Equivariant cohomology, localisation and moment graphs

University of Melbourne
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Wednesday Problem Sheet

1. Let $T = (S^1)^n$ be a compact torus, and X a T-space. Show that one has an equality of Euler characteristics

$$\chi(X) = \chi(X^T).$$

(You may need to impose some finiteness assumption on X.)

2. Let $T = (\mathbb{C}^\times)^n$ be an algebraic torus, and X a T-space. Let $K = (S^1)^n \subset T$ denote the compact subtorus. Show that one has a canonical isomorphism

$$H_T^\bullet(X) \cong H_K^\bullet(X)$$

where we view X as a K-space by restriction.

3. Let X and Y be G-spaces and let $\pi : X \to Y$ be a G-equivariant locally trivial bundle with contractible fibres. Show that π induces an isomorphism $H_G^\bullet(Y) \xrightarrow{\sim} H_G^\bullet(X)$.

4. Let V be a complex vector space equipped with a linear action of an algebraic torus T, and let $X \subset V$ be a closed subvariety. Show that for a generic subtorus $T' \subset T$ of codimension 1 one has $X^{T'} = X^T$. Can you describe those subtori for which one has $X^{T'} \neq X^T$?

5. Make $\mathbb{P}^1\mathbb{C}$ into a $\mathbb{C}^\times$-variety via $\lambda \cdot [x : y] = [x : \lambda y]$. Calculate the $\mathbb{C}^\times$-equivariant cohomology of $\mathbb{P}^1\mathbb{C}$. Hence show that $\mathbb{P}^1\mathbb{C}$ is equivariantly formal with respect to this action. (*Hint:* Find an appropriate covering of $\mathbb{P}^1\mathbb{C}$ by $\mathbb{C}^\times$-stable subsets and use the Mayer-Vietoris sequence.)

6. Let T be an algebraic torus and χ a character of T. Recall that we denote by $\mathbb{C}_\chi$ the corresponding one-dimensional T-module.

a) Calculate the T-equivariant cohomology of $\mathbb{C}_\chi^n \setminus \{0\}$.

b) Conclude that if X is any T-space which admits a map $X \to \mathbb{C}_\chi^n \setminus \{0\}$ then $H_T^\bullet(X)$ is annihilated by a power of χ.

(The second statement is an important step in establishing the localisation theorem.)

7. (This exercise explores the relationship between cohomology and equivariant cohomology and requires some knowledge of spectral sequences.) Let T denote an algebraic torus and X a (nice) T-space.

a) Show that the map $ET \times_T X \to ET/T = BT$ is a locally trivial fibre bundle with fibre X. Conclude that there is a spectral sequence

$$E_2 = H^\bullet(X) \otimes H_T^\bullet(pt) \Rightarrow H_T^\bullet(X).$$

b) Show that the quotient map $ET \times X \to ET \times_T X$ is a locally trivial fibre bundle with fibre T. Conclude that there is a spectral sequence

$$E_2 = H^\bullet(T) \otimes H_T^\bullet(X) \Rightarrow H^\bullet(X).$$

c) Describe the differentials in b) explicitly in terms of the S_T-module structure on $H_T^\bullet(X)$. Hence prove that if $H_T^\bullet(X)$ is a free S_T-module then

$$H^\bullet(X) = H_T^\bullet(X)/(S_T^+ \cdot H_T^\bullet(X))$$

(as claimed in lectures).

d) Show that $H_T^\bullet(X)$ is a free S_T-module if and only if the spectral sequence in a) degenerates at E_2. (This is often taken as the definition of equivariant formality.)

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