{\underline{\varepsilon}} = c_1 c_s c_1 \cdots c_u$

$\deg(c_{\underline{\varepsilon}}) = -d + 2|\underline{\varepsilon}|$

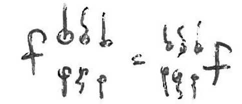
using



its of the form

$1\otimes\cdots\otimes 1 = c_{bot} = c_{00\cdots0}$

$c_{top} = c_{11\cdots1}$ $fc_{top} = c_{top}f$



**Exercise:** Write $fc_{\underline{\varepsilon}}$ in right $R$ basis for $f$ linear.

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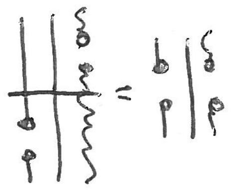
More properties of BSBim: $BS(\underline{w})$ is a commutative ring! $R\underset{R^s}{\otimes} R \underset{R^t}{\otimes} \cdots \underset{R^u}{\otimes} R$ termwise mult.

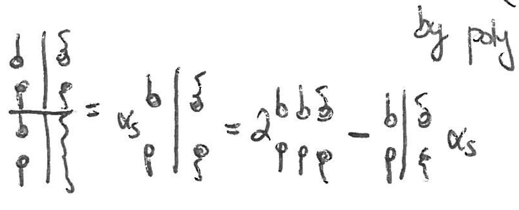
grading is wrong though b/c $c_{bot} = id \in \deg -d$. $\deg(a\cdot b) = \deg a + \deg b + d$.

In 01 sequences mult = stacking, i.e. $\varphi_{\underline{\varepsilon}}(c_{bot})\cdot\varphi_{\underline{\eta}}(c_{bot}) = (\varphi_{\underline{\varepsilon}}\circ\varphi_{\underline{\eta}})(c_{bot})$ ($\varphi_{\underline{\varepsilon}}$ all commute) by poly argument.

**Ex:**



**Ex:**



$BS(\underline{w})$ also has a trace $Tr : BS(\underline{w}) \longrightarrow R$ take coeff of $c_{\underline{\varepsilon}}$ (this is canonical...)

and a pairing $BS(\underline{w}) \times BS(\underline{w}) \longrightarrow R$

$\langle a,b\rangle = Tr(ab)$

**Ex:**

**Ex:** $\langle c_{top}, c_{top}\rangle = \prod \alpha_{s_i}$

Claim: $\langle c_{\underline{\varepsilon}}, c_{1-\underline{\varepsilon}}\rangle = 1$ and $\langle c_{\underline{\varepsilon}}, c_{1-\underline{\varepsilon}'}\rangle = 0$ for $\underline{\varepsilon}' < \underline{\varepsilon}$ in some order

$\Rightarrow$ NONDEGENERATE TO DEGREE 0.

This pairing is invariant: $\deg\langle a,b\rangle = \deg a + \deg b$ ; $\langle af, b\rangle = \langle a, bf\rangle = \langle a,b\rangle f$ ; $\langle fa, b\rangle = \langle a, fb\rangle$ (nothing)

Some abstract bullshit: $\mathrm{Hom}_{\text{Right } R\text{-mod}}(B,R) = \mathbb{D}B$, its an $R$-bimodule. $\mathbb{D}(B(1)) = \mathbb{D}B($

$\mathrm{Hom}_{R\text{-Bim}}(B,\mathbb{D}B)^0$ = space of invariant forms on $B$

isom $\longleftrightarrow$ nondegen to degree 0

Thus $BS(\underline{w})$ has nondegen form $\Longrightarrow$ $BS(\underline{w})$ is self-dual. What about other SBim?

<u>Easy Exercise</u>: $R_w$ std bim is self-dual.

Thus SCT $\Longrightarrow$ $B_w \overset{\oplus}{\subset} BS(\underline{w})$ so $\mathbb{D}B_w \overset{\oplus}{\subset} \mathbb{D}BS(\underline{w})$, ! property $\Longrightarrow$ $B_w \cong \mathbb{D}B_w$

rex $\nearrow$ $\boxed{\text{All indecomp SBim are self-dual}}$ $\Longrightarrow$ $\exists$ nondegen invt form.

What is this form? Clearly $B_w$ inherits a form from $BS(\underline{w})$.

<u>Exercise</u>: $c_{bot} \in B_w$ (use support filtration). Thus restricted form is nonzero.
$c_{top} \in B_w$ (use duality)

<u>Cor</u>: Suppose S. Conj, so that $\mathrm{End}(B_w)^0 = \mathbb{R}$. Then any nonzero invt form is the unique nondegen invt form (up to $\mathbb{R}^\times$).

We'll prove this en route to proving S. Conj later. Example to come

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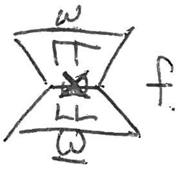
Second description of elts of $BS(\underline{w})$, also parametrized by $2^d$ terms...

Any elt is $\varphi(c_{bot})$ for some $\varphi \in \mathrm{End}(BS(\underline{w}))$. Well, $\varphi = \sum$  $f_{..}$

What is $\varphi(c_{bot}) = ?$

<u>Prop</u>: For each $x$ appearing as a subexp. of $\underline{w}$, $\exists!$ subexp $\mathrm{can}_x$ s.t.

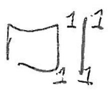
$LL_{\mathrm{can}_x}: BS(\underline{w}) \to BS(\underline{x})$ sends $c_{bot}$ to something nonzero!

It has maximal degree and is minimal in the Bruhat path dominance order.

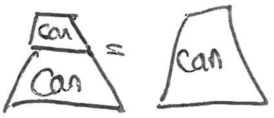
Moreover, $LL_{\mathrm{can}_x}(c_{bot,\underline{w}}) = c_{bot,\underline{x}}$

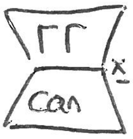
<u>Pf</u>: rex move sends $c_{bot} \to c_{bot}$. So must have only ups!

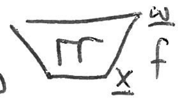
If U0  If D0  $\mapsto 0$ Exercise $\Rightarrow$ the rest of the prop

If U1  If D1  $\mapsto 0$

**Rmk:** $LLcan_x$ depends on a choice of rex move, but two choices differ by a term ③
which kills $c_{bot}$.

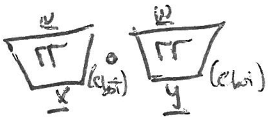
**Rmk:** 

So $\varphi(c_{bot}) = \sum$ 

$f(c_{bot}) = \sum$ 

$f \; (c_{bot,\underline{x}})$ finally, given by $Z^d$ terms.

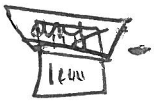
upwards light leaves are injective

In this description, what is multiplication 

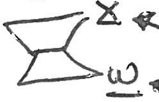
$= ?$ Not obvious!

COB to description 1?

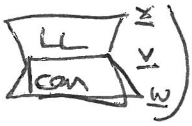
One nice fact:

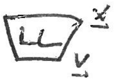
**Claim:** If $\varphi(c_{bot}) = 0$ then coeff of 

is zero, i.e. its in lower terms $\mathcal{D}^{<w}$.

$\varphi = \sum$ 

← anything
← red exp

**Pf:** $\varphi(c_{bot}) = \sum \Big($ coeff of 

$\Big) \cdot$ symbol 

↖ lin indep.

in particular, this coeff is zero

Will use later.

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TIME FOR QUESTIONS