EQUIVARIANT COHOMOLOGY, LOCALISATION AND MOMENT GRAPHS

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1. Lecture 1

Suppose that X is a nice space (manifold, algebraic variety) acted on by a Lie group G. Suppose one is interested in some aspect of the topology of X. Surely the fact that X is a G-space should give you some leverage?! It does, and one such lever is equivariant cohomology.

Equivariant cohomology is a "cohomology theory for G-spaces". That is, equivariant cohomology is a contravariant functor $H_G(-)$ from the category of G-spaces to graded rings. It turns out that if G is a Lie group, then $H_G^\bullet(X)$ is determined by $H_T^\bullet(X)$, where T is a maximal torus of G.

Now there is a very rich dictionary between the topology of X and properties of $H_T^\bullet(X)$ as a module over $H_T^\bullet(pt)$ (which in contrast to ordinary cohomology is an interesting ring; in fact a polynomial ring). This dictionary is so rich that it often permits one to calculate the (ordinary) cohomology of X using relatively simple commutative algebra. This is the subject of this course.

1.1. Conventions.

Throughout:

(1) "space" will mean "CW complex" and
(2) $H^\bullet(-)$ will denote cohomology with $\mathbb{Q}$-coefficients.
(3) G will denote a topological group but nothing will be lost if one assumes that G is a Lie group. The key examples to have in mind are $G = \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}$, $(\mathbb{C}^\times)^r$, S^1, $GL_n(\mathbb{C})$.

1.2. Principal bundles and classifying spaces.

Definition 1.1. (1) A *bundle* is a map $\pi : E \to B$.

(2) A bundle $\pi : E \to B$ is *locally trivial* (with fibre F) if it is locally isomorphic to the projection map $U \times F \to U$.
(3) A *(left) principal G-bundle* is a a bundle such that
   (a) E is a (left) G-space,
   (b) π is locally isomorphic to the projection $G \times U \to U$ with trivial G-action on U. (In particular, π is equivariant with respect to the trivial action of G on B.)
(4) A (left) G-space X is a *free* G-space if the quotient map

$$\pi : X \to G \setminus X$$

is a principal G-bundle.

Exercise 1.2. (1) Show that the projection map
$$\mathbb{C}^2 \setminus \{0\} \to \mathbb{P}^1\mathbb{C}$$
is a principal $\mathbb{C}^*$-bundle.
(2) Find ten (good) examples of principal bundles "in nature"!

If $\pi : E \to B$ is a principal G-bundle and $\phi : B' \to B$ is any map then we can form the Cartsian square

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "\pi'"] & E \arrow[d, "\pi"] \\ B' \arrow[r] & B \end{tikzcd}$$

and one can check (do it!) that E' is a principal G-bundle. We say that $\pi' : E' \to B'$ is the *pull-back* of E under f and write $\pi' = f^*\pi$.

The following theorem might look a bit miraculous when you first see it.

Theorem 1.3. *Suppose that $\pi_0 : E_0 \to B_0$ is a principal G-bundle and that B is a CW-complex of dimension $n \le \infty$. If B is connected and $\pi_i(B_0) = 0$ for $1 \le i \le n$ then we have a bijection*
$$\begin{array}{ccc} [B, B_0] & \tilde{\to} & \left\{ \begin{array}{c} \textit{principal } G\textit{-bundles} \\ \textit{over } B \end{array} \right\} / \cong \\ f & \mapsto & f^*\pi_0. \end{array}$$

Very sketchy proof. Given a G-bundle $\pi : E \to B$ on G it is enough to find an G-equivariant map $\widetilde{f} : E \to E_0$ in which case the π is automatically the pull-back under the induced map $f = G \setminus \widetilde{f} : B \to B_0$. One then argues that one can find a cell-decomposition of E compatible with the G-action. Compatible means: the map to B is cellular, and the bundle is trivial over each cell in B. We use the following fact constantly: to give an equivariant map $\pi : C \times G \to E_0$ is equivalent to giving a map $C \to E_0$. We map the 0-cells of B arbitrarily into E and extend it to a map to the preimage of the zero skeleton in E to E_0 using G-equivariance. To extend this map to a map on the 1-skeleton we use that E to is path-connected. To extend this map to the 2-skeleton we use 1-connectedness etc. Finally, one can show that any two such maps are homotopic (again using n-connectedness). □

Remark 1.4. (1) This shows that (if $\pi_0 : E_0 \to B_0$ exists with $\pi_i(E_0) = 0$ for all $i \ge 0$) the functor which assigns to any space B the set of G-bundles over B up to isomorphism is representable in the homotopy category.
(2) It is nice to think about this result for specific examples where EG has a simple description. For example, the case $G = \mathbb{Z}$ is done in the exercises.

