Mind your P and Q-symbols
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`https://people.mpim-bonn.mpg.de/geordie/Hecke.pdf`
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Errata and addenda by Darij Grinberg

The following are corrections and comments on the honours thesis "Mind your P and Q-symbols" by Geordie Williamson.

8. Errata

- **page v:** It is not quite correct that "there is no one source that explains why the Hecke algebra of the symmetric group is a cellular algebra". In fact this is proved in Mathas's [24, Theorem 3.20] through the Murphy cellular basis. What you seemingly intended to say here is that no source explains why the Kazhdan–Lusztig basis is cellular.

- **page 1, just before Proposition 1.1.2:** "the number of simple transpositions" $\rightarrow$ "the number m of simple transpositions". (This is to clarify that the transpositions are counted with multiplicity – e.g., the expression $s_1 s_2 s_1$ contains three, not two, simple transpositions.)

- **page 2, proof of Theorem 1.1.5:** It should be said that you WLOG assume that $i < j$ here.

- **page 2, proof of Theorem 1.1.5:** After "We first verify that $\operatorname{Im}(\varphi) \subset N(wt)$.", add "Let $(p, q) \in N(w)$.".

- **page 2, proof of Theorem 1.1.5:** "either $p = i$ and $q = j$" should be "either $p = i$ or $q = j$".

- **page 3, proof of Theorem 1.1.5:** The first paragraph on this page (where you prove injectivity of φ) is confusing and unnecessary complicated. Instead you can argue as follows: The value $\varphi(p, q)$ of the map φ always belongs to $N(t)$ in the case when $t(p) > t(q)$ (since $(p, q) \in N(t)$ in this case), but never belongs to $N(t)$ in the case when $t(p) < t(q)$ (since $t(t(p)) = p < q = t(t(q))$ entails that $(t(p), t(q)) \notin N(t)$ in this case). Hence, two values $\varphi(p_1, q_1)$ and $\varphi(p_2, q_2)$ of the map φ cannot be equal unless they come from the same case (i.e., unless the two inequalities $t(p_1) < t(q_1)$ and $t(p_2) < t(q_2)$ either both hold or both fail). But two values of φ coming from the same case cannot be equal unless the inputs are equal. Thus, φ is injective.

- **page 3, proof of Theorem 1.1.5:** The last paragraph of this proof can be simplified: The extra consideration of the $k = m$ case is unnecessary. Indeed, u_{m+1} is defined as an empty product and thus equals the identity permutation *id*. Hence, $u_{m+1}(i) < u_{m+1}(j)$ (because $i < j$). But $u_1(i) = w(i) > w(j) = u_1(j)$. Hence, there exists some $k \in \{1, 2, \ldots, m\}$ such that $u_k(i) > u_k(j)$ but $u_{k+1}(i) < u_{k+1}(j)$. From here, proceed as you do.

- **page 3, Corollary 1.1.7:** Replace "$w \in Sym$" by "$w \in Sym_n$".

- **page 4, proof of Lemma 1.2.2:** "Hence the exchange condition" $\to$ "Hence the left-hand version of the exchange condition".

- **page 5, proof of Proposition 1.2.1:** In (1.2.9), replace "u_2" by "u_1".

- **page 6, proof of Corollary 1.2.3:** Replace "u_n" by "u_2" on the first line of page 6.

- **page 6, proof of Theorem 1.2.4:** "by relation (i)" $\to$ "by relation (1.2.11a)".

- **page 7, proof of Lemma 1.3.1:** Replace "u_n" by "u_m" twice in the last paragraph of this proof.

- **page 7, before Proposition 1.3.2:** "$i_1, i_2, \ldots i_s$" should be "$i_1, i_2, \ldots, i_s$".

- **page 7, before Proposition 1.3.2:** "$i_1 < i_2 \cdots < i_s$" should be "$i_1 < i_2 < \cdots < i_s$".

- **page 7, proof of Proposition 1.3.2:** "and $t_m \in T$ such that $w_{m-1} = vt_m$" should be "and $t_{m-1} \in T$ such that $w_{m-1} = w_m t_{m-1} = vt_{m-1}$".

- **page 8, proof of Lemma 1.3.4:** "If $xr < yr$" should rather be "If $xr \leq yr$".

- **page 8, proof of Lemma 1.3.4:** "subexpression or" $\to$ "subexpression of".

- **page 8, §1.4:** "*right descent set* of a permutation $w \in Sym_n$" $\to$ "*right descent set* of w" (since you already have said "Given a permutation $w \in Sym_n$" at the beginning of this sentence).

- **page 8, §1.4:** "$\mathcal{L}(s_1 s_2) = s_1$, $\mathcal{R}(s_1 s_2) = s_2$" should be "$\mathcal{L}(s_1 s_2) = \{s_1\}$, $\mathcal{R}(s_1 s_2) = \{s_2\}$" (you forgot the set-braces).

- **page 8, Lemma 1.4.3:** The "$\supset$" signs should be "$\supseteq$" signs. Likewise in the proof.

- **page 10, §2.1:** After "A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$", add "of nonnegative integers".

- **page 10, §2.1:** In "where $m = l(\lambda)$ and $\lambda_i = 0$ for $i > l(\lambda)$", replace the "and" by "since". (The "$\lambda_i = 0$ for $i > l(\lambda)$" part is not a requirement but a consequence of $m = l(\lambda)$.)

- **page 10, §2.1:** *"Ferrer's diagram"* $\rightarrow$ *"Ferrers diagram"*.

