Lecture 0.3 Categorifications of braid groups.

X space $\rightsquigarrow$ $\pi(X)_{\leq 1}$ "fundamental groupoid" $\uparrow$ isoclasses $\pi(X)_{\leq 0}$

objects: points of X morphisms: paths $x \to y$ / homotopy.

$\mathrm{End}(x) = \pi_1(X, x)$.

$\rightsquigarrow$ $\pi(X)_{\leq 2}$ "fundamental 2-groupoid"

objects: points •

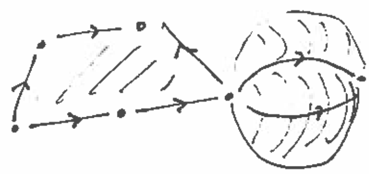
morphisms paths $\underset{x}{\bullet} \longrightarrow \underset{y}{\bullet}$

2-morphisms

$\rightsquigarrow$ $\pi(X)$ fundamental ∞-groupoid

Goal: diagrammatic description of $\pi(X)_{\leq 2}$ "2-version of group presentation".

Assume X is a 2-complex:

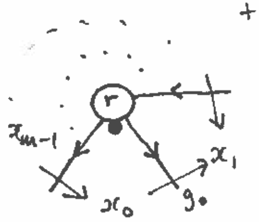


~~objects~~: 0-cells: X_0.

1-cells: arrows $\overset{y}{\bullet} \overset{g}{\longleftrightarrow} \overset{x}{\bullet}$. X_1. "colours". (without loops), (then we don't need tokens).

2-cells: loops $x_0 \longrightarrow x_1 \longleftarrow x_2 \longrightarrow \dots \longrightarrow x_m = x_0$. $r \in X_2$.

To describe $\mathscr{C}$ "we dualize":

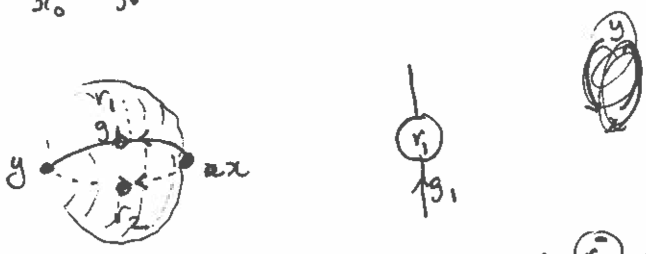


+ reflection

morphisms are composed out of relators.

$y \uparrow x$

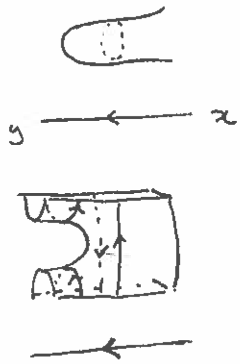
identity on g_1.



Example: $\underset{x}{\bullet} \longrightarrow \bullet \longleftarrow \underset{y}{\bullet}$

endomorphisms of identity

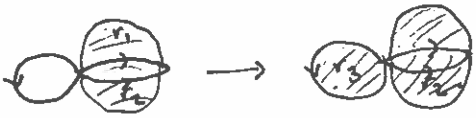
Modulo relations: 



"cancelling spiders": 

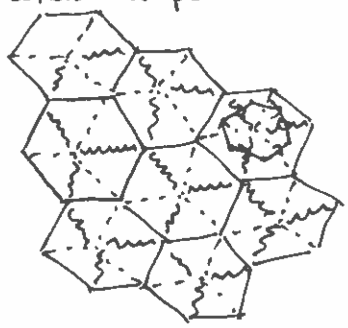
<u>Thm (?)</u>: $\mathcal{C}_X \cong \pi(X)_{\leq 2}$ (equivalence of 2-cats). See Fenn "Techniques of geometric topology". Beautiful book!!

<u>Exercise</u>: Explore the functor

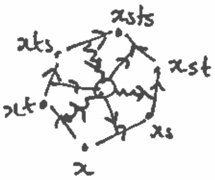


Now assume that W is ~~finite~~ rank 3 Coxeter group.

Consider the 2-complex X — "dual Coxeter complex".

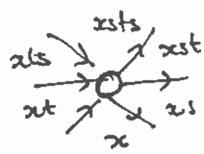


- objects $w \in W$. X_0
- X_1: $x \longrightarrow xs$ if $x < xs$.
- X_2:



$\rightsquigarrow$ $\pi(X)_{\leq 2}$ is equivalent to the category with objects $w \in W$.

morphisms $xs \downarrow x$

2-morphisms: planar diagrams ~~of~~ with generators: 

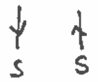
Also: $\mathrm{End}(x) = \pi_2(|(W,S)|, x) = \begin{cases} \mathbb{Z} & \text{if } W \text{ is finite} \\ 0 & \text{if } W \text{ is infinite.} \end{cases}$

This explains where EW thm comes from.

Any closed diagram in W $\rightsquigarrow$ element of $\mathcal{P}_{\text{dual Coxeter graph}}$.

Braid groups:

Now consider ~~&~~ the variant $\mathcal{B}$ ~~where we forget~~ the same category where we forget the labels in regions.

- one object
- generators: 
- morphisms  $2m_{st}$-valent vertices.
- relations 

Zamo.

Then $\mathcal{B} \cong$ 2-groupoid of B_W.

Rmk: Gives a simple criterion for a braid group to act on a category.

Proof uses the following result of Deligne, which we'd like to remove!