The above theorem (and remark) motivates the following definition:

Definition 1.5. A *universal (G-)bundle* is a right[1] principal G-bundle $\pi_G : EG \to BG$ such that EG is contractible. The space EG (which is well-defined up to homotopy) is the *classifying space* of G.

The following theorem guarantees the existence of universal bundles:

[1]The switch from left to right bundles is to make the discussion of equivariant cohomology easier. It can be ignored.

Theorem 1.6. *For any topological group G there exists a universal G-bundle $\pi_G : EG \to BG$.*

Remark 1.7. The construction of such a bundle (as the geometric realisation of a certain simplicial G-space) is due to Milnor and is often referred to as the *Milnor construction*. If one has a particular case in mind there are usually a number of possible constructions. Some such possibilities are discussed in the exercises.

Example 1.8.
(1) If $G = \mathbb{Z}$ then G acts freely by translations on $\mathbb{R}$ with quotient S^1. Because $\mathbb{R}$ is contractible we can take $\pi_{\mathbb{Z}} : \mathbb{R} \to S^1$ as a universal bundle in this case.
(2) If $G = \mathbb{C}^\times$ then we might be tempted to take $G = \mathbb{C}^\times$. But $\pi_1(\mathbb{C}^\times) \neq 0$. What about $\mathbb{C}^2 \setminus \{0\}$? No, $\pi_3(\mathbb{C}^2 \setminus \{0\}) \neq 0$. What about $\mathbb{C}^n \setminus \{0\}$? No, $\pi_{2n-1}(\mathbb{C}^n \setminus \{0\}) \neq 0$. Hence we are force to take $\mathbb{C}^\infty \setminus \{0\}$ (the direct limit of all these possibilities). This works (showing directly that $\mathbb{C}^\infty \setminus \{0\}$ is a fun exercise!) and so we can take
$$\pi_{\mathbb{C}^\times} : \mathbb{C}^\infty \setminus \{0\} \to \mathbb{P}^\infty\mathbb{C}$$
as our universal bundle.

1.3. **Characteristic classes.** Fix a group G. By the above theorem, given a G-bundle $\pi : E \to X$ is the same thing as given a homotopy class of maps
$$f \in [X, BG].$$
If we fix generators $\xi_1, \dots, \xi_n \in H^*(BG)$ we get well-defined elements $f^*\xi_1, \dots, f^*\xi_n \in H^*(X)$. There are the *characteristic classes* of π.

Example 1.9. If $G = \mathbb{C}^\times$ then we can take $BG = \mathbb{P}^\infty\mathbb{C}$ and $H^\bullet(\mathbb{P}^\infty\mathbb{C}) = \mathbb{Q}[x]$ where x is the fundamental class of $\mathbb{P}^1$. Given a $CM^\times$-bundle on $\pi : E \to X$, then we have $\pi = f^*\pi_{\mathbb{C}^*}$ for some map $f : X \to \mathbb{P}^\infty\mathbb{C}$ (well-defined up to homotopy). The *Chern class* of π is $c_1(\pi) = f^*x \in H^2(X)$.

Further examples include the Chern classes of a vector bundle (the case of $G = GL_n$) and the Stiefel-Whitney classes (the case $G = GL_n(\mathbb{R})$ with cohomology coefficients $\mathbb{Z}/2\mathbb{Z}$).

1.4. **Equivariant cohomology.** Equivariant cohomology is a cohomology theory for G-spaces. An essential feature is that $H^\bullet_G(X) = H^\bullet(G \setminus X)$ if X is a free G-space. In general the relationship between $H^\bullet_B(X)$ and $H^\bullet(X)$ is somewhat subtle (this will be a big part of next lecture).

The definition is motivated by the following fact:

Exercise 1.10. If X is a free G-space, and Y is any G-space, then the diagonal G-action on $X \times Y$ is free.

Notation 1.11. If X is a right G-space and Y is a right G-space write
$$X \times_G Y := X \times Y/(xg, y) \sim (x, gy).$$
For want of a better name we call $X \times_G Y$ the *balanced product* of X and Y.

Now, suppose we have a G-space X. Then in the previous sections we have seen that we always have a (right) G-space EG which is contractible, and has free G-action. Hence one might expect the "G-homotopy type" of X and $EG \times X$ to be the same. If this is true then one is forced into the following definition.

Definition 1.12. Given a G-space X its *G-equivariant cohomology* is the graded vector space

$$H^\bullet_G(X) = H^\bullet(EG \times_G X).$$

Here are some basic properties of equivariant cohomology:

(1) It is independent of the choice of EG. Indeed, if $E'G$ denotes another choice then one can check that
$$(EG \times E'G) \times_G X \to EG \times_G X$$
is a locally trivial $E'G$-bundle (and similarly for the other projection). Hence one has canonical isomorphisms
$$H^\bullet(EG \times_G X) \cong H^\bullet((EG \times E'G) \times_G X) \cong H^\bullet(E'G \times_G X).$$

(2) By definition
$$H^\bullet_G(pt) = H^\bullet(EG \times_G pt) = H^\bullet(BG).$$
We will see that this is an interesting algebra, even for relatively simple groups G.

