A MASLOV INDEX FOR NON-HAMILTONIAN SYSTEMS

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Abstract. We extend the definition of the Maslov index to a broad class of non-Hamiltonian dynamical systems. To do this, we introduce a family of topological spaces — which we call *Maslov–Arnold spaces* — that share key topological features with the Lagrangian Grassmannian, and hence admit a similar index theory. This family contains the Lagrangian Grassmannian, and much more. We construct Maslov–Arnold spaces that are dense in the Grassmannian, and hence are much larger than the Lagrangian Grassmannian (which is a submanifold of positive codimension). The resulting index is then used to study eigenvalue problems for non-selfadjoint reaction–diffusion systems.

Contents

1. Introduction

From the viewpoint of dynamical systems, the Maslov Index provides a way of distinguishing trajectories in a (nonlinear) Hamiltonian system. It achieves this by isolating curves of subspaces in the tangent bundle to the phase space along the underlying trajectory. The Hamiltonian structure affords a restriction to Lagrangian subspaces of each tangent space, and this structure means that such curves can be classified according to an integer index. This is known widely as the Maslov Index.

The Maslov Index has been applied in the Calculus of Variations [19] and to study the stability of nonlinear waves [5, 9, 10, 11, 12, 17]. In the former case, Arnold [2] showed that the Morse Index Theorem could be proved using the Maslov Index. In a related paper [3], he showed how this use of the Maslov Index is a natural generalization of Sturm–Liouville Theory, which relates numbers of zeroes of an eigenfunction for a self-adjoint operator on an interval (or the real line) to the place that the associated eigenvalue takes within the ordering of the (real) spectrum. This gives the connection between these two applications of the Maslov Index and it is the generalization of Sturm–Liouville Theory, afforded by the Maslov Index, to operators arising in the linearization of systems of partial differential equations (PDEs) in one space dimension that motivates the work in this paper.

To date, the need for the underlying system to be Hamiltonian has enforced a restriction to PDEs that lead to self-adjoint operators when linearized at the wave. Looking in the context of reaction-diffusion systems of PDEs, this means that the nonlinearity must be of gradient type. It has been adapted recently to a class of systems, known as skew-gradient, by transforming the problem to one that is Hamiltonian [9]. It has also been successfully applied to other PDEs that are conservative, such as the Nonlinear Schrödinger Equation [15, 16, 17], and various water wave problems [5]. In each of these cases, it can, however, be shown that there is some hidden Hamiltonian (symplectic) structure in the linearized problem, see for instance [10].

The question we pose here is whether this restriction to underlying Hamiltonian structure can be weakened in order to open up a greater range of applications. The idea we pursue is to look for subspaces of the Grassmannian of half-dimensional subspaces of the tangent space that have the needed topological properties. We then see how these are connected with the underlying PDEs we wish to study.

We take $\mathbb{R}^{2n}$ as the ambient (phase) space. The Grassmannian of n-dimensional subspaces of $\mathbb{R}^{2n}$ is denoted $Gr_n(\mathbb{R}^{2n})$. Such an n-dimensional subspace is Lagrangian, with respect to a particular symplectic form ω on $\mathbb{R}^{2n}$, if ω vanishes on it. The space of Lagrangian subspaces, known as the Lagrangian Grassmannian, is then denoted $\Lambda(n)$.

The Maslov index is an integral homotopy invariant defined for continuous paths in the Lagrangian Grassmannian $\Lambda(n)$. This is well defined because $H^1(\Lambda(n); \mathbb{Z}) = \mathbb{Z}$. Moreover, one can explicitly identify the generator of $H^1(\Lambda(n); \mathbb{Z})$, and hence interpret the index of a curve as a signed count of its intersections with a fixed Lagrangian subspace, which we can choose to encode the boundary conditions for the eigenvalue equation.

Our strategy is to find a large subset of $Gr_n(\mathbb{R}^{2n})$ that still has the topological features necessary to define a Maslov-like index. To that end, we introduce the concept of a *Maslov–Arnold space* — see Definition 3.1 — and show that every Maslov–Arnold space admits a generalized Maslov index, which counts intersections of subspaces and hence can be used to count eigenvalues for differential operators. We then construct a family of Maslov–Arnold spaces that are open, dense submanifolds of the Grassmannian. (The Lagrangian Grassmannian is a closed submanifold of codimension $n(n-1)/2$, so our index is defined for a much larger class of subspaces.)

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2. Motivation

Many interesting physical phenomena are described by systems of reaction–diffusion equations. These have the form

$$u_t = Du_{xx} + F(u), \tag{1}$$

where $u(x,t) \in \mathbb{R}^n$, $D = \operatorname{diag}(d_1, \ldots, d_n)$ with all $d_i > 0$, and $F\colon \mathbb{R}^n \to \mathbb{R}^n$. Given a steady state $\bar{u}(x)$, i.e. a solution to $D\bar{u}_{xx} + F(\bar{u}) = 0$, it is natural to ask whether or not it is stable to small perturbations.

