LECTURE 4.2 — HL for SBim — 1

Recall our grand induction: $B_x \overset{\oplus}{\subset} BS(\underline{x})$, has induced form, has $L = \rho\circ$ / $B_x B_s \overset{\oplus}{\subset} BS(\underline{x})B_s$ ($\overline{B_x}$), has induced form, has $L_\varsigma = \rho\circ + \mathrm{id}_{B_x}\rho\circ$

Spse "everything" for $< xs$. WTS

$$hL(x;s)_\varsigma \;\; \varsigma\geq 0 \xrightarrow[\substack{\text{limiting argument} \\ HR(x)\Rightarrow HR(x;s)_{\varsigma>>0}}]{} HR(x;s)_\varsigma \;\; \varsigma\geq 0 \xRightarrow[\varsigma=0]{} \begin{matrix} S(xs) \\ hL(xs) \\ HR(xs)\end{matrix}$$

How to finish the loop?

$S(<xs) \Longrightarrow$ (2) F_x has diagonal miracle, so $F_x^j = \bigoplus_{z\in()} B_z(\overset{\circ}{j}) \overset{\oplus}{\subset} F_{\underline{x}}^j = \bigoplus BS(\underline{y})(\overset{\circ}{j})$, $\underline{y} = x_{\underline{\ }j}$

$\Downarrow$

(1) $\begin{matrix} hL(<xs) \\ HR(<xs)\end{matrix} \Longrightarrow$ (3) RoHR(x) $:=$ $F_x^j(-j)$ satisfies HR, hL using induced form, L.

$\Downarrow$

$\begin{matrix} hL(<x;s) \\ HR(<x;s)\end{matrix} \Longrightarrow$ (4) We know a LOT about $F_xF_s = B_xB_s \rightarrow \boxed{\quad\quad} \rightarrow$
enough to deduce $hL(x;s)_\varsigma$

Need to explain 3 things: (1) How we get RoHR (its a similar argument)
(2) Why F_xF_s has anything to do w/ $hL(x;s)_\varsigma$
(3) Using facts about F_xF_s to deduce $hL(x;s)_\varsigma$

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(1) **Prop 1:** $\begin{matrix} S(\leq x) \\ HR(y;s) \\ y<x \end{matrix} \Longrightarrow$ RoHR(x)

Pf: $F_x \overset{\oplus}{\subset} F_yF_s$ so $\overset{\text{L-stable}}{F_x^j} \overset{\oplus}{\subset} F_y^jB_s \oplus F_y^{j-1}(1)$

$y = xs < x.$

$\Rightarrow F_x^j(-j) \overset{\oplus}{\subset} \underbrace{F_y^jB_s(-j)}_{\text{has HR?? Not really, but...}} \oplus \overset{\text{Has HR}}{F_y^{j-1}(-(j-1))}$

$$F_y(-j) = \bigoplus B_z = B^\uparrow \oplus B^\downarrow \quad \text{where} \quad B^\uparrow = \bigoplus_{zs>z} B_z \qquad B^\downarrow = \bigoplus_{zs<z} B_z$$

Now $\overline{B^\uparrow B_s}$ has HR by $HR(y;s)$. OTOH $B^\downarrow B_s \cong B^\downarrow(1) \oplus B^\downarrow(-1)$ L-stable decomp

Should think that form on $B^\downarrow B_s$ is nondeg but pairs $B^\downarrow(1)$ against $B^\downarrow(-1)$. Regardless, its clear that $\langle\, ,\rangle_{B^\downarrow B_s}\big|_{B^\downarrow(1) \text{ or } B^\downarrow(-1)} = 0$ for degree reasons (V has hL $\Rightarrow V(k)$ has $(\,)_L = 0$)

The map $F_x^j(-j) \longrightarrow B^\downarrow(1) \oplus B^\downarrow(-1)$ lands entirely inside $B^\downarrow(1)$ (NO maps $B_z \to B_{z'}(-1)$ we have S. Conj)

and won't contribute to the Lefschetz pairing!

$\overline{F_x^j(-j)} \xrightarrow{\text{project}} \overline{B^\uparrow B_s} \oplus \overline{F_y^{j-1}(-(j-1))}$ is an isometry of Lef forms ②

also injective – the map $F_x^j(-j) \to B^\downarrow(1)$ is in the "maxl ideal", so the rest of the projection must be injective (split)

So $\overline{F_x^j(-j)}$ is L-stable summand of HR w/ balanced gdl dim $\Rightarrow$ HR.

Key idea: F_yF_s didn't have HR, wasn't perverse (at least, until you look at homotopy) but Lefschetz forms don't care! Image had HR. Similar tricks to come.

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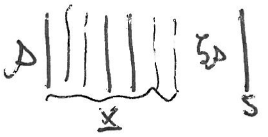
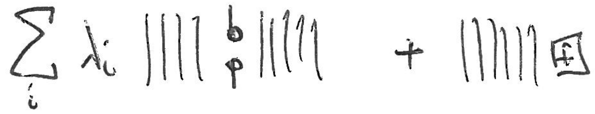
② Consider L_ζ on $BS(\underline{x})B_s$



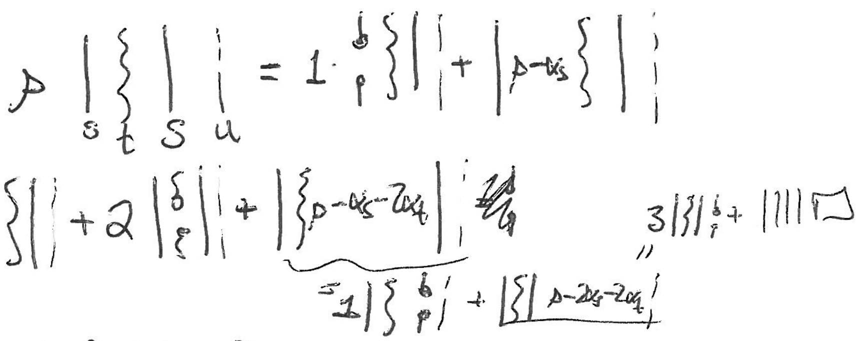
$xs > x$

Multiplication by this is equal to the endomorphism



for some scalars λ_i and linear poly f.