(3) If X is a free G-space then
$$EG \times_G X \to X$$
is a locally trivial EG-bundle. Hence $H^\bullet_G(X) = H^\bullet(G \setminus X)$.

(4) In ordinary cohomology the final map $X \to pt$ tells us that $H^\bullet(X)$ is a $H^\bullet(pt) = \mathbb{Q}$-algebra, which we already knew. In equivariant cohomology this becomes more interesting: $H^\bullet_G(X)$ is a graded $H^\bullet_G(pt)$-algebra. Much of this course will be concerned with translation topological facts about X into algebraic facts about the $H^\bullet_G(pt)$-module $H^\bullet_G(X)$.

(5) Given a subgroup $H \hookrightarrow G$ one can take $EH = EG$ (under some assumptions on H which we won't spell out). Thus one obtains map
$$BH = EG/H \to EG/G = BG$$
and hence
$$H^\bullet_H(pt) \leftarrow H^\bullet_G(pt). \tag{1.1}$$
(In fact one has a canonical such map for any homomorphism $H \to G$ of topological groups. How might one construct it?)

(6) Under the above assumptions, if X is an H-space then one has a commutative diagram
$$\begin{tikzcd} EG \times_G G \times_H X \arrow[r, "\sim"] \arrow[d] & EG \times_H X \arrow[d] \\ EG \times_G pt \arrow[r] & EG \times_H pt \end{tikzcd}$$
This gives the *induction isomorphism*:
$$H^\bullet_G(G \times_H X) \cong H^\bullet_H(X).$$
This isomorphism is compatible (via (1.1)) with the $H^\bullet_G(pt)$- and $H^\bullet_H(pt)$-module structures.

(7) If X is a trivial G-space (that is, the action map $G \times X \to X$ is the projection) then
$$EG \times_G X = EG/G \times X$$
and hence $H^\bullet_G(X) = H^\bullet_G(pt) \otimes_{\mathbb{Q}} H^\bullet(X)$ by the Künneth formula.

2. Lecture 2

We start with a motivating example. If X is a reasonable S^1-space then one has an equality of Euler characteristics
$$\chi(X) = \chi(X^{S^1}).$$
The classic example of this is S^1-action on S^2 by rotation about a fixed axis.

One might ask: can one lift this equality to cohomology? Note that of course $H^\bullet(X) \ncong H^\bullet(X^{S^1})$ and so we expect a somewhat subtle answer. Localisation in T-equivariant cohomology gives a satisfactory answer.

2.1. **Borel's picture.** As we mentioned in the introduction, if G is a Lie group then the equivariant cohomology $H^\bullet_G(X)$ is determined by $H^\bullet_T(X)$, where T is a maximal torus of G. For this reason, for the rest of this course we will concentrate on the case where G is a torus. That is, either $G = (S^1)^n$ or $G = (\mathbb{C}^\times)^n$ (the *compact*, resp. *algebraic* case).

We will mostly be concerned with the case where $G = T = (\mathbb{C}^\times)^n$ is an algebraic torus. So for concreteness assume that this is the case. Our first goal is to obtain a concrete, canonical description of $H^\bullet_T(pt)$.

We have seen that we can take $ET = (\mathbb{C}^\infty \setminus \{0\})^n$ with diagonal action of T. Hence
$$H^\bullet_T(pt) = H^\bullet(BT) = H^\bullet((\mathbb{P}^\infty)^n) = \bigotimes_{i=1}^{n} \mathbb{Q}[x_i] = \mathbb{Q}[x_1, \dots, x_n] \tag{2.1}$$
(the third isomorphism follows by the Künneth theorem). This is certainly concrete, but is not canonical. (It depended on our fixed isomorphism $T \cong (\mathbb{C}^\times)^n$.)

A more canonical picture is as follows. Recall that a character of T is, by definition, a homomorphism of algebraic groups $\chi : T \to \mathbb{C}^*$. Because every homomorphism $\mathbb{C}^\times \to \mathbb{C}^\times$ is of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$, we have an isomorphism $\mathrm{Hom}(\mathbb{C}^\times, \mathbb{C}^\times) = \mathbb{Z}$. It follows that
$$X(T) = \mathrm{Hom}(T, \mathbb{C}^*)$$
is a free $\mathbb{Z}$-module of rank n.