The linear stability of $\bar{u}$ is determined by the spectrum of the linearized operator

$$\mathcal{L} = D\frac{d^2}{dx^2} + \nabla F(\bar{u}). \tag{2}$$

The eigenvalue equation $\mathcal{L}v = \lambda v$ can be written as a $2n \times 2n$ system

$$\frac{d}{dx}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & D^{-1} \\ \lambda I - \nabla F(\bar{u}) & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}. \tag{3}$$

If $F = \nabla G$ for some function $G\colon \mathbb{R}^n \to \mathbb{R}$, then $\nabla F = \nabla^2 G$ is symmetric, hence $\mathcal{L}$ is formally selfadjoint, and the system (3) is Hamiltonian. In this case the state $\bar{u}$ has a well defined Maslov index, which can be shown to equal the number of positive eigenvalues of $\mathcal{L}$.

Here we consider the case that F does not have a gradient structure, hence (3) is not Hamiltonian with respect to the standard symplectic form. This generalization is critical because reaction–diffusion equations are primarily studied for their propensity to support patterns and other permanent structures. As was shown by Turing [22] (see below for an in-depth discussion), a fundamental mechanism for generating such patterns requires that ∇F has competing terms, thus ensuring that F is not a gradient. In the literature, equations of the form (1) for which a stable equilibrium can be destabilized in the presence of diffusion are called *activator–inhibitor* systems.

Yanagida [24, 25] initiated the study of a broad class of activator–inhibitor systems called *skew-gradient*, for which $F = Q\nabla G$ with $Q = \operatorname{diag}\{q_i\}$, $q_i = \pm 1$. Chen and Hu subsequently showed how to define the Maslov index of a standing wave and how to use it as a tool in stability analysis for the skew-gradient case [8, 9]. Cornwell and Jones extended these ideas to traveling waves in [10, 11]. In both cases, the parity of the Maslov index is shown to determine the sign of the derivative of the Evans function [1] at $\lambda = 0$; cf. [5, 12]. The results in the aforementioned works hinged on the fact that the eigenvalue equation for $\mathcal{L}$ preserves the manifold of Lagrangian planes for a non-standard symplectic form. In contrast to the Hamiltonian case, the index might be non-monotone in its parameters, and $\mathcal{L}$ might possess complex eigenvalues. Nonetheless, a nonzero Maslov index can still be used to prove instability. (Jones used the same idea to prove an instability criterion for standing waves in nonlinear Schrödinger-type equations [17].) The index can also be used to prove stability in a particular case if the above concerns are addressed. For example, the Maslov index was used to prove stability of both standing and traveling waves in a doubly-diffusive FitzHugh–Nagumo equation [9, 11].

For a general (not necessarily gradient or skew-gradient) system, (3) will generate a family of n-dimensional subspaces in $\mathbb{R}^{2n}$, which need not be Lagrangian. The Grassmannian $Gr_n(\mathbb{R}^{2n})$ of n-planes in $\mathbb{R}^{2n}$ has first cohomology group $H^1(Gr_n(\mathbb{R}^{2n}); \mathbb{Z}) = \mathbb{Z}_2$, and so the Maslov index does not admit an extension to this space. A $\mathbb{Z}_2$ index theory was recently developed in [14], and used to study bifurcations of heteroclinic orbits in non-Hamiltonian systems. However, we are interested in defining an *integral* index that is defined more generally than the Maslov index.

As mentioned in the introduction, we remedy this situation by introducing Maslov–Arnold spaces. Using these spaces, and the resulting indices, we can study the spectrum of the non-selfadjoint operator $\mathcal{L}$ defined in (2). In Section 4 we prove that (for a particular choice of Maslov–Arnold space) the generalized index is monotone with respect to the spatial variable x. As a result, we are able to conclude that

$$\#\{\text{positive real eigenvalues of } \mathcal{L}\} \geq \#\{\text{conjugate points}\}, \tag{4}$$

as long as the system (3) leaves the Maslov–Arnold space invariant. Therefore, the existence of a conjugate point gives a sufficient condition for the instability of the steady state $\bar{u}$.

Consequently, to apply this machinery, we need verifiable invariance conditions for our Maslov–Arnold spaces, to ensure the index is defined and hence (4) holds. As an illustration of our method, we completely analyze systems with constant coefficients, corresponding to linearization about homogeneous equilibria $\bar{u}$, and give some results for systems where the pairwise products of the diffusion coefficients are large.

3. Maslov–Arnold spaces

In this section we introduce Maslov–Arnold spaces, the main object of study in this paper. We start with definitions and basic properties in Section 3.1. In Section 3.2 we construct a large family of Maslov–Arnold spaces that are open, dense subsets of the Grassmannian, and in Section 3.3 we describe the two-dimensional case in detail. Finally, in Section 3.4 we construct a Maslov–Arnold space that contains $\Lambda(2)$ and is dense in $Gr_2(\mathbb{R}^4)$. (The spaces constructed in Section 3.2 are dense in $Gr_n(\mathbb{R}^{2n})$ but do not contain all of $\Lambda(n)$.) This space, which we call the "Fat Lagrangian

Grassmannian," does not make an appearance in later sections, where we study systems of reaction–diffusion equations, but is of theoretical interest, and motivates our construction of Maslov–Arnold spaces that do not contain $\Lambda(n)$.