- **page 10, §2.2:** In the description of the bumping algorithm, it is worth explaining that we regard a tableau T of shape λ as containing infinitely many rows, with all but the first $l(\lambda)$ of them being empty. Thus, if an entry is bumped out of the last nonempty row of T, then it is inserted into the next row, which is empty, and finds rest in that row (which is no longer nonempty). Thus, the new box created is the first box in the (formerly empty) $(l(\lambda)+1)$-st row.

- **page 11, §2.2:** It should be said that the new box is counted as part of the bumping route (despite the definition of the bumping route sounding like it isn't).

- **page 11, §2.2:** "Similarly (i,j) is *strictly below* (k,l) if $i < k$ and *weakly below* if $i \leq k$" $\rightarrow$ "Similarly (i,j) is *strictly below* (k,l) if $i > k$ and *weakly below* if $i \geq k$".

  Also, it is worth saying that "below" and "to the left" mean "weakly below" and "weakly to the left" unless qualified differently.

- **page 11, Lemma 2.2.1:** This lemma is too crowded. To make it clearer, I would break it up into three parts:

  > (a) The bumping route of $T \leftarrow x$ moves to the left. That is, if x_i and x_{i+1} are (consecutive) elements in the bumping sequence then x_{i+1} is weakly left of x_i in T.
  >
  > (b) Furthermore, if $x < y$, then the bumping route of $T \leftarrow x$ is strictly left of the bumping route of $(T \leftarrow x) \leftarrow y$, and the new box of $T \leftarrow x$ is strictly left and weakly below the new box of $(T \leftarrow x) \leftarrow y$.
  >
  > (c) Furthermore, if $x < y$, then the bumping route of $(T \leftarrow y) \leftarrow x$ is weakly left of the bumping route of $T \leftarrow y$, and the new box of $(T \leftarrow y) \leftarrow x$ is strictly below and weakly left of the new box of $T \leftarrow y$.

  The three paragraphs of the proof correspond precisely to these three parts (a), (b) and (c). And this correspondence is important, because the very first sentence of the proof ("If the new box of $T \leftarrow x$ is in the first row then there is nothing to prove") clearly makes sense only for part (a), not for parts (b) and (c).

- **page 11, proof of Lemma 2.2.1:** In the first paragraph of the proof, replace "$T_{i+1,y}$" by "$T_{i+1,j}$" twice (on the same line).

- **page 11, proof of Lemma 2.2.1:** In the third paragraph of the proof, replace "$x_1, \ldots x_j$" by "$x_1, \ldots, x_j$".

- **page 11, proof of Lemma 2.2.1:** In the third paragraph of the proof, replace "and let p be as above" by "and let p be the minimum of k and j".

- **page 11, proof of Lemma 2.2.1:** In the third paragraph of the proof, replace "bumping sequence of $T (\leftarrow y) \leftarrow x$" by "bumping route of $(T \leftarrow y) \leftarrow x$".

- **page 12, proof of Lemma 2.2.2:** "an an element" $\to$ "an element".

- **page 12, §2.3:** In the displayed equation "$\varnothing \leftarrow w = (\ldots ((\varnothing \leftarrow w_1) \leftarrow w_2) \ldots) \leftarrow w_n$", the "$w_n$" should be a "$w_i$".

- **page 12, §2.3:** On the last line of this page, replace "the w corresponds" by "that w corresponds".

- **page 12, §2.3:** On the last line of this page, replace "Shensted" by "Schensted".

- **page 13, proof of Theorem 2.3.1:** "pair of tableau" $\to$ "pair of tableaux".

- **page 13, proof of Theorem 2.3.1:** This proof is missing a noticeable part: You only showed that the reverse procedure undoes the Robinson–Schensted correspondence, but it should also be proved that the reverse procedure can be applied to a pair (P, Q) of same-shape standard tableaux (even if we don't know a-priori that this pair is the image of a permutation under the Robinson–Schensted correspondence) and can then be undone by the Robinson–Schensted correspondence. (Unless you already know that the number of pairs (P, Q) is $\leq |Sym_n|$, which allows you to skip this step thanks to the pigeonhole principle.) This requires checking, among other things, that the reverse bumping algorithm, when applied to a tableau S and an outside corner c of S, always produces a tableau T and a number x which satisfy $T \leftarrow x = S$ and produce the new box c.

- **page 14, first line:** "we define $(\lambda \cup \mu)_i =$" should be "we define a new partition $\lambda \cup \mu$ by $(\lambda \cup \mu)_i =$".

  Also, "for $1 \leq i \leq \max\{m, n\}$" should be "for $i \geq 1$" (otherwise the formulation allows for nonzero parts beyond $\max\{m, n\}$). Generally, it is better to use the infinite form of partitions here (i.e., replace "$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$" by "$\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ and $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$"), as this will keep the letter n free for its later use for the size of λ.