Any $\chi \in X(T)$ allows us to view $\mathbb{C}^\times$ as a T-space via χ which we will denote $\mathbb{C}^\times_\chi$. Consider the quotient map
$$\pi_\chi : ET \times_T \mathbb{C}^*_\chi \to ET \times_T pt = BT.$$
This is easily seen to be a $\mathbb{C}^*$-bundle on BT and hence has a first Chern class $c_1(\pi_\chi) \in H^\bullet(BT)$. Extending multiplicatively we obtain a map
$$\begin{aligned} \mathrm{ch} : S(X(T)_{\mathbb{Q}}) &\to H^\bullet(BT) = H^\bullet_T(pt). \\ \chi &\mapsto c_1(\pi_\chi). \end{aligned}$$
Here $S(X(T)_{\mathbb{Q}})$ denotes the symmetric algebra on $X(T)_{\mathbb{Q}} := X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, which we view as a graded algebra with $X(T)_{\mathbb{Q}}$ in degree 2. (Recall that the Chern class of a $\mathbb{C}^*$-bundle is of degree 2.)

Theorem 2.1. ch *is an isomorphism.*

Remark 2.2.
(1) The above theorem is true for any commutative algebraic group.
(2) The analogue of the above theorem (an even the extension to commutative algebraic groups) is valid with $\mathbb{Z}$ coefficients.
(3) With coefficients in $\mathbb{C}$ one can phrase the above as a canonical identification

$$\mathrm{ch} : \mathcal{O}(\mathrm{Lie}\, T) \xrightarrow{\sim} H_T(\mathrm{pt}; \mathbb{C}).$$

where $\mathcal{O}(\mathrm{Lie}\, T)$ denotes the regular functions on the Lie algebra of T. This allows one to view the equivariant cohomology of spaces as quasi-coherent sheaves on $\mathrm{Lie}\, T$. This is both a suggestive and useful way of looking at things.

Sketch proof. In view of the isomorphism (2.1) One just needs to check that a basis of $X(T)$ is mapped to a generating set of $H^\bullet(BT)$. However, one can choose an isomorphism $T = (\mathbb{C}^*)^n$ compatible with the choice of basis of $X(T)$, and one is reduced to checking that the first Chern class of $\mathbb{C}^\infty \setminus \{0\} \to \mathbb{P}^\infty\mathbb{C}$ generates $H^\bullet(\mathbb{P}^\infty\mathbb{C})$. Hopefully you did this in the exercises! □

Note that the isomorphism ch is both concrete and canonical. From now on we will use ch to identify $H^\bullet_T(pt)$ and $S(X(T)_\mathbb{Q})$.

Having described $H^\bullet_T(pt)$ we now turn to the next simplest T-spaces, namely homogenous spaces. Any homogenous space T-space X is of the form

$$X \cong T/T'$$

for some closed subgroup $T' \subset T$. By the induction isomorphism we have

$$H^\bullet_T(T/T') = H^\bullet_T(T \times_{T'} pt) = H_{T'}(pt) = S(X(T')_\mathbb{Q})$$

which we view as a $H^\bullet_T(pt) = S(X(T)_\mathbb{Q})$-module via the pullback homomorphism $X(T) \to X(T')$.

Example 2.3. One should keep the two extreme examples in mind:
(1) If $X = pt$ then $T = T'$ gives Borel's picture.
(2) If $X = T$ then $T' = 1$ and $X(T')$ is of rank 0. In this case $H^\bullet_T(T) = S(X(T')) = \mathbb{Q}$ which we already knew, because T acts freely on T.
(3) A very important case in what follows is the case when X is one-dimensional. In this case $X \cong \mathbb{C}^*$ and the action of T is given by a character $\chi : T \to \mathbb{C}^*$ (which is well-defined up to ± 1). Recall that we denote such a T-space by $\mathbb{C}^\times_\chi$. In this case we have an exact sequence

$$T' = \ker \chi \hookrightarrow T \overset{\chi}{\twoheadrightarrow} \mathbb{C}^\times.$$

Taking characters we get

$$H^\bullet_T(\mathbb{C}^*_\chi) = X(T')_\mathbb{Q} = X(T)_\mathbb{Q}/(\chi).$$

Let's take a step back for a moment. From the above we see that the bigger the T-orbit, the smaller it's T-equivariant cohomology. This motivates the following

Slogan: T-equivariant information is concentrated at "small" orbits.

2.2. **The localisation theorem.** As in the previous section $T \cong (\mathbb{C}^\times)^n$ is an algebraic torus. From now on we abbreviate

$$S_T := S(X(T)_\mathbb{Q}) = H^\bullet_T(pt).$$

We now come to an all-important (and somewhat mysterious) condition:

Definition 2.4. A T-space X is *equivariantly formal* if $H^\bullet_T(X)$ is a free S_T-module.

We probably won't have time to do this condition justice.