3.1. **Preliminaries and definitions.** As noted in the introduction, the Maslov index for curves in the Lagrangian Grassmannian has two features that make it useful for stability analysis:

1) There exists a cohomology class $\alpha_0 \in H^1(\Lambda(n);\mathbb{Z})$ so that the Maslov index of any continuous loop $\gamma\colon S^1 \to \Lambda(n)$ equals the canonical pairing $\langle \alpha_0, [\gamma]\rangle \in \mathbb{Z}$.
2) If $\gamma\colon S^1 \to \Lambda(n)$ is a sufficiently generic smooth loop, then its Maslov index is equal to a signed count of intersections between γ and the train of a fixed Lagrangian plane P, where the *train* of an n-plane P in $\Lambda(n)$ is the set $\mathcal{Z}_P \cap \Lambda(n)$, with

$$\mathcal{Z}_P := \big\{ W \in Gr_n(\mathbb{R}^{2n}) : W \cap P \neq \{0\} \big\}. \tag{5}$$

In other words, the Maslov index is a well-defined homotopy invariant of loops and it counts intersections of Lagrangian planes. The last property is essential — in our setup intersections of Lagrangian planes correspond to solutions to eigenvalue equations (with appropriate boundary conditions), and so the Maslov index can be used to count unstable eigenvalues for the linear operator (2).

Before moving further, we make precise the meaning of "sufficiently generic." The subset

$$\mathcal{Z}_P^1 := \big\{ W \in Gr_n(\mathbb{R}^{2n}) : \dim(W \cap P) = 1 \big\} \subseteq \mathcal{Z}_P$$

is a smooth submanifold of $Gr_n(\mathbb{R}^{2n})$ with one-dimensional normal bundle ν. We say a smooth map $\gamma\colon S^1 \to Gr_n(\mathbb{R}^{2n})$ is *sufficiently generic* if $\gamma(S^1) \cap \mathcal{Z}_P = \gamma(S^1) \cap \mathcal{Z}_P^1$ and all of these intersections are transverse.

Given a subset $\mathcal{M} \subseteq Gr_n(\mathbb{R}^{2n})$ and an n-plane $P \in Gr_n(\mathbb{R}^{2n})$, we call $\mathcal{Z}_P \cap \mathcal{M}$ the train of P in $\mathcal{M}$. A *co-orientation* of the train is an orientation of the restricted line bundle $\nu|_{\mathcal{M}\cap\mathcal{Z}_P^1}$. Given a sufficiently generic curve $\gamma\colon S^1 \to \mathcal{M} \subseteq Gr_n(\mathbb{R}^{2n})$ and a co-orientation of $\mathcal{M} \cap \mathcal{Z}_P^1$, the *intersection number* of γ with the train $\mathcal{M} \cap \mathcal{Z}_P$ is defined to be the finite sum

$$\sum_{\substack{t\in S^1\\ \gamma(t)\in\mathcal{Z}_P}} \operatorname{sgn}(t)$$

where $\operatorname{sgn}(t) = 1$ (resp. -1) if the induced linear isomorphism $T_tS^1 \to \nu_{\gamma(t)}$ is orientation preserving (reversing).

Definition 3.1. *A rank n Maslov–Arnold (MA) space $(\mathcal{M}, P, \alpha)$ consists of*

- *a connected subset $\mathcal{M} \subseteq Gr_n(\mathbb{R}^{2n})$,*
- *a rank n vector space $P \in Gr_n(\mathbb{R}^{2n})$, and*
- *a cohomology class $\alpha \in H^1(\mathcal{M};\mathbb{Z})$ of infinite order,*

where $\mathcal{Z}_P \cap \mathcal{M}$ has a co-orientation such that for any sufficiently generic smooth loop $\gamma\colon S^1 \to \mathcal{M}$, the intersection number with $\mathcal{Z}_P$ equals the pairing $\langle \alpha, [\gamma]\rangle$.

The idea is that the paths of interest lie in $\mathcal{M}$, and thus have a Maslov-type index, while the subspace P, through its train, provides the mechanism for evaluating this index. Note the ironic fact that P itself is not required to be an element of $\mathcal{M}$. When it is unlikely to cause confusion, we will sometimes refer to an MA space $(\mathcal{M}, P, \alpha)$ simply by the underlying space $\mathcal{M}$ or by $(\mathcal{M}, P)$.

For any continuous loop $\gamma\colon S^1 \to \mathcal{M}$, we define the *(generalized) Maslov index of* γ *with respect to* P by

$$\operatorname{Mas}(\gamma; P) = \langle \alpha, [\gamma] \rangle. \tag{6}$$

For the hyperplane Maslov–Arnold spaces constructed below, the index has a simple geometric interpretation as a winding number in $\mathbb{R}P^1$. This gives us a practical method for computing the index, and also allows us to define it for continuous paths $\gamma\colon [a,b] \to \mathcal{M}$ with distinct endpoints.

It follows from [2] that $(\Lambda(n), P, \alpha_0)$ is a Maslov–Arnold space for any $P \in \Lambda(n)$, and $\alpha_0 \in H^1(\Lambda(n); \mathbb{Z}) \cong \mathbb{Z}$ either one of the two generators. Moreover, the symplectic form defining $\Lambda(n)$ determines a canonical choice of α_0, called the *Maslov class.* With this choice of generator we call $(\Lambda(n), P, \alpha_0)$ a *classical Maslov–Arnold space.* On the other hand, $Gr_n(\mathbb{R}^{2n})$ cannot be an MA space if $n \geq 2$, because $H^1(Gr_n(\mathbb{R}^{2n}); \mathbb{Z}) \cong \mathbb{Z}_2$ contains no cohomology classes of infinite order. In the case $n = 1$, it is easy to see that $\Lambda(1) = Gr_1(\mathbb{R}^2) = \mathbb{R}P^1 \cong S^1$. This is the home of classical Sturm–Liouville theory, which is often approached through studying the angle of a path in S^1.