- **page 14:** "ommiting" $\to$ "omitting".
- **page 14, Lemma 2.4.1:** Here you are writing saturated chains of partitions $\varnothing = \lambda_0 \subset \lambda_1 \subset \cdots \subset \lambda_n$ as $\lambda_1 \subset \lambda_2 \subset \cdots \subset \lambda_n$ (that is, you are omitting λ_0). This is perfectly valid (λ_0 is $\varnothing$ and thus is uninteresting) but should probably be explained.
- **page 14, Lemma 2.4.1:** The notation $w\left(i\right)$ here is rather misleading, as it would normally mean the image of i under w (that is, the same as w_i).
- **page 14, proof of Lemma 2.4.1:** "we defined we defined" $\to$ "we defined".
- **page 14, proof of Lemma 2.4.1:** "be adding" $\to$ "by adding".
- **page 14, proof of Lemma 2.4.1:** "$Shape(P^{(1))}$" should be "$Shape\left(P^{(1)}\right)$".
- **page 15, first sentence:** It should be said that $\left(i,j\right)$ denotes the cell in the i-th row (counted from the bottom) and the j-th column (counted from the left).
- **page 15, first sentence:** It is also worth being explicit: $w\left(i,j\right)$ is the word that remains if we start with $w_1 w_2 \ldots w_j$ and remove all letters larger than i.
- **page 15:** "row-bumping the $\left(i,j\right)^{th}$ partial permutation into the empty set" $\to$ "row-bumping the $\left(i,j\right)^{th}$ partial permutation $w\left(i,j\right)$ into the empty tableau". Or you can just say $Shape\left(\varnothing \leftarrow w\left(i,j\right)\right)$, since you have that nice notation for it.
- **page 15:** "label them with the empty set" $\to$ "label them with the empty tableau".
- **page 16, first paragraph:** An "$\subset$" sign is missing in "$r\left(n,0\right) \subset r\left(n,1\right) \subset \cdots r\left(n,n\right)$".
- **page 16, proof of Lemma 2.5.1:** Add a period at the end of the displayed relation.
- **page 16, proof of Lemma 2.5.1:** In "or $w\left(i+1,j\right) = w_1 w_2 \ldots w_l$ (i+1) $w_{l+1} \ldots w_k$", the "i+1" should be "$i+1$" (that is, in mathmode).
- **page 16, proof of Lemma 2.5.1:** "for some $l \leq j$" $\to$ "for some $l \leq k$".
- **page 16, proof of Lemma 2.5.1:** At the very end, "$r\left(i+1,j\right) \subset r\left(i,j\right)$" should be "$r\left(i,j\right) \subset r\left(i+1,j\right)$".

- **page 16, possibility 2:** An easier and less confusing argument would proceed as follows: "In this case clearly $\mu = \lambda$ and $\rho = \nu$ but $\lambda \subset \nu$ and $\mu \subset \rho$. From $\lambda \subset \nu$, we obtain $\nu = \lambda \cup \nu = \mu \cup \nu$ (since $\lambda = \mu$) and thus $\rho = \nu = \mu \cup \nu$."

- **page 16, possibility 3:** An easier and less confusing argument would proceed as follows: "In this case clearly $\nu = \lambda$ and $\rho = \mu$ but $\lambda \subset \mu$ and $\nu \subset \rho$. From $\lambda \subset \mu$, we obtain $\mu = \mu \cup \lambda = \mu \cup \nu$ (since $\lambda = \nu$) and thus $\rho = \mu = \mu \cup \nu$."

- **page 17, possibility 4:** Replace "w_k" by "w_j" three times in this argument.

- **page 17, possibility 4:** After "must be in the same location as i in $T(\mu)$", I would add "(since $T(\nu) = T(\lambda) \leftarrow w_j$)" for clarity.

- **page 17, possibility 4:** "places w_k in the box previously occupied" $\to$ "bumps i out of the box previously occupied". (The entry that bumps i is not necessarily w_k.)

- **page 17, possibility 4:** "(the row number of i in μ)" $\to$ "(the row number of i in $T(\mu)$)".

- **page 17, possibility 4:** I find the last two sentences of this argument rather hard to follow. I would instead argue as follows:

  By construction, $T(\rho) = T(\mu) \leftarrow w_j$ and $T(\nu) = T(\lambda) \leftarrow w_j$. Again by construction, the tableaux $T(\lambda)$ and $T(\nu)$ can be obtained from $T(\mu)$ and $T(\rho)$ (respectively) by deleting the entry i.

  Note that there is a single box in $\rho \setminus \mu$ (since $T(\rho) = T(\mu) \leftarrow w_j$). This box does not belong to μ, and thus does not belong to ν (since $\mu = \nu$).

  Let (s, s') be the box of $T(\mu)$ that contains the entry i. Thus, this box lies in the s-th row. Moreover, the tableaux $T(\mu)$ and $T(\lambda)$ differ only in the box (s, s') (since $T(\lambda)$ can be obtained from $T(\mu)$ by deleting the entry i, which lies in the box (s, s')). Hence, $\mu_s = \lambda_s + 1$, whereas $\mu_t = \lambda_t$ for all $t \neq s$.

  Now, consider the bumping route of $T(\mu) \leftarrow w_j$ (which produces $T(\rho)$) and the bumping route of $T(\lambda) \leftarrow w_j$. If the former bumping route did not contain the box (s, s'), then it would be identical with the latter bumping route (since the tableaux $T(\mu)$ and $T(\lambda)$ differ only in the box (s, s'), and thus the insertion of w_j into both tableaux will proceed exactly identically unless it hits this specific box); but this is impossible, since the former bumping route contains the single box in $\rho \setminus \mu$ (because it produces the tableau $T(\rho)$) whereas the latter bumping route does not (since this box does not belong to ν). Hence, the former bumping route must contain the box (s, s'). The entry bumped from this box is, of course, i (since this is the entry of $T(\mu)$ in this box). After being bumped, this entry i is moved into the next (i.e., $(s+1)$-st) row, where it finds its rest at the end of the row (it cannot bump any further entry, since i is larger than all other entries of $T(\mu)$). Thus, the new box of $T(\mu) \leftarrow w_j$ lies in the $(s+1)$-st row. Since $T(\mu) \leftarrow w_j = T(\rho)$, this new box must be the single box in $\rho \setminus \mu$, and thus we conclude that the single box in $\rho \setminus \mu$ lies in the $(s+1)$-th row. Hence, $\rho_{s+1} = \mu_{s+1} + 1$, whereas $\rho_t = \mu_t$ for all $t \neq s+1$.

