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An Iwahori-Whittaker model for the Satake category

<u>Motivation</u>: representations of p-adic groups

G ~~semisimple of adjoint type~~ split reductive gp scheme over $\mathbb{F}_p$ (split)

<u>Satake isomorphism</u>: spherical affine Hecke algebra

$$\left\{ f : G(\mathbb{F}_p((t))) \to \mathbb{C} \;\middle|\; \begin{array}{l} f(g_1 h g_2) = f(h) \quad \forall g_1, g_2 \in G(\mathbb{F}_p[[t]]) \\ f \text{ nonzero only on finitely many double cosets} \end{array} \right\}$$ with <u>convolution</u>

$$\downarrow \wr$$

$$\mathbb{C}[X_*(T)]^{W_f}$$

$$\wr\,|$$

$$\mathbb{C} \otimes_{\mathbb{Z}} K^0(\mathrm{Rep}(G_k^\vee))$$

Langlands dual gp.

<u>Geometric counterpart</u>: Satake equivalence.

$\mathbb{F}$ alg. closure of $\mathbb{F}_p$ $\rightsquigarrow$ $G_{\mathbb{F}}$ reductive gp over $\mathbb{F}$

$G_{\mathcal{K}}$ = ind-gp scheme over $\mathbb{F}$ which represents the functor

$$R \mapsto G_{\mathbb{F}}(R((t)))$$

$G_{\mathcal{O}}$ = gp scheme over $\mathbb{F}$ which represents $R \mapsto G_{\mathbb{F}}(R[[t]])$

→ Affine Grassmannian: $Gr_G = G_{\mathcal{K}} / G_{\mathcal{O}}$ (fpqc quotient)

ind-projective ind-scheme of ind-finite type, over $\mathbb{F}$.

If k finite field of char. ℓ or finite extension of $\mathbb{Q}_\ell$ with $\ell \neq \mathrm{char}(\mathbb{F}_q)$

$\rightsquigarrow$ $\mathrm{Perv}_{G_{\mathcal{O}}}(Gr_G, k)$ = étale $G_{\mathcal{O}}$-equivariant k-perverse sheaves on Gr_G

- Lusztig, Mirković-Vilonen: this category is <u>monoidal</u> for convolution product
- Drinfeld: there is a commutativity constraint for this monoidal product.

<u>Thm</u> (Ginzburg, Mirković-Vilonen)

There exists an equivalence of monoidal categories

$$\mathrm{Perv}_{G_O}(\mathrm{Gr}_G, k) \simeq \mathrm{Rep}(G_k^\vee)$$

where $G_k^\vee$ is the Langlands dual split reductive gp over k.

<u>Back to p-adic groups</u>

Whittaker module: $U^+ \subset G$ positive maxl unipotent subgp.

$$U^+/(U^+,U^+) \xleftarrow{\sim} \prod_{\alpha \text{ simple root}} U_\alpha \simeq \prod_{\alpha \text{ simple root}} G_a$$

Fix $\psi : \mathbb{F}_p \longrightarrow \mathbb{C}^\times$ non trivial.

$\rightsquigarrow$ We set $\chi : \begin{cases} U^+(\mathbb{F}_p((t))) \longrightarrow \mathbb{C}^\times \\ u \longmapsto \prod_\alpha \psi(\mathrm{Res}(\underset{\mathbb{F}_p((t))}{u_\alpha})) = \psi(\sum_\alpha \mathrm{Res}(u_\alpha)) \end{cases}$

<u>Whittaker module</u>: $\mathbb{W} = \left\{ f : G(\mathbb{F}_p((t))) \to \mathbb{C} \;\middle|\; \begin{array}{l} f(gx) = f(g) \text{ if } x \in G(\mathbb{F}_p[[t]]) \\ f(ng) = \chi(n) f(g) \text{ if } n \in U^+(\mathbb{F}_p((t))) \\ + f \text{ nonzero only on finitely many double cosets} \end{array} \right\}$

<u>Fact</u>: $\mathbb{W}$ is a free module of rk 1 over the spherical affine Hecke algebra. (right module)

(cf. Casselman-Shalika formula)

<u>Q</u>: How can one get a geometric version of this claim?

<u>Back to geometry</u> (Same setting as before)

We might want to consider perverse sheaves on Gr with equivariance condition wrt action of U_K^+.

<u>Pb</u>: The U_K^+-orbits on Gr_G are both of <u>infinite dimension</u> and <u>infinite codimension</u> $\Longrightarrow$ No "elementary" theory of perverse sheaves adapted to this setting.

