Lecture 5.X

ALGEBRAIC QUANTUM GEOMETRIC SATAKE

First, the $\mathfrak{sl}_2$ example. Rep $\mathfrak{sl}_2$ is a $\otimes$-cat. But it's more than that.

$$\mathrm{Rep}\,\mathfrak{sl}_2 \circlearrowright \mathrm{Rep}\,\mathfrak{sl}_2 = \underset{\oplus V_{2n}}{\mathrm{Even}} \oplus \underset{\oplus V_{2n-1}}{\mathrm{Odd}}$$ (no maps in between!)

$\left.\begin{matrix}\mathrm{Odd} \to \mathrm{Odd} \\ \mathrm{Even} \to \mathrm{Even}\end{matrix}\right] \otimes \mathrm{Even}$

$\left.\begin{matrix}\mathrm{Even} \to \mathrm{Odd} \\ \mathrm{Odd} \to \mathrm{Even}\end{matrix}\right] \otimes \mathrm{Odd}$

really a 2-cat w/ 2 objects r, b

$\mathrm{Hom}(r,r) = \mathrm{Even}$ etc

$\mathrm{Hom}(r,b) = \mathrm{Odd}$

(Even is special b/c it has V_0
r, b are indistinguishable)

Fund $\subset$ Rep $\mathfrak{sl}_2$ is sub-2-cat gen by ${}_bV_r$ and ${}_rV_b$. How to draw?

2-colored temperley lieb, as before!

$= -2$.

Now fix Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ for $m_{st} = \infty$ dihedral gp. $S = \{s, t\}$ is not finitary, affine weyl gp.
$\{s\}$ and $\{t\}$ are maxl finitary, Weyl gp

$\mathbb{M}\mathbb{S}\mathrm{Bim}$ & ing $\subset \mathbb{S}\mathbb{S}\mathrm{Bim}$ generated by ${}_{R^t}R_{R^s}$ ${}_{R^s}R_{R^t}$. Way back on tuesday we showed/stated
(R^I, R^J)-bim
for $I, J \subseteq S$

Thm: $\mathrm{Fund} \xrightarrow{\sim_0} \mathbb{M}\mathbb{S}\mathrm{Bim}$ induces an isom $\mathrm{Hom} \to \mathrm{Hom}^0$
(understands all idempotents)

$$\begin{tikzcd} \mathrm{Fund} \arrow[r, "\sim_0"] \arrow[d, phantom, "\cap"] & \mathbb{M}\mathbb{S}\mathrm{Bim} \arrow[d] \\ \mathrm{Kar}(\mathrm{Fund}) = \mathfrak{sl}_2\text{-rep} \arrow[r, "\sim_0"] & \mathrm{Kar}(\mathbb{M}\mathbb{S}\mathrm{Bim}) = \mathbb{S}\mathbb{S}\mathrm{Bim} \end{tikzcd}$$

$\Rightarrow$ ← not so hard.

This is the Algebraic Satake Equivalence (connection to geom later)

Can go further: q-deformation over $\mathbb{Z}[q, q^{-1}]$ w/ $\bigcirc = -[2]$

Cartan matrix $\begin{pmatrix} 2 & -q-q^{-1} \\ -q-q^{-1} & 2 \end{pmatrix}$ over $\mathbb{Z}[q, q^{-1}]$, can still define $\mathbb{S}\mathrm{Bim}^0$.

Thm: When q generic (i.e. in $\mathbb{Q}(q)$), $\mathrm{Fund}_q \xrightarrow{\sim_0} \mathbb{M}\mathbb{S}\mathrm{Bim}^0$

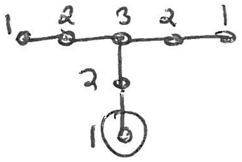
This is Quantum AS.

Let's generalize: $\mathfrak{g}$ $\mathbb{C}$x s.s l.a, reps are graded by $\Lambda_{wt}/\Lambda_{rt} = \Omega$
$V_\lambda \mapsto \bar{\lambda}$.

Why? $\underset{\text{gives central character}}{\Omega} \cong Z(G^{sc})^{*}$. Also $\cong \pi_1(G^{adj})^*$ (Some isom non-canon, have duals, and dual gp)

So $\mathrm{Rep}\,\mathfrak{g} \circlearrowright \mathrm{Rep}\,\mathfrak{g}$ is a 2-cat w/ objects $x \in \Omega$. V_λ sends $x \mapsto x + \bar{\lambda}$.