Recall that we haven't yet discussed the relation between $H^\bullet_T(X)$ and $H^\bullet(X)$. One important consequence (certainly not the only one) of equivariant formality is the following theorem:

Theorem 2.5. *If X is equivariantly formal then one has an isomorphism of graded rings*

$$H^\bullet(X) \cong H^\bullet_T(X)/(S_T^+ H^\bullet_T(X))$$

where S_T^+ denotes the ideal of elements of positive degree.

Example 2.6. The following spaces are equivariantly formal:

(1) spaces for which $H^i(X) = 0$ for odd i (parity vanishing),
(2) smooth complex projective varieties with algebraic T-actions, (a consequence of Deligne's proof that the Leray-Serre spectral sequence degenerates for smooth morphisms between complex algebraic varieties),
(3) if T is a compact torus and X is a compact symplectic T-variety with Hamiltonian T-action (proved by Kirwan).

We now turn to the first part of the localisation theorem. One needs some kind of finiteness conditions on X, which we won't worry about for this lecture: we want to first give feeling for the localisation theorem. So assume that X is a complex algebraic variety with an algebraic T-action.

Consider the inclusion

$$i : X^T \hookrightarrow X.$$

Theorem 2.7. *Suppose that X is equivariantly formal and satifies* (4.1)*. Then the restriction map*

$$i^* : H^\bullet_T(X) \to H^\bullet_T(X^T) = H^\bullet(X^T) \otimes_\mathbb{Q} S_T.$$

is injective and becomes an isomorphism after tensoring with $Q_T = Quot S_T$.

Remark 2.8. (1) Because $H^\bullet(X)$ is free over S_T we have

$$\dim H^\bullet(X) = \operatorname{rank}_S H^\bullet_T(X) = \dim_Q(H^\bullet_T(X) \otimes_{S_T} Q_T) = \operatorname{rank}_S H^\bullet_T(X^T) = \dim H^\bullet(X^T).$$

which is interesting when compared with the equality $\chi(X) = \chi(X^T)$. It says that for an equivariantly formal space the sums of the even and odd Betti numbers of $H^\bullet(X)$ and $H^\bullet(X^T)$ agree.

(2) Note that i^* is a homomorphism of rings. In many cases, this gives a relatively simple description of the ring structure on $H^\bullet_T(X)$.

The second part of the localisation theorem gives a description of the image of i^*. Given any subtorus $T' \subset T$ one has a commutative diagram of inclusions

$$\begin{tikzcd} X^T \arrow[rr] \arrow[dr] & & X \\ & X^{T'} \arrow[ur] & \end{tikzcd}$$

and hence a commutative diagram of S_T-modules

$$\begin{tikzcd}
H_T^\bullet(X^T) & & H_T^\bullet(X) \arrow[ll, "i^*"'] \arrow[dl] \\
& H_T^\bullet(X^{T'}) \arrow[ul, "i_T^*"] &
\end{tikzcd}$$

Hence the image of i^* is certainly contained in the image of $i_{T'}^*$ for all $T' \subset T$.

Somewhat amazingly, the image of i^* is precisely the intersection of these images:

Theorem 2.9. *Under the assumptions of Theorem 2.7 one has*

$$\operatorname{Im} i^* = \bigcap_{T' \subset T} \operatorname{Im} i_{T'}^*$$

where the intersection runs over all (connected) subtori $T' \subset T$ of codimension 1.

We will finish this lecture with an example, which illustrates the power of this theorem.

2.3. **Example:** $\mathbb{P}^n\mathbb{C}$**.** Let $T = (\mathbb{C}^\times)^{n+1}$ and define $e_i \in X(T)$ by $e_i(\lambda_0, \dots, \lambda_n) = \lambda_i$. Let $X = \mathbb{P}^n\mathbb{C}$ with T-action in homogenous coordinates given by

$$(\lambda_0, \lambda_1, \dots, \lambda_n) \cdot [x_0 : \dots : x_n] = [\lambda_0 x_0 : \dots : \lambda_n x_n]$$

If we let $U_i = \{x = [x_0 : \dots : x_n] \in \mathbb{P}^n\mathbb{C} \mid x_i \neq 0\}$ then $U_i \cong \mathbb{A}^n$ via $x \mapsto (x_0/x_i, \dots, x_n/x_i)$. Under this isomorphism T acts with characters

$$(e_0 - e_i, e_1 - e_i, \dots, e_n - e_i).$$

Clearly this action has no fixed points outside of 0, and hence T has $(n+1)$-fixed points on X given by $[1 : 0 : \dots : 0]$, $[0 : 1 : \dots 0]$, $\dots$, $[0 : 0 : \dots : 1]$. Because X is a smooth projective variety it is equivariantly formal, and the first part of the localisation theorem gives an injection

$$H_T^\bullet(\mathbb{P}^n\mathbb{C}) \hookrightarrow \bigoplus_{i=0}^n S_T.$$

We now want to use the second part of the localisation theorem to determine its image. Let $T' \subset T$ be a codimension 1 torus. Then the calculation of the action of T on U_i above show that show that $U_i^{T'} = U_i^T = \{0\}$ unless T' is the subtorus given by $e_i = e_j$ for some $j \neq i$. In this case

$$X^{T'} = (n-1)\text{-points} \cup \mathbb{P}^1\mathbb{C}$$

where T acts on $\mathbb{P}^1\mathbb{C}$ by the character $\pm(e_i - e_j)$.