- **page 17, after the four possibilities:** "(since i is greater than all the w_i)" $\rightarrow$ "(since i is greater than each of $w_1, w_2, \ldots, w_k$)" (beware of overusing the letter i).

- **page 17, after the four possibilities:** "Hence $\rho_i = \lambda_i$ when $i \neq 1$ and $\rho_1 = \lambda_1 + 1$" $\rightarrow$ "Hence $\rho_t = \lambda_t$ when $t \neq 1$ and $\rho_1 = \lambda_1 + 1$" (again, beware of overusing the letter i).

- **page 17, local rule 3:** "then let i be the unique integer such that $\mu_i = \lambda_i + 1$. Then, $\rho_j = \mu_j$ if $j \neq i+1$ and $\rho_{i+1} = \mu_{i+1} + 1$" $\rightarrow$ "then let s be the unique integer such that $\mu_s = \lambda_s + 1$. Then, $\rho_t = \mu_t$ if $t \neq s+1$ and $\rho_{s+1} = \mu_{s+1} + 1$"

- **page 17, local rule 4:** "$\rho_j = \lambda_j$ if $j \neq 1$" $\rightarrow$ "$\rho_t = \lambda_t$ if $t \neq 1$" (you are overusing the letter j this time).

- **page 17, §2.6:** The first paragraph is misadvertising Knuth equivalence. When it comes to **deciding** whether two words have the same P-symbol, the easiest way to proceed is to compute their P-symbols using Robinson–Schensted insertion. Knuth equivalence does not provide a decision procedure (unless you mean the inefficient "map out the whole equivalence class by search and backtracking" method).

- **page 17, §2.6:** After "three adjacent elements", I would add "of w" (as opposed to three consecutive integers).

- **page 18:** "and write $u \equiv w$" $\rightarrow$ "and write $v \equiv w$".

- **page 18, proof of Proposition 2.6.1:** "show that elementary" $\rightarrow$ "show that an elementary".

- **page 18, proof of Proposition 2.6.1:** After "assume that T has one row", I would add "(or zero rows, in which case we pretend that T has one empty row)".

- **page 18, proof of Proposition 2.6.1:** In the "seven possibilities", it should be said that the "$t_{i-1} <$" part of the chain of inequalities should be understood as void if $i = 1$. Likewise, the case $m = 0$ is counted towards possibility 1.

- **page 18, proof of Proposition 2.6.1:** The "seven possibilities" are not a complete list. There are in fact some further possibilities:

  7a) $t_{m-1} < x < y < z < t_m$:

$$T \leftarrow zxy = \begin{array}{|c|c|c|c|c|c|} \hline t_1 & t_2 & \cdots & t_{m-1} & x & y \\ \hline z \\ \cline{1-1} t_m \\ \cline{1-1} \end{array} = T \leftarrow xzy.$$

  8) $t_{i-1} < x < y < t_i < \cdots < t_m < z$ and $i < m$:

$$T \leftarrow zxy = \begin{array}{|c|c|c|c|c|c|c|} \hline t_1 & t_2 & \cdots & x & y & \cdots & z \\ \hline t_i & t_j \\ \cline{1-2} \end{array} = T \leftarrow xzy \qquad \left(t_j := t_{i+1}\right).$$

  8a) $t_{i-1} < x < y < t_i < \cdots < t_m < z$ and $i = m$ (thus $t_{m-1} < x < y < t_m < z$):

$$T \leftarrow zxy = \begin{array}{|c|c|c|c|c|c|} \hline t_1 & t_2 & \cdots & t_{m-1} & x & y \\ \hline t_m & z \\ \cline{1-2} \end{array} = T \leftarrow xzy.$$

  Furthermore, some of the possibilities need to be subdivided further:

- **page 18, proof of Proposition 2.6.1:** In possibility 1, there should not be a "t_i" in the second row of the tableau.

- **page 18, proof of Proposition 2.6.1:** In possibility 5, the definition of t_j needs to be qualified slightly: We do indeed set $t_j := t_{i+1}$ if $i + 1 < k$; otherwise we set $t_j := z$.

- **page 19, proof of Proposition 2.6.1:** In possibility 7, you need to require $i < m$, since the output looks different in the case $i = m$ (see possibility 7a above).

- **page 19, proof of Proposition 2.6.1:** "In cases 1, 2 and 3" $\to$ "In cases 1, 2, 3, 7a, 8 and 8a".

- **page 19, Lemma 2.6.3:** Add "for any positive integer v" at the end of this lemma.

- **page 20, proof of Lemma 2.6.3:** "as $r_1, r_2 \ldots r_p$" $\to$ "as $r_1, r_2, \ldots, r_p$".

- **page 20, proof of Lemma 2.6.3:** Missing $<$ sign in "$v = v_1 < v_2 \cdots < v_r$".

- **page 20, proof of Lemma 2.6.3:** For the equivalence "$r_i v_i \equiv v_{i+1} r_i'$" to hold for all $i \leq r$, you need to set $v_{r+1} := \varnothing$. Alternatively, you may want to state this equivalence for $i \leq r - 1$ only, and then add $r_r v_r \equiv r_r'$ and $r_i = r_i'$ for all $i > r$ as a last step. But it is perhaps best to proceed differently: Set $v_i := \varnothing$ for all $i > r$, and then argue that $r_i v_i \equiv v_{i+1} r_i'$ holds for all $i \geq 1$ (including the cases $i < r$ and $i \geq r$).