How to visualize Ω as a <u>set</u>: (for ADE)



affire DD w/ marked extension $\tilde{\Gamma} \supset \Gamma$

②

Vertices labeled 1 enumerate Ω.

vertex $\underset{\tilde{\Gamma}}{\overset{\in}{s}} \leftrightarrow \omega_s \leftrightarrow V_{\omega_s} \mapsto \overline{\omega_s}$. vertex $\odot \mapsto V_0 \to \bar{0}$. $\otimes$ makes gp.

Ex: $\widehat{\mathfrak{sl}_n}$

enumerate w/ $\Lambda^0 V_{std}, \Lambda^1 V_{std}, \dots, \Lambda^{n-1} V_{std}$. $\mathbb{Z}/n\mathbb{Z}$.

Def: $\mathrm{Fund}_{\mathfrak{g}} = \bigotimes_{i \in \Gamma} V_{\omega_i}$

Def: $\mathbb{M}\mathbb{S}\mathrm{Bim}_{W_{aff}}$ Ob: $I \overset{f}{\subset} \tilde{S}$ w/ $W_I \cong W_{fin}$

Mor: gen by

${}_J R_I^{I \cap J}$

note: $I = \tilde{S} \setminus i$
$J = \tilde{S} \setminus j$
$I \cap J = \tilde{S} \setminus i,j$

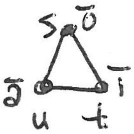
Thm ([illegible] in type $\tilde{A}$ explicitly [new proof !!] or for general using Geom Satake): $\mathrm{Fund}_{\mathfrak{g}} \xrightarrow{\sim} \mathbb{M}\mathbb{S}\mathrm{Bim}_{W_{aff}}$ AS

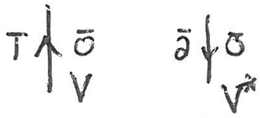
Thm (type $\tilde{A}$): $\mathrm{Fund}_{q,\mathfrak{sl}_n} \xrightarrow{\sim} \mathbb{M}\mathbb{S}\mathrm{Bim}$ for the q-deformed Cartan matrix QAS

(non-sym matrices w/ m odd make much harder to define Frob structures consistently.)

$$\left(\begin{array}{ccccc|c} 2 & -1 & & & & -q^{-1} \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & 2 & & -q \\ \hline -q & & & & -q^{-1} & 2 \end{array}\right)$$

Ex: $\mathfrak{sl}_3$



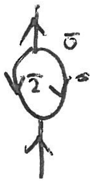


also

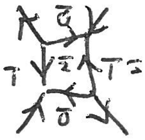
$\Lambda^2 V = V^*$
$\updownarrow$
$V \otimes V$

Relations:

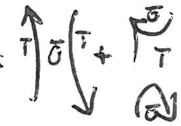
$= -3$ or $-[3]$



$+[2]$ or $+2$

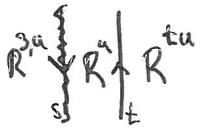


$=$



Now look in
$$\begin{array}{c} \\ \\ \\ \end{array}\left(\begin{array}{ccc} s & t & u \\ 2 & -1 & -q^{-1} \\ -1 & 2 & -q \\ -q & -q^{-1} & 2 \end{array}\right)$$

$\bar{0} \leftrightarrow \tilde{S} \setminus s$



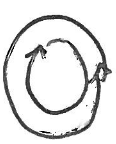
$\mapsto$

using Frob structure on

$R^{t} \underset{R^{st}}{\overset{R^{tu}}{}}$

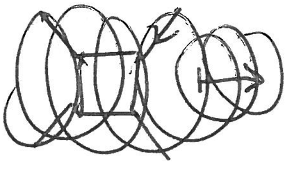
$$\begin{tikzcd} & R^{st} \\ R^t & \otimes \arrow[u] & R^t \\ & R^{tu} & \end{tikzcd}$$

circle

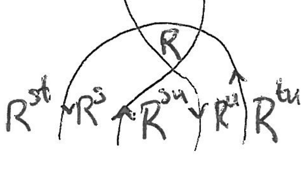


is $\partial_t \partial_s \left(\Delta^t_{tu} \right) \underset{q=1 \text{ case}}{=} \partial_t \partial_s (\alpha_t(\alpha_t + \alpha_u)) = -3$ multiply, take $\partial_t \partial_s$.