In the exercises we have calulcated the equivariant cohomology of $\mathbb{P}^1\mathbb{C}$ and we conclude that if T' is given by $e_i = e_j$ then image of

$$H_T^\bullet(X^{T'}) \overset{\text{ß}_{T'}^*}{\hookrightarrow} \bigoplus_{i=0}^n S_T$$

consists of those $(f_0, \dots, f_n)$ such that $f_i - f_j$ is divisible by $e_i - e_j$. Hence

$$H_T^\bullet(\mathbb{P}^n) = \{(f_i)_{i=1}^n \in S_T^{n+1} \mid (e_i - e_j) | (f_i - f_j) \text{ for all } i \neq j\}.$$

Generalising this situation leads to the theory of moment graphs.

3. Third lecture

3.1. **Idea of the proof of the localisation theorem.** Each flavour of the localisation theorem requires some sort of finiteness assumption. Our assumption will be:

$$\begin{gathered} X \text{ has covering by finitely many } T\text{-stable subsets,} \\ \text{each of which admits an equivariant closed embedding} \\ \text{into a complex vector space with linear } T\text{-action.} \end{gathered} \tag{3.1}$$

For example this theorem is satisfied for any normal algebraic T-variety by a theorem of Sumihiro. Analogues are true for actions of compact Lie groups on manifolds.

Both parts of the localisation theorem will following from the following statement and a result of commutative algebra.

Proposition 3.1. *Suppose that X is equivariantly formal and (4.1) is satisfied. For any subtorus $T' \subset T$*

$$H_T^\bullet(X^{T'}) \stackrel{i_{T'}^*}{\to} H_T^\bullet(X^T)$$

becomes an isomorphism after inverting those characters which restrict non-trivially to T'.

Remark 3.2. If we take $T' = \{1\} \subset T$ this implies the first part of the localisation theorem. Indeed, by assumption $H_T^\bullet(X)$ is free, and so injects into any localisation.

Sketch proof. We have a long exact sequence

$$\dots \to H_T^\bullet(X^{T'}, X^T) \to H_T^\bullet(X^{T'}) \stackrel{i_{T'}^*}{\to} H_T^\bullet(X^T) \to H_T^{\bullet+1}(X^{T'}, X^T) \to \dots$$

and it is enough to show that $H_T^{\bullet+1}(X^{T'}, X^T)$ is annihilated by $s \in S_T$ which is a product of characaters which restrict non-trivially to T'.

However this follows from our assumption (4.1), Mayer-Vietoris, a trick and the following "local" result: if V is a T-module, $Y \subset V$ is a subset such that $Y \cap V^{T'} = \emptyset$ then $H_T^\bullet(Y)$ is annihilated by a power of

$$s_{V,T'} = \chi_1 \dots \chi_l$$

where the $\chi_1, \dots, \chi_l$ denote those characters which both occur in V and do not contain T' in their kernel. $\square$

To finish the proof we need to recall the following fact from commutative algebra. Let M be a finitely generated free (or more generally reflexive) S_T-module. Given any prime ideal $\mathfrak{p} \subset S_T$ let $M_\mathfrak{p}$ denote the localisation of M at $\mathfrak{p}$. Let $i : M \to M_{(0)}$ and $i_\mathfrak{p} : M_\mathfrak{p} \to M_{(0)}$ denote the canonical maps (here (0) denotes the zero ideal, so $M_{(0)} = M \otimes_{S_T} Q_T$). We have a commutative diagram

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & M_{(0)} \\ & M_\mathfrak{p} \arrow[ur, "i_\mathfrak{p}"'] & \end{tikzcd}$$

and we can try to use the maps $i_\mathfrak{p}$ to find M inside its localisation.

Theorem 3.3. *We have*

$$\operatorname{Im} i = \bigcap \operatorname{Im} i_\mathfrak{p}$$

where the intersection takes place over all height one prime ideals $\mathfrak{p} \subset S_T$.

This is true for more general rings and (reflexive) modules over them. Note that in our case, "height one" means the same as "principal".