Definition 3.2. *Given a pair of equal rank Maslov–Arnold spaces, we say* $(\mathcal{M}_1, P_1, \alpha_1)$ *extends* $(\mathcal{M}_2, P_2, \alpha_2)$ *if* $\mathcal{M}_1 \supseteq \mathcal{M}_2$, $P_1 = P_2$, *and* $i^*(\alpha_1) = \alpha_2$, *where* $i\colon \mathcal{M}_2 \hookrightarrow \mathcal{M}_1$ *is subspace inclusion.*

To extend the definition of the classical Maslov index, it is natural to look for proper extensions of the classical Maslov–Arnold spaces. This is possible when $n = 2$.

Theorem 3.3. *There exists a rank two Maslov–Arnold space* $(\mathcal{F}, P, \alpha)$, *with* $\mathcal{F}$ *dense in* $Gr_2(\mathbb{R}^4)$, *that extends the classical Maslov–Arnold space* $(\Lambda(2), P, \alpha_0)$, *where* $P \in \Lambda(2)$ *and* α_0 *is the Maslov class.*

Therefore, one can assign an integer index to every continuous loop in $\mathcal{F}$, and for a loop in $\Lambda(2)$ it coincides with the classical Maslov index. This new index has the advantage of being much more broadly defined, since $\mathcal{F}$ is dense in $Gr_2(\mathbb{R}^4)$, whereas $\Lambda(2)$ is a hypersurface.

However, the space $\mathcal{F}$ given by Theorem 3.3 is not a submanifold of the Grassmannian. It will be seen in the proof (which we postpone to Section 3.4) that it does not contain an open neighbourhood of $\Lambda(2)$, which makes it difficult to use in practice. Although $\mathcal{F}$ is left invariant by the flow of any Hamiltonian system with Lagrangian initial data (because $\Lambda(2)$ is), an arbitrarily small perturbation of the system may cause its trajectories to leave $\mathcal{F}$, in which case the index is no longer defined.

It turns out this undesirable behaviour is inevitable for extensions of the classical Maslov–Arnold spaces.

Theorem 3.4. *If* $(\mathcal{M}, P, \alpha)$ *is an extension of the classical Maslov–Arnold space* $(\Lambda(n), P, \alpha_0)$, *and* $\mathcal{M}$ *is strictly larger than* $\Lambda(n)$, *then* $\mathcal{M}$ *is not a smooth submanifold of* $Gr_n(\mathbb{R}^{2n})$.

In other words, the only smooth MA space that extends $\Lambda(n)$ is $\Lambda(n)$ itself. Therefore the requirement that an MA space contain $\Lambda(n)$ is overly restrictive. By dropping this requirement, we are able to produce a large family of open, dense MA spaces, each the complement of a closed, codimension two real variety. Moreover, there is considerable freedom in the choice of variety to remove. The particular construction we use in applications will be dictated by properties of the differential equation and boundary conditions under consideration. The general construction is given in Section 3.2, and in Section 4 we show how to choose an MA space that is suited to the study of reaction–diffusion equations.

Proof of Theorem 3.4. We start by reviewing the spectral flow interpretation of the Maslov index, as given by Robbin and Salamon [21].

Equip $\mathbb{R}^{2n}$ with the standard inner product product $\langle \cdot, \cdot \rangle$ and define a complex structure $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ by $\omega(v,w) = \langle v, Jw \rangle$. Then the Lagrangian subspace P has a Lagrangian complement $Q := J(P) = P^{\perp}$, so $\mathbb{R}^{2n} = Q \oplus P$. We define the coordinate neighbourhood $U_P \subseteq Gr_n(\mathbb{R}^{2n})$ to be the set of n-planes in $\mathbb{R}^{2n}$ that intersect Q trivially and can therefore be represented as graphs of linear maps from P to Q. We have

$$U_P = \{gr(B) : B \in \mathrm{Hom}(P,Q)\} = \{gr(JA) : A \in \mathrm{Hom}(P,P)\} \cong \mathrm{Hom}(P,P) \tag{7}$$

where we have abused notation and denoted by $J\colon P \to Q$ the restriction of J. In this coordinate neighbourhood

$$\begin{aligned} \Lambda(n) \cap U_P &\cong \{A \in \mathrm{Hom}(P,P) : A = A^T\} \\ \mathcal{Z} \cap U_P &\cong \{A \in \mathrm{Hom}(P,P) : \det A = 0\}. \end{aligned}$$

The co-orientation of the train in $\Lambda(n)$ is such that under the identification (7), the index of a path $\gamma\colon I \to U_P \cap \Lambda(n)$ counts the difference between the number of positive eigenvalues of the symmetric matrices $\gamma(1)$ and $\gamma(0)$. That is, the Maslov index of γ equals the *spectral flow* of the corresponding family of symmetric matrices; see [21, Theorem 2.3].

Now let $\phi\colon \mathcal{M} \cap U_P \to \mathrm{Hom}(P,P)$ be the composition of inclusion with the diffeomorphism (7). To prove the theorem, it suffices to find a path $\gamma\colon I \to \mathcal{M} \cap U_P$ that does not intersect the train $\mathcal{Z}_P$, such that $\phi(\gamma(0))$ and $\phi(\gamma(1))$ are symmetric matrices with different numbers of positive eigenvalues.