× Solution ① : Replace Gr_G by Drinfeld's compactification of the stack of principal U-bundles on a curve : cf. Frenkel-Gaitsgory-Vilonen

②

× Solution ② = learn derived algebraic geometry : cf Gaitsgory & students

× Solution ③ : replace the Whittaker condition by a simpler condition, more adapted to perverse sheaves: Iwahori-Whittaker condition

Iwahori-Whittaker perverse sheaves

$$\begin{array}{ccc} I^+ & \subset & G_{\mathcal{O}} \quad \text{Iwahori subgp} \\ \downarrow & & \downarrow{\scriptstyle t\mapsto 0} \\ B^+ & \subset & G \end{array}$$

I_u^+ = pro-unip radical = inverse image of U^+

We define $\chi : I_u^+ \longrightarrow \mathbb{G}_a$ as follows:

$$I_u^+ \longrightarrow U^+ \longrightarrow\!\!\!\!\to U^+/(U^+,U^+) \simeq \prod_{\substack{\alpha \text{ simple} \\ \text{root}}} \mathbb{G}_a \xrightarrow{+} \mathbb{G}_a$$

Artin-Schreier local system : $AS : \begin{cases} \mathbb{G}_{a,\mathbb{F}} \longrightarrow \mathbb{G}_{a,\mathbb{F}} \\ x \longmapsto x^p - x \end{cases}$

Galois covering of Galois gp $\mathbb{F}_p$.

We fix $\psi : \mathbb{F}_p \longrightarrow k^\times$

$\rightsquigarrow \quad \mathcal{L}_{AS} = \left(AS_* \underline{k}_{\mathbb{G}_a} \right)_\psi$

Iwahori-Whittaker perverse sheaves on Gr_G:

$Perv_{IW}(Gr_G, k)$ = étale k-perverse sheaves on Gr_G s.th.

$$a^*(\mathcal{F}) \simeq \chi^*(\mathcal{L}_{AS}) \boxtimes \mathcal{F}$$

where $a : I_u^+ \times Gr_G \longrightarrow Gr_G$ action map.

simple objects: $\{ I_u^+\text{-orbits on } Gr_G \} \longleftrightarrow X_*(T)$

$$I_u^+ \cdot L_\lambda \longleftarrow\!\!\!| \quad \lambda$$

but the orbit attached to λ supports an IW local system iff $\lambda \in X_*(T)^{++}$: strictly dominant weights

$\rightsquigarrow \{ \text{simple objects in } Perv_{IW}(Gr_G, k) \} \longleftrightarrow X_*(T)^{++}$

$$IC_\lambda^{IW} \longleftarrow\!\!\!| \quad \lambda$$

Thm (Bezrukavnikov-Gaitsgory-Mirković-R.-Rider)

Assume that there exists $\xi \in X_*(T)$ s.t. $\langle \xi, \alpha \rangle = 1$ $\forall \alpha$ simple root.

Then we have an equivalence of categories

$$\begin{cases} \mathrm{Perv}_{G_O}(Gr_G, k) \xrightarrow{\sim} \mathrm{Perv}_{IW}(Gr_G, k) \\ \mathcal{F} \longmapsto IC^{IW}_{\xi} \overset{G_O}{\star} \mathcal{F} \end{cases}$$

Moreover, IC_λ is sent to $IC^{IW}_{\lambda+\xi}$ for all $\lambda \in X_*(T)^+$

Rmk: earlier proof by Arkhipov-Bezrukavnikov-Braverman-Gaitsgory-Mirković in the case of char. 0 coeff^ts.

Important ingredients: Our categories are highest weight (cf. MV / BGS) + modular reduction

Application:

From the theory of parity sheaves: for $\lambda \in X_*(T)^+$ we have

$\mathcal{E}_\lambda \in D^b_{(G_O)}(Gr_G, k)$ indec. parity sheaf with $\mathcal{E}_\lambda|_{Gr^\lambda_G} \simeq \underline{k}_{Gr^\lambda_G}[\dim Gr^\lambda_G]$

constructible wrt stratification by G_O-orbits

⇝ Consider ${}^pH^0(\mathcal{E}_\lambda) \in \mathrm{Perv}_{(G_O)}(Gr_G, k) = \mathrm{Perv}_{G_O}(Gr_G, k)$ (Mirković-Vilonen)

Corollary: Satake$({}^pH^0(\mathcal{E}_\lambda))$ is a tilting $G^\vee_k$-module.

Important ingredient: $\mathrm{Perv}_{IW}(Gr_G, k)$ is a highest weight category ⇝ we have tilting objects

On the other hand the theory of parity sheaves of JMW applies in $D^b_{IW}(Fl_G, k)$.

Fact: The relevant I_u^+-orbits have constant parity on each connected component of Gr_G

⟹ the tilting IW perverse sheaves are parity sheaves!

Rmks: • property conjectured by JMW

• if ℓ good for G then in fact $\mathcal{E}_\lambda$ is perverse.

• Allows to reprove the fact that (tilting) ⊗ (tilting) is tilting

Other application of the IW construction: $\mathcal{O}^{asph} \simeq \mathrm{Perv}_{IW}(Fl_G, k)$