It should be clear now how to proceed: if an ideal $\mathfrak{p}$ doesn't contain any characters then $H_T(X)_\mathfrak{p} = H_T(X^T)_\mathfrak{p}$. Hence we only need to consider ideals generated by a character χ. If we let T' denote its kernel then the above results gives us isomorphisms

$$\begin{tikzcd} H_T(X) \arrow[rr] \arrow[dr] & & H_T(X)_{(0)} = H_T(X^T)_{(0)} \\ & H_T(X)_{(\chi)} = H_T(X^{T'})_{(\chi)} \arrow[ur] & \end{tikzcd}$$

and Proposition 3.1 gives the localisation theorem. (Could explain this better!)

4. Moment graphs

Let X be a T-variety.

Definition 4.1. a T-fixed point $x \in X$ is *attactive* if there exists $\mathbb{C}^* \subset T$ such that

$$\lim_{\mathbb{C}^* \ni \lambda \to 0} \lambda \cdot z = x$$

for all z in some neighbourhood of x.

Now let X be a smooth projective T-variety. Let us assume that

$$T \text{ has finitely many fixed points and one-dimensional orbits on } X, \tag{4.1}$$

and

$$\text{every } T\text{-fixed point is attractive.} \tag{4.2}$$

Because X is smooth there exists an open affine T-stable neighbourhood U of $x \in X$ and a T-equivariant isomorphism

$$U \xrightarrow{\sim} T_x X.$$

Exercise 4.2. Using this isomorphism, show that assumption (4.1) implies

$$\begin{array}{c}\text{For any fixed point } x \in X^T, \text{ the characters of } T \\ \text{which occur in } T_x X \text{ are pairwise linearly independent.}\end{array} \tag{4.3}$$

For any subtorus $T' \subset T$ the T-action on $X^{T'}$ factors over T/T' and hence, for any $T' \subset T$ of codimension 1, this implies (using (4.3)) that $X^{T'}$ is a disjoint union of the closures of one-dimensional orbits of T on X.

We can encode this structure in an edge labelleded graph Γ with

(1) vertices $\mathcal{V}$ in bijection with X^T,
(2) edges $\mathcal{E}$ corresponding to one-dimensional orbits, with each edge incident to the T-fixed points in its closure.
(3) each edge E is labelled with a character $\chi_E \in X(T)$ for which there is an equivariant isomorphism with $E \cong \mathbb{C}^*_{\chi_E}$. (This character is well-defined up to ± 1.)

Definition 4.3. $(\mathcal{V}, \mathcal{E}, E \mapsto \chi_E)$ is the *moment graph* of the T-variety X.

Remark 4.4. The terminology "moment graph" comes from the following. If we instead consider the compact subtorus $K \subset T$ the K-action on X admits a moment map $\mu : X \to \mathfrak{k}^*$ where $\mathfrak{k}$ denotes the Lie algebra of K. The image of X under μ yields a convex polytope with vertices $\mu(X^T)$. Moreover, if $x, y \in X^T$ lie in the closure of a one-dimensional orbit E then $\mu(x) - \mu(y)$ is a multiple of $\alpha_E \in \mathfrak{k}^*$. Hence the moment map μ gives a nice "realisation" of the moment graph inside $\mathfrak{k}^*$.

We can specialise the localisation theorem as follows:

Theorem 4.5. *The T-equivariant cohomology $H_T(X)$ is equal to the ring*

$$\{(f_x) \in \bigoplus_{x \in VC} S_T \mid \begin{array}{c} \chi_E \textit{ divides } f_x - f_y, \\ \textit{whenever } x \textit{ and } y \textit{ lie on a common edge } E \end{array} \}$$

Remark 4.6. The localisation theorem implies that the subring of $\bigoplus_{x \in VC} S_T$ given by the conditions of the theorem is free. Moreover, Poincaré duality for equivariant cohomology implies that it is self-dual (up to a shift) over S_T. These two conditions are often far from obvious a priori.

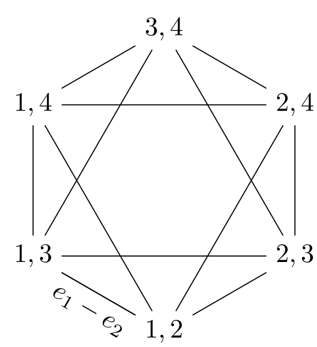
5. Moment graphs for Grassmannians and flag varieties

We begin with an exercise:

Exercise 5.1. Let $\mathcal{G}r(k, n)$ denote the Grassmannian of k-planes in $\mathbb{C}^n$. This has a $T = (\mathbb{C}^\times)^n$-action induced from the obvious T-action on $\mathbb{C}^n$. Show that the moment graph of $\mathcal{G}r(k, n)$ has the following description:

(1) vertices are given by k-subsets $I \subset \{1, \dots, n,$
(2) two vertices $I_1 \neq I_2$ are joined by an edge if and only if $|I_1 \cap I_2| = (k - 1)$ in which this edge is labelled by $e_i - e_j$, where i and j are the two elements in the symmetric difference of I_1 and I_2.