- **page 20, §2.7:** After "$i+1$ lies strictly below and weakly left of i in P", it is worth adding "(actually the "weakly left" part follows from the "strictly below" part, since P is a tableau)". This is tacitly used in the proof of Proposition 2.8.2.
- **page 21, proof of Proposition 2.7.1:** "row Row Bumping Lemma" $\to$ "Row Bumping Lemma".
- **page 21:** After "successively down each column", add "(from the leftmost column to the rightmost)".
- **page 21, Lemma 2.7.2:** I would replace "Suppose" by "Let P be a tableau such that".
- **page 21, Lemma 2.7.2:** Replace the "$\subset$" sign by "$\subseteq$", as I think (not sure about this!) that you use "$\subset$" for proper subsets.
- **page 21, Lemma 2.7.2:** Add a period before "then".
- **page 21, proof of Lemma 2.7.2:** "fill a diagram" $\to$ "fill the diagram" (twice). Also, this is a bit misleading: You don't want an arbitrary filling (such a filling would not be unique), but rather a tableau. (You use this when you tacitly argue that the entries $1, 2, \ldots, c_1$ lie in the first column rather than further right.)
- **page 21, proof of Lemma 2.7.2:** Again, replace the "$\subset$" signs by "$\subseteq$".
- **page 21, proof of Lemma 2.7.2:** In "$c_1+1, c_1+2, \ldots, c_2$", replace "$c_2$" by "$c_1+c_2$".
- **page 21, proof of Lemma 2.7.2:** "contained in $\mathcal{D}(S_\lambda)$" $\to$ "containing $\mathcal{D}(S_\lambda)$".
- **page 22, first paragraph:** In "$\lambda_1+\cdots+\lambda_k \trianglelefteq \mu_1+\cdots+\mu_k \trianglelefteq \pi_1+\cdots+\pi_k$", the "$\trianglelefteq$" signs should be "$\leq$".
- **page 22, second paragraph:** "in a partition or tableau" $\to$ "in a partition λ or tableau" (since you refer to λ in the same sentence).
- **page 22, before Lemma 2.8.1:** After "It turns out that the dominance order", add "on the partitions of a given integer n" (you are not saying anything about partitions of different n's here, even though you defined dominance for them as).
- **page 22, Lemma 2.8.1:** "weight" $\to$ "size".
- **page 22, proof of Lemma 2.8.1:** "operations" $\to$ "operation".

- **page 23, first paragraph:** "otherwise $\mu_i = \mu_{i-1} = \lambda_{i-1} \geq \lambda_i$" $\to$ "otherwise $\mu_i = \mu_{i-1} \geq \lambda_{i-1} \geq \lambda_i$".

- **page 23, second paragraph:** This argument does not work as given. For example, let $\lambda = (6, 4, 3, 1)$ and $\mu = (5, 5, 2, 2)$. Then, $i = 1$ and $j = 4$ and therefore $\mu' = (6, 5, 2, 1)$. But we don't have $\mu' \trianglelefteq \lambda$, and thus we cannot apply the induction hypothesis.

  One way to correct the argument is by changing the definition of j: We let j be the smallest integer $j > i$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_j \leq \mu_1 + \mu_2 + \cdots + \mu_j$. Then, it is easy to see that $\lambda_j < \mu_j$ (otherwise, j would not be the smallest) and $\lambda_{j+1} \geq \mu_{j+1}$ (otherwise, we would have $\lambda_1 + \lambda_2 + \cdots + \lambda_{j+1} < \mu_1 + \mu_2 + \cdots + \mu_{j+1}$, contradicting $\mu \trianglelefteq \lambda$), so that $\mu_{j+1} \leq \lambda_{j+1} \leq \lambda_j < \mu_j$. Thus, (j, μ_j) is an outside corner of μ. Define μ' in the way you suggest (using this new j) and show that $\sum \left|\lambda_k - \mu'_k\right| < n$ and $\mu' \trianglelefteq \lambda$. Then use the induction hypothesis.

  In all of this, you should view λ and μ as infinite sequences, so that the case $l(\mu) > l(\lambda)$ case disappears.

- **page 23, proof of Proposition 2.8.2:** Replace the "$\subset$" sign by "$\subseteq$".

- **page 23, proof of Proposition 2.8.2, Case 1:** "below" $\to$ "strictly below".

- **page 24, proof of Proposition 2.8.2, Case 2:** "below" $\to$ "strictly below" (three times).

- **page 24, Note 2:** This is a bit imprecise: Skew shapes cannot be arbitrary sets of tiles; if the shape is not a set difference of two Young diagrams, then slides might not preserve the P-symbol of the reading word. I don't know whether this makes the description of the construction here false.

- **page 24, Note 2:** "w_m" should be "w_n".

- **page 24, Note 2:** "Shützenberger" $\to$ "Schützenberger".

- **page 26, §3.1, first paragraph:** "algebra of A" $\to$ "algebra over A".

- **page 26, §3.1:** "in Theorem 1.2.11" $\to$ "in Theorem 1.2.4".

- **page 26, §3.1:** "are reduced expression" $\to$ "are reduced expressions".

- **page 26, §3.1:** In the long(ish) computation, I would replace "$T_{i_1} T_{i_2} \ldots T_{i_m}^2$" by "$T_{i_1} T_{i_2} \ldots T_{i_{m-1}} T_{i_m}^2$" to make it clearer.

