Representations of reductive algebraic groups 1

Basic notation: k alg. closed field

G connected reductive alg. gp (e.g. $G = GL(n,k)$)

$B \subset G$ Borel subgp (eg $B = \{$lower triangular inv. matrices$\}$)

$T \subset B$ maxl torus (ex $T = \{$diagonal inv. matrices$\}$)

$\mathbb{X} = X^*(T)$ weights

(e.g. $\mathbb{X} = \mathbb{Z}^n$ with $(\lambda_1, \dots, \lambda_n) \leftrightarrow (\mathrm{diag}(t_1, \dots, t_n) \mapsto \prod_i t_i^{\lambda_i})$)

$\mathcal{R} = \mathcal{R}(G,T) \subset \mathbb{X}$ root system

(e.g. $\{(0, \dots, 0, \pm 1, 0, \dots, 0, \mp 1, 0, \dots, 0)\} \subset \mathbb{Z}^n$)

$\mathcal{R}^+$ = positive roots = T-weights in $\mathrm{Lie}(G)/\mathrm{Lie}(B)$

(e.g. $\{(0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0)\} \subset \mathcal{R}$)

$\mathcal{R}^s$ = simple roots

(e.g. $\{(0, \dots, 0, 1, -1, 0 \dots 0)\} \subset \mathcal{R}^+$)

W_f = Weyl group of (G,T) (e.g. $W_f = \mathfrak{S}_n$)

$\mathbb{X}^+ = \{\lambda \in \mathbb{X} \mid \forall \alpha \in \mathcal{R}^+, \langle \lambda, \alpha^\vee \rangle \geq 0\}$: dominant weights

(e.g. $\{(\lambda_1, \dots, \lambda_n) \mid \lambda_1 \geq \dots \geq \lambda_n\}$)

$\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+} \alpha \in \frac{1}{2}\mathbb{X}$.

$w_0 \in W_f$: longest element

Induction

H (affine) k-alg. gp, $K \subset H$ closed subgroup

$\mathrm{Rep}(H)$ = category of alg H-modules ($= \mathcal{O}(H)$-comodules)

$\mathrm{Rep}(K)$ = ——— K-modules

$$\mathrm{Ind}_K^H : \begin{array}{ccc} \mathrm{Rep}(K) & \longrightarrow & \mathrm{Rep}(H) \\ V & \longmapsto & (\mathcal{O}(H) \otimes_k V)^K \end{array}$$

Here K acts on $\mathcal{O}(H) \otimes V$ via $k \cdot (f \otimes v) = f(-k) \otimes k \cdot v$

H acts on $\mathcal{O}(H) \otimes V$ via $h \cdot (f \otimes v) = f(h^{-1}-) \otimes v$

Basic properties:

① Frobenius reciprocity: Ind_K^H is right adjoint to

$$\mathrm{For}_K^H : \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(K)$$

② Transitivity: if $K_1 \subset K_2 \subset H$ then

$$\mathrm{Ind}_{K_1}^H \simeq \mathrm{Ind}_{K_2}^H \circ \mathrm{Ind}_{K_1}^{K_2}$$

③ Tensor identity: if $V_1 \in \mathrm{Rep}(K)$ and $V_2 \in \mathrm{Rep}(H)$ then

$$\mathrm{Ind}_K^H\left(V_1 \otimes \mathrm{For}_K^H(V_2)\right) \xrightarrow{\sim} \mathrm{Ind}_K^H(V_1) \otimes V_2$$

Rmk: ① implies that Ind_K^H sends injectives to injectives

$\Rightarrow$ there are enough injective objects in $\mathrm{Rep}(H)$.

Then one can consider the derived functor

$$R\mathrm{Ind}_K^H : D^+\mathrm{Rep}(K) \longrightarrow D^+\mathrm{Rep}(H)$$

and all the formulas above hold also at the derived level.

Classification of simple G-modules

U = unipotent radical of B.

Then $B = T \ltimes U$

$\Rightarrow$ any character of T extends uniquely to B.