For example, here is a picture of the moment graph of $\mathcal{G}r(2, 4)$:



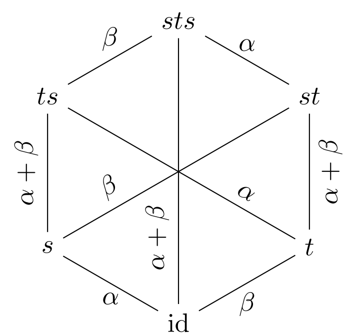
(We have ommited all but one edge label.)

Another exercise (which requires the basic structure theory of algebraic groups) is the following:

Exercise 5.2. Let G denote a connected complex reductive group and let $T \subset G$ denote a maximal torus and Borel subgroup of G. Let W be the Weyl group of (G, T). Recall that one has a bijection between the roots R of (G, T) and reflections in W which we denote by $X(T) \ni \alpha \mapsto s_\alpha \in W$. Show that the moment graph of G has:

(1) show that $(G/B)^T = W$ and hence the vertices may be canonically identified with the Weyl group W of (G, T),
(2) edges joining $w_1, w_2 \in W$ if and only if there exists a reflection $t \in W$ with $tw_1 = w_2$, in which case the reflection is labelled by the root corresponding to t.

For example, if $G = GL_3(\mathbb{C})$ then the moment graph of G/B looks like:



Let us try to give an idea of why this is true for $G = GL_n(\mathbb{C})$. Let U^- denote the lower unitriangular matrices, and B the upper triangular matrices. Then one can check directly that multiplication defines an open immersion

$$U^- \times B \to GL_n(\mathbb{C}).$$

It follows that U^- gives a chart around the point $B/B \in G/B$. Now $U^- \cong \mathbb{A}^{n(n-1)/2}$ and the T-conjugation action U^- induces linear action on $\mathbb{A}^{n(n-1)/2}$ with weights $e_i - e_j$ for $i > j$ (the negative roots). Note that the one-dimensional orbits with B/B in their closure are all given by root subgroups for negative roots.

Now one can cover G/B by subgroups of the form wU^-/B (Bruhat decomposition) and hence $(G/B)^T = W$. Similarly one can check that

$$\lim_{\lambda \to \infty} u_\alpha(\lambda) \cdot B/B = s_\alpha B/B$$

for any negative root α (here $u_\alpha(\lambda)$ denotes the one-parameter root subgroup corresponding to α). It follows that $x, y \in (G/B)^T$ are connected by a one-dimensional T-orbit if and only if $x = s_\alpha y$ for some root α in which case T-acts on this one-dimensional orbit by $\pm\alpha$.

6. Directions of current research

Let us mention the following topics:

(1) The Bruhat decomposition

$$G/B = \bigsqcup_{w \in W} BwB/B$$

gives canonical classes $c_w := [\overline{BwB/B}]$ in the (equivariant) cohomology ring. They may be described in a completely combinatorial manner on the moment graph. One also knows that one has a formula:

$$c_x c_y = \sum a_z c_z$$

for some polynomials $a_z \in S_T$ (these determine the structure constants of $H^\bullet(G/B)$ under the specialisation $S_T \to \mathbb{Q}$). What are these structure constants? Even in the case of the Grassmannian they are very complicated. The moment graph approach has recently received much attention after the work of Knutsen-Tao which sheds new light on an old combinatorial description (using so called Littlewood-Richardson coefficients) of these structure constants in the case of the Grassmannian.

(2) One can introduce the notion of a sheaf on the moment graph, and hence handle other sheaves. In particular, this allows one to calculate the equivariant intersection cohomology of Schubert varieties in a purely combinatorial manner (Braden-MacPherson algorithm). Here the localisation theorem can be rephrased as saying that the global sections of a given sheaf are given as an intersection over $\mathbb{P}^1\mathbb{C}$ situations.

(3) The idea of "localising" a sheaf on the moment graph has a parallel in representation theory. Given a complex semi-simple Lie algebra $\mathfrak{g}$ it is a difficult (and important) question to determine the characters of simple highest weight modules. Using BGG reciprocity this can be translated into the question of calculating the characters of indecomposable projective modules in the BGG category $\mathcal{O}$. Now projective modules admit deformations over $S(\mathfrak{h})$ and can also be localised on the moment graph, and hence one obtains a purely formal proof (an idea of Soergel, Fiebig) that projective modules correspond to Braden-MacPherson sheaves on the moment graph which (using deep facts from complex algebraic gometry) can be shown to calculate the stalks of intersection cohomology complexes, which are known to be given by Kazhdan-Lusztig polynomials.

(4) These ideas can also be pursued in positive characteristic. Parity sheaves, Fiebig's proof of Lusztig's conjecture. (So we get back to the question of the simple modules for $GL_n(\mathbb{F}_q)$!)