- **page 27, §3.2:** "a ring homomorphism" $\to$ "an A-algebra homomorphism".

- **page 27, §3.2:** I'd replace "representations afforded by $H_n(q)$" by "representations of $H_n(q)$", since "afforded by" suggests modules to me (i.e., I would refer to a representation of $H_n(q)$ on some A-module M as a "representation afforded by M", not "by $H_n(q)$").

- **page 27, Lemma 3.3.1:** It would be best to add the extra condition "$\ell(x) \leq m$" under the summation sign (i.e., to replace the summation sign by "$\sum\limits_{\substack{x \leq w; \\ \ell(x) \leq m}}$"). This makes the lemma a little bit stronger (the proof is very easy to adapt), and this extra strength is needed in the proof of Proposition 3.3.2.

- **page 28, proof of Lemma 3.3.1:** I'd replace "(since $r_1 r_2 \ldots r_{i+1} < r_1 r_2 \ldots r_i$)" by the more detailed explanation "(by (3.1.2), since $r_1 r_2 \ldots r_{i+1} < r_1 r_2 \ldots r_i$)".

- **page 28, proof of Lemma 3.3.1:** It is also worth explaining how you obtain the "$qT_{r_1} \ldots \widehat{T_{r_p}} \ldots \widehat{T_{r_q}} \ldots T_{r_m}$" term. (Namely, you rewrite $T_{r_1 \ldots r_{i+1}}$ as $T_{r_1} \ldots \widehat{T_{r_p}} \ldots \widehat{T_{r_q}} \ldots T_{r_{i+1}}$ using the fact that $r_1 \ldots \widehat{r_p} \ldots \widehat{r_q} \ldots r_{i+1}$ is a reduced expression for $r_1 r_2 \ldots r_{i+1}$; then you multiply the extra factors $T_{r_{i+2}}, \ldots, T_{r_m}$ on the right of this.)

- **page 28, proof of Proposition 3.3.2:** The long displayed computation could use a bit more explanation: To get from the first line to the second, you use the fact that $r_m r_{m-1} \ldots r_1$ is a reduced expression for w^{-1} (by Lemma 1.13) and thus we have $T_{w^{-1}} = T_{r_m} T_{r_{m-1}} \cdots T_{r_1}$.

- **page 28, proof of Proposition 3.3.2:** On the last line of this proof, "$\left(q^{-1}T_{r_1}\right)\left(q^{-1}T_{r_1}\right)$" should be "$\left(q^{-1}T_{r_1}\right)\left(q^{-1}T_{r_2}\right)$".

- **page 28, proof of Proposition 3.3.2:** The claim in the last sentence of this proof is not quite obvious: Why cannot T_w emerge from any other subexpressions? This becomes clear once Lemma 3.3.1 is strengthened as I suggested above (adding the $\ell(x) \leq m$ condition under the summation sign), since then we see that any subexpression of length $< m$ can only produce T_x terms with $\ell(x) < m$.

- **page 29:** "Applying * yields" $\to$ "Let $r = s_j \in S$. Applying * to (3.1.2) yields".

- **page 29:** "Now Lemma 1.4.2 shows that" $\to$ "Now Corollary 1.3.3 shows that".

- **page 29:** "If is straightforward" $\to$ "It is straightforward".

- **page 29, proof of Proposition 3.4.1:** In the first sentence, it is not clear why you can extend ι multiplicatively like this (what if two equal $T_{i_1} T_{i_2} \ldots T_{i_m}$s correspond to different $\iota\left(T_{i_1}\right)\iota\left(T_{i_2}\right)\cdots\iota\left(T_{i_m}\right)$'s?).

Instead I would proceed as follows:

Base-changing the standard A-algebra structure on $H_n(q)$ using the ring homomorphism $\overline{} : A \to A$, we obtain a new A-algebra structure on $H_n(q)$, which is given explicitly by

$$a \rightharpoonup h = \overline{a} \cdot h \qquad \text{for all } a \in A \text{ and } h \in H_n(q)$$

(where the "$\rightharpoonup$" symbol means the action of A on the new A-algebra $H_n(q)$, whereas the "$\cdot$" symbol means the original action of A on $H_n(q)$). Alternatively, if you view an A-algebra as a ring M equipped with a ring homomorphism from A to its center $Z(M)$, then this new A-algebra structure on $H_n(q)$ is the composition $A \to A \to H_n(q)$, where the left arrow is the involution $\overline{} : A \to A$ and the right arrow is the ring homomorphism corresponding to the original A-algebra structure on $H_n(q)$. Either way, let us denote this new A-algebra structure on $H_n(q)$ by $\overline{H_n(q)}$. As a ring, it is still the old $H_n(q)$, but the action of A is different (in that any polynomial $F(q)$ now acts as $\overline{F(q)} = F(q^{-1})$).