To begin, observe that the image of the tangent map $d\phi_P\colon T_P\mathcal{M} \to T_0 \mathrm{Hom}(P,P) = \mathrm{Hom}(P,P)$ must contain some non-zero $B \in \mathrm{im}(d\phi_P) \subseteq \mathrm{Hom}(P,P)$ such that $B = -B^T$. Therefore B admits an orthogonal eigenspace decomposition $\mathbb{R}^{2n} = \oplus_{\lambda \geq 0} V_\lambda$ with purely imaginary eigenvalues $\pm\lambda \in i\mathbb{R}$. Since B is non-zero, there exists $\lambda_0 \neq 0$ for which $V_{\lambda_0} \neq \{0\}$. Let Π be the orthogonal projection onto V_{λ_0} and let $\Pi' := I_{2n} - \Pi$. Then the path in $\mathrm{im}(d\phi_P)$ defined by

$$d\phi_P(\tilde{\gamma}(t)) := \Pi' + \cos(\pi t)\Pi + \sin(\pi t)B$$

is non-degenerate for all $t \in [0,1]$ and has endpoints I_{2n} and $\Pi' - \Pi$, which are non-degenerate and symmetric, with different numbers of positive eigenvalues.

To complete the proof, choose an arbitrary Riemannian metric on $\mathcal{M}$, and let $\exp_P\colon T_P\mathcal{M} \to \mathcal{M}$ denote the exponential map at P. Since $(\phi \circ \exp)(0) = 0$ and $d(\phi \circ \exp)(0) = d\phi_P$, Taylor's theorem gives the uniform estimate $\phi(\exp(v)) = d\phi_P(v) + \mathcal{O}(|v|^2)$ for small $v \in T_P\mathcal{M}$. It follows that $\epsilon^{-1}\phi\big(\exp(\epsilon\tilde{\gamma}(t))\big) = d\phi_P(\tilde{\gamma}(t)) + \mathcal{O}(\epsilon)$, and so for sufficiently small ϵ the path

$$t \mapsto \phi^{-1}\Big(\epsilon^{-1}\phi\big(\exp(\epsilon\tilde{\gamma}(t))\big)\Big)$$

has the desired property. □

3.2. **Hyperplane Maslov–Arnold spaces.** Let $V \cong \mathbb{R}^{2n}$ and denote by $\bigwedge^n(V)$ the nth degree exterior product of V, which is a vector space of dimension $\binom{2n}{n}$. The projective space $P\left(\bigwedge^n(V)\right)$ is the set of the one dimensional subspaces of $\bigwedge^n(V)$. Given a non-zero n-vector $\xi \in \bigwedge^n(V)$, we denote by $[\xi] \in P(\bigwedge^n(V))$ the span of ξ. The Plücker embedding maps $Gr_n(V)$ into $P(\bigwedge^n(V))$, sending $\operatorname{span}\{v_1, \dots, v_n\}$ to $[v_1 \wedge \cdots \wedge v_n]$. We will sometimes abuse notation and simply identify $Gr_n(V)$ with its image $G \subseteq P(\bigwedge^n(V))$. Observe that G equals the subset of $[\xi] \in P(\bigwedge^n(V))$ for which ξ is decomposable as a product of vectors in V.

Let $V^* := \operatorname{Hom}(V, \mathbb{R})$ denote the dual vector space of V. For $k \geq 1$, each $\omega \in \bigwedge^k(V^*)$ corresponds to a skew-symmetric multilinear map $\omega\colon V^k = V \times \cdots \times V \to \mathbb{R}$. There is a canonical isomorphism $\bigwedge^k(V^*) \cong \bigwedge^k(V)^*$, so elements $\omega \in \bigwedge^k(V^*)$ are also in one-to-one correspondence with linear maps $\tilde{\omega}\colon \bigwedge^k(V) \to \mathbb{R}$. Both interpretations of $\bigwedge^k(V^*)$ will be important in what follows.

Each non-zero n-form $\omega \in \bigwedge^n(V^*)$ represents a point $[\omega] \in P(\bigwedge^n(V^*))$, which corresponds by projective duality to a hyperplane $H_\omega = H_{[\omega]} \subseteq P\left(\bigwedge^n(V)\right)$, namely

$$H_\omega = \left\{[\xi] \in P\Big(\bigwedge^n(V)\Big) : \tilde{\omega}(\xi) = 0\right\}. \tag{8}$$

Conversely, a hyperplane $H \subseteq P\left(\bigwedge^n(V)\right)$ determines a unique one-dimensional space of n-forms $[\omega] \in P(\bigwedge^n(V^*))$ such that $H = H_\omega$.

If the hyperplane H_ω is intersected with G, we get

$$G \cap H_\omega = \left\{[v_1 \wedge \cdots \wedge v_n] \in P\Big(\bigwedge^n(V)\Big) : \omega(v_1, \dots, v_n) = 0\right\}, \tag{9}$$

and by the Plücker embedding this corresponds to

$$G \cap H_\omega \cong \big\{\operatorname{span}\{v_1, \dots, v_n\} \in Gr_n(V) : \omega(v_1, \dots, v_n) = 0\big\}. \tag{10}$$

For instance, if $n = 2$ and ω is a non-degenerate two-form (i.e. a symplectic form), then $G \cap H_\omega$ is the Lagrangian Grassmannian Λ_ω.