$\rightsquigarrow$ For $\lambda \in \mathbb{X}$ we have $H^0(\lambda) := \mathrm{Ind}_B^G(k_B(\lambda))$.

Basic properties:

1) $H^0(\lambda)$ is finite dimensional for all $\lambda \in \mathbb{X}$.
   Moreover, $H^0(\lambda) \neq 0 \iff \lambda \in \mathbb{X}^+$.

2) For all $\lambda \in \mathbb{X}^+$, $H^0(\lambda)$ admits a unique simple submodule, denoted $L(\lambda)$

3) The assignment $\lambda \mapsto L(\lambda)$ induces a bijection

$\mathbb{X}^+ \xrightarrow{\sim} \{$isomorphism classes of simple G-modules$\}$.

Rmks: 1) The classification of simple modules is due to Chevalley. It is independent of the field k!

2) If $\mathrm{char}(k)=0$ one has in fact $L(\lambda) = H^0(\lambda)$ $\forall \lambda \in \mathbb{X}^+$.

3) $L(\lambda)^* \simeq L(-w_0\lambda)$

Weyl character formula

Recall that any alg. T-module is the direct sum of its weight spaces: $V = \bigoplus_{\mu \in \mathbb{X}} V_\mu$ with $V_\mu = \{v \in V \mid \forall t \in T, t\cdot v = \mu(t) v\}$

$\rightsquigarrow$ if $\dim(V) < \infty$ we define its character as

$$\mathrm{ch}(V) = \sum_{\mu \in \mathbb{X}} \dim(V_\mu)\cdot e^\mu \in \mathbb{Z}[\mathbb{X}].$$

Theorem (Weyl character formula): for $\lambda \in \mathbb{X}^+$ we have

$$\mathrm{ch}(H^0(\lambda)) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\sum_{w\in W} (-1)^{\ell(w)} e^{w(\rho)}}$$

Rmk: Once again the answer is independent of k!
(But the structure of $H^0(\lambda)$ will depend on k...)

Consequence: ch induces an isomorphism of rings

$$[\mathrm{Rep}^{fd}(G)] \xrightarrow{\sim} \mathbb{Z}[\mathbb{X}]^{W_f}$$

Steinberg tensor product theorem

Now we assume that $\mathrm{char}(k) = p > 0$.

Frobenius twist: $G^{(1)}$ (or $\dot{G}$) defined by

$\mathcal{O}(\dot{G}) = k \otimes_k \mathcal{O}(G)$ with $\begin{array}{c} k \xrightarrow{\sim} k \\ x \longmapsto x^p \end{array}$ (with Hopf alg. structure induced by that of $\mathcal{O}(G)$)

We have $\mathrm{Fr}: G \longrightarrow \dot{G}$

corresp. to $\left\{ \begin{array}{ccc} \mathcal{O}(\dot{G}) & \longrightarrow & \mathcal{O}(G) \\ \lambda \otimes f & \longmapsto & \lambda f^p \end{array} \right.$

Ex: 1) $T \to \dot{T}$ induces $X^*(\dot{T}) \hookrightarrow \mathbb{X}$ with image $p\mathbb{X}$

$\rightsquigarrow$ We will identify $X^*(\dot{T})$ with $\mathbb{X}$ in such a way that the pullback under $T \to \dot{T}$ corresponds to

$$\begin{aligned} \mathbb{X} &\hookrightarrow \mathbb{X} \\ \lambda &\mapsto p\lambda \end{aligned}$$

2) If $G = \mathrm{Spec}(k) \times_{\mathrm{Spec}(\mathbb{F}_p)} G_0$ for some $\mathbb{F}_p$-gp scheme G_0 (which is always the case!) then we have

$$\mathcal{O}(\dot{G}) = k \otimes_k (k \otimes_{\mathbb{F}_p} \mathcal{O}(G_0)) \cong \mathcal{O}(G)$$

