3.2 Hodge theory and Lefschetz linear algebra

Goal: axiomatize the properties satisfied by the real singular / de Rham cohomology of smooth projective varieties. We will then show that these properties hold for Soergel bimodules $\overline{B_x} := B_x \otimes_{\mathbb{R}} \mathbb{R}$.

Fix: $H = \bigoplus H^i$ a finite dimensional graded $\mathbb{R}$-vector space.

$\langle -,- \rangle : H \times H \to \mathbb{R}$ a symmetric non-degenerate graded form.

$\langle H^i, H^j \rangle = 0$ if $i \neq -j$.

Hence if $b_i = \dim H^i$ then $b_i = b_{-i}$ $\forall i \in \mathbb{Z}$.

Example: If M is a compact manifold of dimension $2n$ set $H^i = H^{i+n}(M;\mathbb{R})$ (singular or de Rham cohomology). We let $\langle -,- \rangle$ be the intersection pairing $\langle \omega_1, \omega_2 \rangle = \int_M \omega_1 \wedge \omega_2$. This is symmetric if $H^i(M;\mathbb{R})$ vanishes for odd i "parity vanishing". The shift is to ensure that degree 0 is the mirror of Poincaré duality.

A Lefschetz operator is a map $L : H^\bullet \to H^{\bullet+2}$ s.t. $\langle Lx, y \rangle = \langle x, Ly \rangle$ for all $x, y \in H$.

Example: With M as above, multiplication by any degree 2 class in $H^2(M;\mathbb{R})$ gives a Lefschetz operator.

Def: A Lefschetz operator L satisfies the hard Lefschetz theorem (hL) if $L : H^{-i} \to H^i$ is an isomorphism.

Exercise: If $\mathfrak{sl}_2(\mathbb{R}) = \mathbb{R}f \oplus \mathbb{R}h \oplus \mathbb{R}e$ then a Lefschetz operator satisfies the hard Lefschetz theorem $\Leftrightarrow$ there exists an action of $\mathfrak{sl}_2(\mathbb{R})$ on H s.t. $e = L$ and $hx = ix$ for all $x \in H^i$, (action is unique).

Example: If $X \subset \mathbb{P}^n\mathbb{C}$ be a smooth projective variety ~~with $H^i(X;\mathbb{R})=0$ for i odd.~~

Then L = multiplication by $c_1(\mathcal{O}(1))$ satisfies the hard Lefschetz theorem.

VERBAL: weak Lefschetz easy

hard Lefschetz, theoreme de Lefschetz vache, ... hard!

If L satisfies (hL) then one has a primitive decomposition

$$H = \bigoplus_{i\geq 0} \underbrace{\Big(\bigoplus_{i \geq j \geq 0} L^j P_L^{-i} \Big)}_{sl_2 \text{ isotypic component}} \qquad \text{where } \underset{\substack{\uparrow \\ \text{"lowest weight vectors"}}}{P_L^{-i}} := \ker L^{i+1} \subset H^{-i}.$$

$\langle -,- \rangle$ pairs H^i and H^{-i}. L^i identifies them.

Lefschetz form: $(\alpha,\beta)_L^{-i} := \langle \alpha, L^i \beta \rangle$ (symmetric).

(hL) $\iff$ non-degeneracy of $(-,-)_L^{-i}$ $\forall i \geq 0$.

Exercise: $(L\alpha, L\beta)_L^{-i+2} = (\alpha,\beta)_L^{-i}$ $i \geq 2$.

(hL): $H^{-i} = P_L^{-i} \oplus LP^{-i-2} \oplus \dots$ is orthogonal wrt Lefschetz form.

Hodge-Riemann bilinear relations: Assume $H^{odd}=0$ or $H^{even}=0$.

Let min be such that $H^{min} \neq 0$ but $H^j = 0$ for $j < min$.

$(H, \langle -,- \rangle, L)$ satisfies the Hodge-Riemann bilinear relations (HR) if the restriction of $(-,-)_L^{min+2i}$ to P_L^{min+2i} is $(-1)^i$-definite.