Another important type of hyperplane, particularly relevant to our theory of Maslov–Arnold spaces, is that corresponding to the train of a fixed subspace, as defined in (5). Given a vector $v \in V$, the contraction map $\iota_v\colon \bigwedge^k(V^*) \to \bigwedge^{k-1}(V^*)$ is defined for each $k \geq 1$ by $(\iota_v\omega)(w_1, \dots, w_{k-1}) := \omega(v, w_1, \dots, w_{k-1})$. Define the kernel of ω by $\ker\omega := \{v \in V : \iota_v\omega = 0\}$.

Lemma 3.5. *Let $\omega \in \bigwedge^n(V^*)$. If $\ker\omega \subseteq V$ has dimension n, then $G \cap H_\omega$ is the train of the subspace $\ker\omega$, i.e.*

$$G \cap H_\omega \cong \mathcal{Z}_{\ker\omega} = \big\{W \in Gr_n(V) : W \cap \ker\omega \neq \{0\}\big\}.$$

Proof. Let $v_1, \dots, v_n$ be a basis of $\ker\omega$, and extend to a basis $v_1, \dots, v_{2n}$ of V, with dual basis $v_1^*, \dots, v_{2n}^* \in V^*$. Then $\omega = c v_{n+1}^* \wedge \cdots \wedge v_{2n}^*$ for some nonzero $c \in \mathbb{R}$. It follows from (9) that $[w_1 \wedge \cdots \wedge w_n] \in G \cap H_\omega$ if and only if $\operatorname{span}\{w_1, \dots, w_n\}$ intersects $\ker\omega$ non-trivially. □

Now consider a pair of hyperplanes H_1 and H_2 corresponding to linearly independent n-forms ω_1 and ω_2.

Proposition 3.6. *Let $M := P(\bigwedge^n(V)) \setminus (H_1 \cap H_2)$. Then M is an orientable manifold with $H^1(M;\mathbb{Z}) \cong \mathbb{Z}$ generated by the Poincaré dual to $H_1 \cap M$.*

Proof. Let $m+1 = \binom{2n}{n}$ and choose an isomorphism $\mathbb{R}^{m+1} \cong \bigwedge^n(V)$ such that H_1 and H_2 are determined by $x_m = 0$ and $x_{m+1} = 0$, respectively. This gives a diffeomorphism

$$M \cong \mathbb{R}P^m \setminus \mathbb{R}P^{m-2}.$$

The surjective map $\mathbb{R}^{m-1} \times S^1 \to \mathbb{R}P^m \setminus \mathbb{R}P^{m-2}$ sending $(x_1, \dots, x_{m+1}) \mapsto [x_1 : \cdots : x_{m+1}]$ descends to a diffeomorphism between M and the Möbius bundle $\mathbb{R}^{m-1} \times S^1/\!\sim$ under the quotient relation $(x,y) \sim (-x,-y)$. Therefore M is orientable (since $\binom{2n}{n}$ is even), and $\pi_1(M) \cong \mathbb{Z}$ is generated by a loop that winds once around the base of the Mobius bundle $S^1/\!\sim\, = \mathbb{R}P^1$. Finally, $M \cap H_1$ is identified with one of the fibres of the Möbius bundle, so it is Poincaré dual to the generator of $H^1(M;\mathbb{Z})$. □

As an immediate corollary, we have the following.

Corollary 3.7. *Given any pair of distinct hyperplanes $H_1, H_2 \subseteq P(\bigwedge^n(V))$, define $\mathcal{M} := G \setminus (H_1 \cap H_2)$. Then $H_1 \cap \mathcal{M}$ represents a cohomology class in $H^1(\mathcal{M};\mathbb{Z})$ which is calculated via a geometric intersection number with $H_1 \cap \mathcal{M}$.*

Proof. The cohomology class is simply the image of the one defined in $H^1(M;\mathbb{Z})$ via the inclusion map. □

We can understand the cohomology class as follows. If ω_1 and ω_2 are the n-forms defining H_1 and H_2, then we get a map

$$\begin{aligned} \phi\colon \mathcal{M} &\longrightarrow \mathbb{R}P^1 \\ \operatorname{span}\{u_1, \dots, u_n\} &\mapsto [\omega_1(u_1, \dots, u_n) : \omega_2(u_1, \dots, u_n)]. \end{aligned} \tag{11}$$

This is well defined because $\omega_1(u_1, \dots, u_n)$ and $\omega_2(u_1, \dots, u_n)$ cannot vanish simultaneously when $\operatorname{span}\{u_1, \dots, u_n\} \in \mathcal{M}$. Given any loop $\gamma\colon S^1 \to \mathcal{M}$, its generalized Maslov index, the intersection number with $H_1 \cap \mathcal{M}$, coincides with the winding number of $\phi \circ \gamma$. We will use this idea below to define the index for non-closed curves in $\mathcal{M}$.

According to Corollary 3.7, any loop in $G \setminus (H_1 \cap H_2)$ has a well defined intersection number with respect to $H_1 \cap G$. If $H_1 \cap G$ is the train of an n-plane, we obtain a Maslov–Arnold space, as in Definition 3.1.