$\rightsquigarrow$ Fr becomes ~~a~~ ~~and~~ gp automorphism of G

e.g. if $G = GL_n(k)$ then $Fr: G \to G$ corresponds to $(a_{ij})_{1 \le i,j \le n} \mapsto (a_{ij}^p)_{1 \le i,j \le n}$. (Then we write $V^{(1)}$ for $Fr^*(V)$)

<u>Restricted dominant weights</u>: $\cancel{X_1(T)}$ $\mathbb{X}_1 = \{ \lambda \in \mathbb{X} \mid \forall \alpha \in \mathcal{R}^s, \ 0 \le \langle \lambda, \alpha^\vee \rangle < p \}$

<u>Theorem</u> (Steinberg)

If $\lambda = \lambda_0 + p\lambda_1$ with $\lambda_0 \in \cancel{X_1(T)}\ \mathbb{X}_1$, $\lambda_1 \in \mathbb{X}^+$ then $L(\lambda) \simeq L(\lambda_0) \otimes Fr^*(\underbrace{\dot{L}(\lambda_1)}_{\text{simple } \dot{G}\text{-module of highest weight } \lambda_1 \in \mathbb{X} = X^*(\dot{T})})$

<u>Rmk</u>: Assume G has simply connected derived subgroup (e.g. $G = GL(n,k)$).

Then any $\lambda \in \mathbb{X}^+$ can be written as

$$\lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + \dots + p^r\lambda_r \quad \text{with } r \geq 0, \ \lambda_i \in \cancel{X_1(T)}\ \mathbb{X}_1$$

(but not necessarily uniquely)

$\rightsquigarrow$ Steinberg's tensor product theorem reduces the description of <u>all</u> simple G-modules to those with <u>restricted</u> highest weight

Examples: 1 $G = SL(2,k)$

$T \simeq k^\times$

$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \longleftrightarrow \lambda$

$\mathbb{X} = \mathbb{Z}$

$\mathcal{R} = \{\alpha, -\alpha\}$ with $\alpha \longleftrightarrow 2$

$\mathbb{X}^+ = \mathbb{Z}_{\geq 0}$

$\cancel{\mathbb{X}_1(T)}$ $\mathbb{X}_1 = \{0, \dots, p-1\}$

Induced modules: $H^0(n) = k[x,y]_n$ (homogeneous polynomials of degree n)

with $SL(2,k)$ acting via $H^0(n) \subset \mathcal{O}(\mathbb{A}^2_k)$ and $SL_2 \circlearrowleft k^2$ naturally.

Weyl character formula: $\operatorname{ch} H^0(n) = \dfrac{e^{n+1} - e^{-n-1}}{e^1 - e^{-1}} = e^n + e^{n-2} + \dots + e^{-n+2} + e^{-n}$

Simple modules: if $n \in \mathbb{X}_1$ then $L(n) = H^0(n)$.

In general, if $n = n_0 + pn_1 + p^2 n_2 + \dots + p^r n_r$ with $n_j \in \mathbb{X}_1$

then $L(\lambda) = L(n_0) \otimes L(n_1)^{(1)} \otimes \dots \otimes L(n_r)^{(r)}$

$\operatorname{ch}(L(\lambda)) = (e^{n_0} + \dots + e^{-n_0})(e^{pn_1} + \dots + e^{-pn_1}) \dots (e^{p^r n_r} + \dots + e^{-p^r n_r})$

e.g. if $p \leq \lambda \leq 2p-2$ then

$$H^0(\lambda) = \begin{array}{|c|} \hline L(2p-2-\lambda) \\ \hline L(\lambda) \\ \hline \end{array}$$

if $\lambda = 2p-1$ then $H^0(\lambda) = L(\lambda)$

if $2p \leq \lambda \leq 3p-2$ then (and $p \neq 2$)

$$H^0(\lambda) = \begin{array}{|c|} \hline L(\lambda-2p) \\ \hline L(4p-\lambda-2) \\ \hline L(\lambda) \\ \hline \end{array}$$