(harder to compute $q \neq 1$)



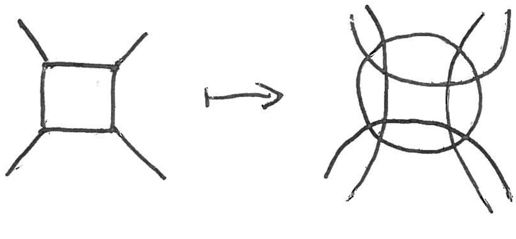
$\mapsto$



where

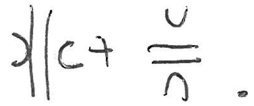
is Ind Ind = Ind Ind
$1 \mapsto 1$.

Tools to do this computation are developed



center is $\begin{array}{c|c|c} R & R^{s} & R \\ \hline R^{t} & R^{st} & R^{t} \\ \hline R & R^{s} & R \end{array}$ can resolve knowing finite Weyl gp stuff

$\parallel$

$JW =$ 

<u>Connection to Geometry</u> Let $G = \check{G}^{ad}$ (though it won't really matter, just easier to state)

$G(K) \equiv G((t))$ think $\mathbb{P}SL_n((t)) = \left(\text{crazy} \right)$

$\pi_0(G(K)) = \pi_1(G) = \Omega$ (this is where $\vee$ came in) think $\begin{pmatrix} t &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix} \begin{pmatrix} t &&& \\ & t && \\ && 1 & \\ &&& 1 \end{pmatrix} \ldots$

$G(O) = G[[t]]$, connected, in $G(K)_0$ but $\begin{pmatrix} t &&& \\ & t && \\ && t & \\ &&& t \end{pmatrix} = \begin{pmatrix} 1 &&& \\ & 1 && \\ && 1 & \\ &&& 1 \end{pmatrix}$ in $\mathbb{P}SL_n$.

Conventional geom satake (sloppily!) says $\mathrm{Perv}_{G(O)}\left(G(K)/G(O)\right) \cong \mathrm{Rep}\, \mathfrak{g}$. as $\otimes$-cat.

But Perv is ~~[illegible]~~ graded, since sheaves supported only on a single component, graded by $\pi_0 = \Omega$

New statement (implied by old) $\mathrm{Perv}_{G(O)\times G(O)}(G(K)) \circlearrowright \mathrm{Perv}_{(G(O))}\left(G(K)/G(O)\right)$

2 cat $\wr\wr$ $\wr\wr$ cat

$\mathrm{Rep}\, \mathfrak{g} \circlearrowright \mathrm{Rep}\, \mathfrak{g}$

Better way of doing it: Spose some other component is $G(K)_0 \cdot g$. (as above for $\mathbb{P}SL_n$)

Then $\mathrm{Perv}_{G(O)\times G(O)}(\text{comp}) = P_{g^{-1}G(O)g \times G(O)}(G(K)_0)$

Let $G_i = g^{-1}G(O)g$ for $i \in \Omega$. $\bigcap G_i = I$ Iwahori.

$G_0 = G(O)$

So 2-cat is $P_{(G_i \times G_j)}(G(K)_0)$ indep of choice of $G^{\vee}$

Now also want to use loop rotation – action of $\mathbb{C}^*$ $t \mapsto \lambda t$.

$H^*_I(pt) = H^*_B(pt) = R_{fin}$ but $H^*_{I\times\mathbb{C}^*}(pt) = R_{aff}$

$H^*_{G(O)}(pt) = H^*_G(pt) = R_{fin}^{W_{fin}}$ but $H^*_{G(O)\times\mathbb{C}^*}(pt) = R_{aff}^{W_{fin}} \rightsquigarrow H^*_{G_i\times\mathbb{C}^*}(pt) = R_{aff}^{W_i}$ $\leftarrow \cong$ to W_{fin}.

Can show all ~~perverse sheaves~~ orbit closures have Bott-Samelson-style resolution $G_i \underset{G_i\cap G_j}{\times} G_j \times \cdots \to G(K)$

so $\mathrm{Kar}(\mathbb{HB}\mathrm{Perv}) = \mathrm{Perv}$. Take Γ_{equiv}, get $\mathbb{HB}\mathrm{Bim}$.