Now, define an A-algebra homomorphism $\iota : H_n(q) \to \overline{H_n(q)}$ by setting $\iota(T_j) := T_j^{-1}$ for each $1 \leq j < n$. It is easy to see that this homomorphism is well-defined, since the elements T_j^{-1} in $\overline{H_n(q)}$ satisfy the same relations (3.1.1) as the elements T_j in $H_n(q)$ (indeed, the first two relations for the T_j^{-1} follow immediately by inverting the corresponding relations for the T_j; the third relation for the T_j^{-1} is saying that $T_j^{-2} = (q-1) \rightharpoonup T_j^{-1} + q \rightharpoonup T_{id}$ in $\overline{H_n(q)}$, but this is equivalent to $T_j^{-2} = \left(q^{-1}-1\right) T_j + q^{-1} T_{id}$ in $H_n(q)$, which can be easily derived from (3.1.1c)). Moreover, ι is an A-algebra homomorphism from $H_n(q)$ to $\overline{H_n(q)}$, thus a ring homomorphism from $H_n(q)$ to $H_n(q)$ (since $\overline{H_n(q)} = H_n(q)$ as rings). It is easy to see that $\iota(T_w) = T_{w^{-1}}^{-1}$ for each $w \in Sym_n$ (indeed, pick a reduced expression $s_{i_1} s_{i_2} \cdots s_{i_m}$ for w and observe that $T_w = T_{i_1} T_{i_2} \cdots T_{i_m}$ and $T_{w^{-1}} = T_{i_m} T_{i_{m-1}} \cdots T_{i_1}$, so that

$$\begin{aligned} \iota(T_w) &= \iota\left(T_{i_1} T_{i_2} \cdots T_{i_m}\right) = \iota\left(T_{i_1}\right) \iota\left(T_{i_2}\right) \cdots \iota\left(T_{i_m}\right) \\ &= T_{i_1}^{-1} T_{i_2}^{-1} \cdots T_{i_m}^{-1} = \left(\underbrace{T_{i_m} T_{i_{m-1}} \cdots T_{i_1}}_{=T_{w^{-1}}} \right)^{-1} = T_{w^{-1}}^{-1} \end{aligned}$$

). Since ι is an A-algebra homomorphism from $H_n(q)$ to $\overline{H_n(q)}$, this entails that

$$\iota\left(\sum_{w \in Sym_n} F_w(q) T_w \right) = \sum_{w \in Sym_n} F_w(q) \rightharpoonup T_{w^{-1}}^{-1} = \sum_{w \in Sym_n} \overline{F_w(q)} T_{w^{-1}}^{-1}$$

for any Laurent polynomials $F_w(q) \in A$. Finally, note that ι is a ring homomorphism, thus a $\mathbb{Z}$-algebra homomorphism. Recall that the $\mathbb{Z}$-algebra $H_n(q)$ is generated by the $n+1$ elements $q, q^{-1}, T_1, T_2, \ldots, T_{n-1}$ (since $A = \mathbb{Z}\left[q, q^{-1}\right]$). Since ι^2 agrees with id on each of these $n+1$ generators (this is easily checked directly[1]), we thus conclude that $\iota^2 = id$ (because ι and thus ι^2 is a $\mathbb{Z}$-algebra homomorphism). Hence, ι is an involution.

- **page 30, proof of Proposition 3.4.2:** "by Lemma 1.1.3" $\to$ "by Proposition 1.1.3".
- **page 30, proof of Proposition 3.4.2:** In the first displayed equation, replace "$\iota\left(T_y\right)^*$" by "$\left(\iota\left(T_y\right)\right)^*$" for greater clarity. A similar change is worth doing in the second displayed equation.
- **page 31, Theorem 3.5.1:** "element C_w" $\to$ "element $C_w \in H_n(q)$".
- **page 31, after Theorem 3.5.1:** "basis" $\to$ "basis of $H_n(q)$".
- **page 31, after Theorem 3.5.1:** "the linear map sending C_w to T_w" $\to$ "the linear map sending T_w to C_w" (you don't know yet that the C_w form a basis, so you cannot define a linear map by its action on the C_w).
- **page 31, Proof of Uniqueness:** I would replace "our expression for $T_{y^{-1}}^{-1}$ in Section 3.3" by "our expression for $T_{y^{-1}}^{-1}$ from Proposition 3.3.2" for better clarity ("in" sounds like "into" here).
- **page 32, Proof of Uniqueness:** In (3.5.1), the "$q_x^{-\frac{1}{2}}$" on the right hand side should be "$q_x^{\frac{1}{2}}$".
- **page 32, Proof of Uniqueness:** In the paragraph after (3.5.1), "degree at least 1" should be "degree at least $\dfrac{1}{2}$".
- **page 32, Proof of Existence:** "w lies in S" $\to$ "w lies in S" (mathmode!).
- **page 32, Proof of Existence:** "there exists $r \in R$" $\to$ "there exists $r \in S$".
- **page 32, Proof of Existence:** "since $C_r = q^{\frac{1}{2}} T_r - q^{\frac{1}{2}} T_{id}$" $\to$ "since $C_r = q^{-\frac{1}{2}} T_r - q^{\frac{1}{2}} T_{id}$".
- **page 33, Proof of Existence:** "and then equating coefficients of T_x in (3.5.3)" $\to$ "and then equating coefficients of T_x in Theorem 3.5.1".

[1]Indeed, ι sends each of these generators to its reciprocal, and thus ι^2 must reciprocate it twice, but of course $\left(x^{-1}\right)^{-1} = x$.

