Equivariant cohomology, localisation and moment graphs

University of Melbourne
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Friday Problem Sheet

1. Let $\mathrm{Gr}(k, n)$ denote the Grassmannian of k-planes in $\mathbb{C}^n$. This has a $T = (\mathbb{C}^\times)^n$-action induced from the obvious T-action on $\mathbb{C}^n$. Show that the moment graph of $\mathrm{Gr}(k, n)$ has the following description:

a) vertices are given by k-subsets $I \subset \{1, \dots, n\}$,

b) two vertices $I_1 \neq I_2$ are joined by an edge if and only if $|I_1 \cap I_2| = (k-1)$ in which case this edge is labelled by $\pm(e_i - e_j)$, where i and j are the two elements in the symmetric difference of I_1 and I_2.

2. Let J denote the $n \times n$ anti-diagonal matrix $J = \begin{pmatrix} 0 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 0 \end{pmatrix}$. Let ω be the symplectic form on $\mathbb{C}^{2n}$ given in the standard basis by $\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$. Let $G = Sp(2n) \subset GL_{2n}(\mathbb{C})$ denote the group of symplectic automorphisms of $\mathbb{C}^{2n}$ and let T denote the subgroup of diagonal matrices in $G = Sp(2n)$. So that $T \cong (\mathbb{C}^\times)^n$ via $(\lambda_1, \dots, \lambda_n) \mapsto \mathrm{diag}(\lambda_1, \dots, \lambda_n, \lambda_n^{-1}, \dots, \lambda_1^{-1})$. Let X denote the Grassmannian of n-dimensional isotropic subspaces of $\mathbb{C}^{2n}$. (Recall that a subspace $V \subset \mathbb{C}^{2n}$ is *isotropic* if the restriction of ω to V is identically zero.)

a) Show that T has 2^n fixed points on X.

b) Describe the moment graph of X.

3. (Requires some knowledge of the structure theory of reductive algebraic groups.) Let G denote a connected complex reductive group and let $T \subset G$ denote a maximal torus and Borel subgroup of G. Let W be the Weyl group of (G, T). Recall that one has a bijection between the roots R modulo ± 1 of (G, T) and reflections in W which we denote by $X(T) \ni \alpha \mapsto s_\alpha \in W$.

a) Show that $(G/B)^T = W$ and hence the vertices may be canonically identified with the Weyl group W of (G, T),

b) Show that there is a one-dimensional orbit joining $w_1, w_2 \in W$ if and only if there exists a reflection $s_\alpha \in W$ with $s_\alpha w_1 = w_2$, in which case the corresponding edge in the moment graph is labelled by α.

(*Hint:* We discussed the case of $G = GL_n$ in lectures.)

4. Keep the notation of the previous section and let Γ denote the moment graph of G/B. In addition let $\Delta \subset R^+ \subset R$ denote the simple and positive roots, and let $\leq$ denote the Bruhat order on W. Let w_0 denote the longest element in W. Recall that a *section* of Γ is a tuple $(f_x) \in \bigoplus_{x \in W} S_T$ such that $f_x - f_{s_\alpha x}$ is divisible by α for all pairs $x \in W$ and $\alpha \in R$. Given an element $f = (f_x) \in \bigoplus_{x \in W} S_T$ define its *support* to be the set $\mathrm{supp}\, f = \{x \in W \mid f_x \neq 0\}$.

a) Define $f^{w_0} = (f^{w_0}_x)$ by $f^{w_0}_{w_0} = \prod_{\alpha \in R^+} \alpha$ and $f^{w_0}_x = 0$ if $x \neq w_0$. Show that f^{w_0} is a section of Γ.

b) Recall that S_T is graded with $X(T)$ in degree 2. Given $x \in W$ show that the space of sections $f \in \bigoplus_{x \in W} S_T$ such that $\deg f = 2\ell(x)$ and $\mathrm{supp}\, f \subset \{y \mid y \geq x\}$ is at most one-dimensional.

c) Given a section $f = (f_x)$ and a root α show $\partial_\alpha f$, defined by $(\partial_\alpha f)_x := \frac{1}{x\alpha}(f_x - f_{xs})$ is also a section. (The operators ∂_α are called BGG operators.)

d) Let f denote a section of degree d with $\mathrm{supp}\, f \subset \{\geq x\}$ and $f_x \neq 0$. Show that if α is a simple root with $s_\alpha x < x$ then the support of $\partial_\alpha f$ is contained in $\{\geq s_\alpha x\}$, $f_{sx} \neq 0$ and $\deg \partial_\alpha f = d - 2$.

e) Use a), b) and d) to conclude that the space of global sections of the moment graph is a free S_T-module of rank $|W|$.

(We already knew that e) is true by equivariant formality. However it is interesting to see how subtle an algebraic proof of this fact is!)

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