Ex: $\rho = \omega_s + \omega_t + \omega_u$



If had ζ, would get $3+\zeta$ for last coeff.

Claim: $\lambda_i > 0$. This was on the exercises. If $wt > w$ then $\langle w\rho, \alpha_t^\vee \rangle > 0$

also $\langle \zeta\rho, \alpha_s^\vee \rangle > 0$ for $\zeta > 0$.

Recall Geordie's Weak Lefschetz Substitute – to show hL on V we want a map of degree $+1$ $V \xrightarrow{\Phi} W$ (has HR), commuting w/ L, s.t. $\langle v, Lw \rangle = \langle \Phi v, \Phi w \rangle$ and Φ injective in degrees <0.

Well, we have $BS(\underline{x})B_s \xrightarrow{\sum_i \text{||||pill}} \bigoplus BS(\underline{y})$. $\underline{y} = \underline{xs} \setminus 1$. Not quite right – stupid scalars muck it up.

$\sum \langle \text{||||pill}(v), \text{||||pill}(w) \rangle = \sum \langle v, \text{||||}\,\text{|||}\, w \rangle$, want $\sum \lambda_i \langle v, \text{||||}\,\text{|||}\, w \rangle$.

2 choices. ① Let $\Phi = \sum \sqrt{\lambda_i}\ \text{||||pill}$ works since $\lambda_i > 0$.

② Let $\Phi = \sum \text{||||pill}$ coeff 1, but renormalize $\langle , \rangle$ on $BS(\underline{y})$ by $\lambda_i > 0$ ← won't screw up HR

Why is Φ injective in degrees <0? B/c Φ is first differential in $F_x F_s$! (or renormalized version) ③

$BS(\underline{x})B_s \xrightarrow{\sum |||\cdot||} \bigoplus_{y \in \underline{x}\setminus 1} BS(\underline{y})$ ↙ $\boxed{\text{no shift, } F_{xs}^1(-1)}$, and cohomology was $R_{xs}(-\ell(xs))$, injective in degrees $< \ell(xs)$ even

But why would $BS(\underline{y})$ have HR. It doesn't... Can do something like in Prop 1.

Thm: $RoHR(x)$ etc. $\Rightarrow hL(x,s)_\zeta$ for $\zeta \geq 0$, $xs > x$, and also $hL(x,s)_\zeta$ for $\zeta > 0$, $xs < x$.

Pf: ① $\zeta > 0$, $xs < x$. Don't need Ro. Comp at all, fix a basis + compute, exercise. Gives hL. HR is a limit exercise

② $\zeta > 0$, $xs > x$. We have diagonal miracle for F_x. $F_x^0 = B_x$

$F_x^1 = \bigoplus_{z<x} B_z(1) = B^\uparrow(1) \oplus B^\downarrow(1)$

$\overline{(F_x F_s)^0} \xrightarrow{\Phi} \overline{(F_x F_s)^1} = \overline{B^\uparrow B_s} \oplus \overline{B^\downarrow B_s} \oplus \overline{B_x}$ $\boxed{\checkmark}$

$\overline{(F_x F_s)^0} = \overline{B_x B_s}$

has HR by $HR(z,s)_\zeta$, $z<x$ — has HR by part ① — has HR wrt L (no ζ term)

Rmk: Can't split $B^\downarrow B_s$ into $B^\downarrow(1) \oplus B^\downarrow(-1)$ as before. Splitting does not commute w/ L_ζ. (What is L_ζ on the RHS?) only have L, want ζs not to contribute

→ maps are 

③ $\zeta = 0$ $xs > x$

$L_\zeta = L$.

$\overline{B_x B_s} \xrightarrow{\Phi} \overline{B^\uparrow B_s} \oplus \overline{B^\downarrow B_s} \oplus \overline{B_x}$

$\overline{B^\downarrow B_s} \cong \overline{B^\downarrow(1)} \oplus \overline{B^\downarrow(-1)}$ ← now L commutes w/ decomp!

Can't quite use the same trick immediately to finish. Before it was a degree 0 map. Now degree 1, then hit $B^\downarrow(-1)$. And will – that's the stuff that splits off in $B_x B_s = B_{xs} \oplus$ rest !!!

Now $F_x F_s \in K^{\geq 0} \Rightarrow$ only neg shift allowed in minimal complex $\Rightarrow$ the $\overline{B^\downarrow}(1)$ term must contract against something in degree 2. We can ignore it.

$\overline{B_x B_s} \xrightarrow{\Phi} \overline{B^\downarrow}(-1) \oplus \left(\overline{B^\uparrow B_s} \oplus \overline{B_x}\right)$ ← has HR for L w/ shift. ← has HR for L

ⓐ If $\overline{B_x B_s} \to \overline{B^\downarrow}(-1)$ nonzero, hL holds. $L^k \Phi(v) \neq 0$ by hL on $\overline{B^\downarrow}$, so $L^k v \neq 0$. deg $v = -k$, The stupid shift!

ⓑ If $\overline{B_x B_s} \to \overline{B^\downarrow}(-1)$ zero, then Φ goes into something w/ HR. $\boxed{\checkmark}$