- **page 34, Proof of Existence:** "in the sum $\sum \mu(z,v)\, q_z^{\frac{1}{2}} q_w^{\frac{1}{2}} P_{x,z}$" $\to$ "in the sum $\sum \mu(z,v)\, q_z^{-\frac{1}{2}} q_w^{\frac{1}{2}} P_{x,z}$".
- **page 35, proof of Theorem 3.6.1:** The formula for $T_r C_r$ is unnecessary: A strong induction needs no induction base. (The sum will just be empty if you have $w = r$.)
- **page 35, proof of Theorem 3.6.1:** Before the last long displayed computation, replace "Now:" by "Now, by (3.5.3) and (3.5.2), we have".
- **page 35, proof of Theorem 3.6.1:** After the last long displayed computation, I would add "(by (3.5.3) again)".
- **page 36, proof of Proposition 3.6.2:** In the third displayed equation, "q_y^{-1}" should be "$q_{y^{-1}}^{-1}$". Same in the fourth displayed equation.
- **page 37, Note 1:** "for all primes powers" $\to$ "for all prime powers".
- **page 37, Note 1:** Another reference for this proof is Theorem 11 in Daniel Bump's notes *Hecke algebras* (`http://sporadic.stanford.edu/bump/math263/hecke.pdf`).
- **page 37, Note 2:** I would replace "End_G" here by "$\mathrm{End}_{\mathbb{C}G}$" just to be on the safe side.
- **page 37, Note 3:** Add a comma after "then, in fact".
- **page 37, Note 3:** Another proof of the Tits deformation theorem can be obtained from: Richard Dipper and Gordon James, *Blocks and idempotents of Hecke algebras of general linear groups*, Proc. London Math. Soc. (3) **54** (1987), pp. 57–82, Theorem 4.3 and §3.1 (vii). (The former theorem entails that the Hecke algebra $H_n(q)$ decomposes as a direct product of matrix rings when it is semisimple, and that the factors are indexed by the partitions of n. Then, §3.1 (vii) shows that the dimensions of these factors are combinatorially given in terms of standard tableaux and therefore independent of q.)
- **pages 37–38, Note 5:** Are you sure that the condition for semisimplicity is correct? According to Theorem 4.3 in the paper by Dipper and James just cited, the right condition for $H_n(q)$ to be semisimple is "$q = 1$ or $q^2, q^3, \ldots, q^n \neq 1$ and $q \neq 0$" (unless $n \leq 2$, in which q is allowed to be 0).
- **page 38, Note 5:** "if $e(z) < n$" $\to$ "if $1 < e(z) \leq n$".
- **page 38, Note 8:** Is the preprint [7] available somewhere?

- **page 39, Proposition 4.1.1:** It is worth pointing out that the g can depend on i_k (i.e., there isn't always one single g for the entire chain).
- **page 40:** In the first table on this page, the rows are indexed T_{id} and T_{id}. Clearly, the second row should be indexed T_{s_1} instead.
- **page 43, Lemma 4.3.2:** Do you want to require $rw < w$ here? Otherwise, (4.3.1) does not apply in the proof.
- **page 43, proof of Lemma 4.3.2:** In the first displayed equation of this proof, "q_{rw}" should be "q_w", whereas "$P_{x,z}$" should be "$P_{rw,z}$".
- **page 46, Proposition 4.5.3:** Replace both "$\supset$" signs by "$\supseteq$" signs. Same in the proof.
- **page 48, §5.1:** The notations in the matrix ring example (after the definition of a cellular algebra) are confusing: It is not a good idea to denote the matrix ring by R, since it plays the role of H (not of R) in the definition of a cellular algebra. It is also not good to use k for the base field, since k appears as an index in some sums. I would denote the base ring by R (this fits with the definition of a cellular algebra) and use $R^{n\times n}$ for the matrix rings.

  (The same issue reappears on page 49 and perhaps later.)
- **page 49:** "the the left module" $\to$ "the left module".
- **page 50, §5.2:** "it to show" $\to$ "is to show".
- **page 51:** After "On the other hand," and before "$\mathcal{R} \cap \{s_i, s_{i+1}\}$ contains", add "if".
- **page 52, proof of Proposition 5.2.3:** "and $\ldots zxy \ldots$ to $\ldots xzy \ldots$" $\to$ "and ...$yzx \ldots$ to $\ldots yxz \ldots$" (otherwise you are listing the same transformation twice).
- **page 52, Corollary 5.2.4:** A comma is missing in "$i_1, i_2, \ldots i_m$".
- **page 57, §5.4:** "the descent of a tableau" $\to$ "the descent set of a tableau".
- **page 58, Corollary 5.4.3:** A comma is missing in "$i_1, i_2, \ldots i_m$".
- **page 65:** "use of Lemma 3.3.2" $\to$ "use of Proposition 3.3.2".
- **page 65:** In the displayed equation between (A.1.2) and (A.1.3), the sum $\sum\limits_{y<w} R_{y,w} T_y$ needs to be multiplied by $q_w^{\frac{1}{2}}$. The same applies to the similar sum on the right hand side of (A.1.3).

- **page 65:** In (A.1.3), the "$\widetilde{T}_w$" inside the sum should be a "$\widetilde{T}_y$".
- **page 65:** In "there is a unique element $C_w = \sum_{y \leq w} \epsilon_y \epsilon_w q_w^{\frac{1}{2}} q_y^{-\frac{1}{2}} \overline{P}_{y,w} \widetilde{T}_w$", the "$\overline{P}_{y,w}$" should be a "$\overline{P_{y,w}}$" (the line should cover the whole term), whereas the "$\widetilde{T}_w$" should be "$\widetilde{T}_y$".. Moreover, "element" should be "ι-invariant element".

  Actually, the "$\overline{P}_{y,w}$"$\mapsto$"$\overline{P_{y,w}}$" (and likewise for $P_{x,w}$) change should be done throughout §A.1.
- **page 67:** "from the first row to the third" $\rightarrow$ "from the first column to the third" perhaps?

  Likewise, "second two rows" $\rightarrow$ "second two columns"?

  Or should perhaps S_λ be transposed here? (I can't really follow this argument.)