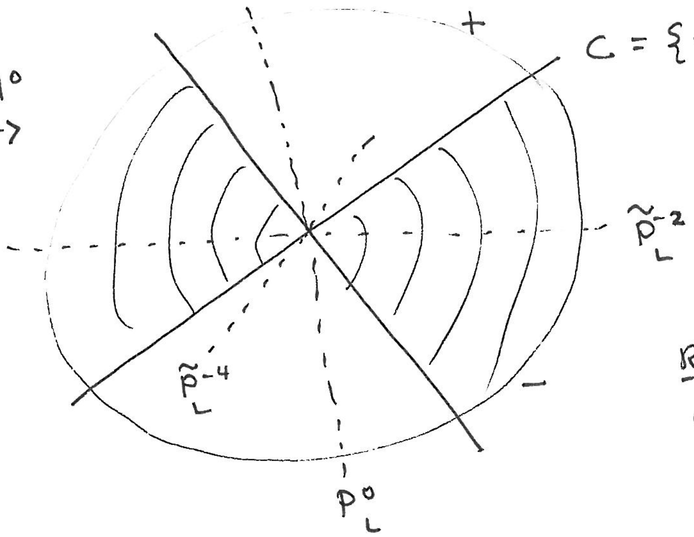
$$H^{min+2i} = \underset{+}{L^i P_L^{min}} \oplus \underset{-}{L^{i-1} P_L^{min+2}} \oplus \dots \oplus \underset{(-1)^i}{P_L^{min+2i}} \quad \text{(orthogonal)} \qquad \Rightarrow \text{(hL)}.$$

$$\iff \text{signature of } (-,-)_L^{min+2i} = \underbrace{\sum_{i\geq 0} (-1)^i \dim P_L^{min+2i}}_{\text{only depends on Betti numbers } \{b_i\}}.$$

Picture: Assume $H^{odd} = 0$ and consider $H^0 = P_L^0 \oplus \dots \oplus L^{-\min/2} P_L^{\min}$. $\tilde{P}_L^{-i} := L^{-i/2} P_L^{-i}$.

FIXED $\to H^0$
$\to \langle -,- \rangle$

$C = \{ v \in H^0 \mid \langle v, v \rangle = 0 \}$ "isotropic cone"



Rmk: dashed spaces move when we move L!

de Cataldo-Migliorini proof strategy: (hL), (HR) in dim n $\xrightarrow[\text{Lefschetz}]{\text{weak}}$ (hL) in dim $n+1$ $\xrightarrow[\text{lemma}]{\text{limit}}$ (HR) in dim $(n+1)$

Limit lemma: Suppose that $[0,\infty) \to \mathrm{Hom}(H, H(2)) : \jmath \mapsto L_\jmath$ is a continuous family of Lefschetz operators satisfying the hard Lefschetz theorem. If there exists $\jmath \in [0,\infty)$ such that $L_\jmath$ satisfies (HR) then all $L_\jmath$ satisfy (HR).

Proof: All $L_\jmath$ satisfy (hL) $\Leftrightarrow$ $(-,-)^{-i}_{L_\jmath}$ is a continuous family of symmetric non-degenerate forms.

Hence all have same signature. Hence all satisfy (HR)! $\square$

Weak Lefschetz substitute: Suppose $H, \langle -,- \rangle_H, L_H, W, \langle -,- \rangle_W, L_W$ are as above with L_H, L_W Lefschetz operators. Suppose we have a map $\phi : H \to W$ of degree 1 s.t.

1) ϕ injective in degrees ≤ -1.
2) $\langle \alpha, L\beta \rangle_H = \langle \phi\alpha, \phi\beta \rangle$.
3) W satisfies (HR).

Then L_H satisfies (hL).

Proof: Fix $0 \neq h \in H^{-i}$, with $i \leq -1$ and consider $\phi(h) \in W^{-i+1}$. Then either

1) $0 \neq L^i(\phi(h)) = \phi(L^i h) \Rightarrow L^i h \neq 0$.

3) $0 = L^i(\phi(h)) \Rightarrow \phi(h) \in P_L^{-i+1} \Rightarrow 0 \neq (\phi(h), \phi(h))_L^{-i+1} = \langle \phi(h), L^{i-1}\phi(h) \rangle = \langle h, L^i h \rangle$.

In any case $\cancel{\phi(h)}$ $L^i h \neq 0$. $\square$