Theorem 3.8. *With notation as above, suppose that $H_1 = H_\omega$, where $\ker\omega \subseteq V$ has dimension n. Then $\mathcal{M}$ is a Maslov–Arnold space with respect to $P = \ker\omega$.*

Proof. It only remains to prove the Poincaré dual of $H_1 \cap \mathcal{M}$ in $\mathcal{M}$ has infinite order. It suffices to identify a loop $\gamma\colon [0,1] \to \mathcal{M}$ with non-zero index.

By iterative application of Lemma 3.9 we can find vectors $v_1, \ldots, v_{n-1}$ so that the contractions $\iota_{v_1\wedge\cdots\wedge v_{n-1}}\omega_1$ and $\iota_{v_1\wedge\cdots\wedge v_{n-1}}\omega_2$ are linearly independent. Therefore, there exist $u_1, u_2 \in V$ such that $\omega_i(u_j, v_1, \ldots, v_{n-1}) = \delta_{ij}$. Consequently, the loop

$$\gamma(t) = \operatorname{span}\{\cos(\pi t)u_1 + \sin(\pi t)u_2, v_1, \ldots, v_{n-1}\}$$

has index one. □

Lemma 3.9. *Let V be a vector space and $k \geq 2$. If $\omega_1, \omega_2 \in \bigwedge^k(V^*)$ are linearly independent, then there exists a vector $v \in V$ such that the contractions $\iota_v\omega_1, \iota_v\omega_2 \in \bigwedge^{k-1}(V^*)$ are linearly independent.*

Proof. Choose a basis $e_1, \ldots, e_n \in V$, with dual basis $e_1^*, \ldots, e_n^* \in V^*$, and expand $\omega_1 = \sum_I a_I e_I^*$ and $\omega_2 = \sum_I b_I e_I^*$, where $I = \{i_1 < \cdots < i_k\}$ are multi-indices and $e_I^* := e_{i_1}^* \wedge \ldots \wedge e_{i_k}^*$. Since ω_1 and ω_2 are linearly independent, there is a pair of multi-indices I, J such that the minor

$$\det \begin{pmatrix} a_I & b_I \\ a_J & b_J \end{pmatrix} \neq 0. \tag{12}$$

If there exists a pair of multi-indices I, J satisfying (12) and an index $i \in I \cap J$, then $\iota_{e_i}\omega_1$ and $\iota_{e_i}\omega_2$ are linearly independent and we are done.

Suppose instead that every pair of multi-indices satisfying (12) has $I \cap J = \varnothing$. For a particular such pair, select $i \in I$ and $j \in J$ and define $I' := I \cup \{j\} \setminus \{i\}$ and $J' := J \cup \{i\} \setminus \{j\}$. Since $k \geq 2$ it follows that each of $I \cap I'$, $I' \cap J$, $I \cap J'$, and $J \cap J'$ is non-empty. Considering the corresponding minors, we deduce that $a_{I'} = a_{J'} = b_{I'} = b_{J'} = 0$ and consequently that $\iota_{e_i+e_j}\omega_1$ and $\iota_{e_i+e_j}\omega_2$ are linearly independent. □

It was observed above that the intersection number of a loop $\gamma\colon S^1 \to \mathcal{M}$ with $H^1 \cap \mathcal{M}$ is equal to the winding number of $\phi \circ \gamma\colon S^1 \to \mathbb{R}P^1$. We can thus define the generalized Maslov index of a non-closed path γ in $\mathcal{M}$ to be the winding number of $\phi\circ\gamma$ through the point $\phi(H_1) = [0:1] \in \mathbb{R}P^1$.

Since $\mathbb{R}P^1 \cong S^1$, it suffices to define the winding number of a continuous path in S^1. This amounts to choosing an (arbitrary) convention for the endpoints, which we do as follows.

Definition 3.10. *Let $\eta\colon [a,b] \to S^1$ be a continuous path. If $\eta(t_0) = 1$ for some $t_0 \in [a,b]$, then there is a unique lift $\theta\colon [a,b] \to \mathbb{R}$ such that $\theta(t_0) = 0$ and $e^{i\theta(t)} = \eta(t)$ for $t \in [a,b]$, and we define*

$$W(\eta) = \left\lfloor \frac{\theta(b)}{2\pi} \right\rfloor - \left\lfloor \frac{\theta(a)}{2\pi} \right\rfloor. \tag{13}$$

If no such t_0 exists we set $W(\eta) = 0$.

It is not hard to see that this is well defined (independent of the choice of t_0). It is clearly additive under concatenation of paths, and if $\eta(a) = \eta(b)$ it reduces to the usual winding number of a loop, $(\theta(b) - \theta(a))/2\pi$. Some consequences of this definition are shown in Figure 1. The path e^{it},

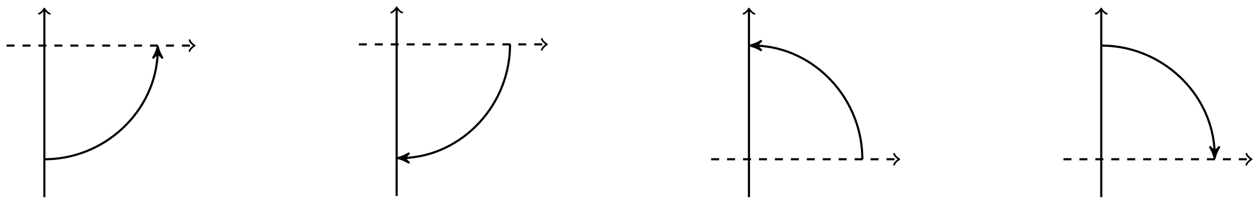

FIGURE 1. Illustrating the winding number, with respect to the point $(1,0)$, for non-closed curves with crossings at their endpoints. Our convention is to count negative crossings at the beginning of a curve and positive crossings at the end.

$-\pi/2 \le t \le 0$ has winding number 1 and e^{-it}, $0 \le t \le \pi/2$ has winding number -1, whereas e^{it}, $0 \le t \le \pi/2$ and e^{-it}, $-\pi/2 \le t \le 0$ both have winding number 0.

A special case is when the path is monotone. If η is C^1 and has the property that $\theta'(t_*) > 0$ whenever $\eta(t_*) = 1$, then the set $\{t_* \in [a,b] : \eta(t_*) = 1\}$ is finite, and

$$W(\eta) = \#\big\{t_* \in (a,b] : \eta(t_*) = 1\big\}. \tag{14}$$

Similarly, if $\theta'(t_*) < 0$ whenever $\eta(t_*) = 1$, then

$$W(\eta) = -\#\big\{t_* \in [a,b) : \eta(t_*) = 1\big\}. \tag{15}$$

3.3. **The two-dimensional case.** We consider in detail the $n = 2$ case, where $\mathcal{M}$ can be described explicitly. The hyperplanes now come in two types. If ω is a non-degenerate 2-form, i.e. a sympectic form, then $H_\omega \cap G$ is the corresponding Lagrangian Grassmanian. If ω is degenerate, then $\ker\omega \subseteq V$ is two-dimensional, and $H_\omega \cap G$ is the train of $\ker\omega$.

Given linearly independent forms $\omega_1, \omega_2 \in \bigwedge^2(V)$, they span a pencil of bilinear forms $x\omega_1 + y\omega_2$, $(x,y) \neq (0,0)$. Consider the homogeneous quadratic polynomial $q(x,y) := \mathrm{Pf}(x\omega_1 + y\omega_2)$, where Pf denotes the Pfaffian. The roots of q correspond to the degenerate two-forms in the pencil. There can be zero, one, two, or infinitely many roots.

Proposition 3.11. *Up to a change of basis transformation of V, there are four possible isomorphism types for $\mathcal{M}$. They are classified by the number of real roots of $q(x,y) := \mathrm{Pf}(x\omega_1 + y\omega_2)$.*

Proof. The Plücker embedding identifies $G \subseteq P(\bigwedge^2(V)) \cong \mathbb{R}P^5$ as a quadric, the so-called Klein quadric, defined by the non-degenerate, split signature symmetric bilinear form

$$B\colon \bigwedge^2(V) \otimes \bigwedge^2(V) \to \bigwedge^4(V) \cong \mathbb{R} \qquad\qquad B(\eta,\xi) = \eta \wedge \xi.$$

We call a linear transformation $A \in GL(\bigwedge^2(V))$ *orthogonal* if it leaves B invariant and *anti-orthogonal* if it sends B to $-B$. Observe that both orthogonal and anti-orthogonal transformations preserve G.

Let $W \subseteq \bigwedge^2(V)$ be the four-dimensional subspace for which $P(W) = H_{\omega_1} \cap H_{\omega_2}$. Since B is non-degenerate, the B-complement of W, $W^\perp := \{u \in \bigwedge^2(V) : B(u,w) = 0,\ \text{for all } w \in W\}$, is two dimensional. Consider the restricted bilinear form $B' := B|_{W^\perp}$. The associate quadratic form

$q'(v) := B(v,v)$ on $W^\perp$ can be identified via duality with $q(x,y)$. By Sylvester's law of inertia, there are six isomorphism possible classes for B' modulo change of basis, and four isomorphism classes modulo multiplication by ± 1. These are classified by the number of roots of $q(x,y)$.

If $W_1, W_2 \subseteq \bigwedge^2(V)$ are four-dimensional subspaces such that $B|_{W_1^\perp}$ is isomorphic to $B|_{W_2^\perp}$, then by Witt's Theorem (see [18, Thm 1.2]) there exists an orthogonal transformation of $\bigwedge^2(V)$ sending W_1 to W_2. Similarly, if $B|_{W_1^\perp}$ is isomorphic to $-B|_{W_2^\perp}$ then there exists an anti-orthogonal transformation sending W_1 to W_2. It follows in either case that $G \setminus P(W_1)$ is isomorphic to $G \setminus P(W_2)$.

Finally we must show that the orthogonal transformation of $\bigwedge^2(V)$ used above can be induced by a linear transformation of V (the anti-orthogonal case is an easy consequence). Denote by $O(B)$ the group of orthogonal transformations of $(\bigwedge^2(V), B)$. The natural homomorphism $\phi\colon SL(V) \to O(B)$ has kernel $\pm I_4$, so since both groups are 15 dimensional, it is a surjection onto the identity component of $O(B)$. It remains to show that for each two-dimensional $U \subseteq \bigwedge^2(V)$, there exists $A \in O(B)$ in each path component of $O(B)$ such that $A(U) = U$.

Choose a basis $e_1, \ldots, e_6 \in \bigwedge^2(V)$ so that $B(e_i, e_j) = (-1)^i \delta_{ij}$, where δ_{ij} is the Kronecker delta. According to [18, Cor 1.1], representatives for the four path components of $O(B)$ are given by the transformations that fix e_1, e_2, e_3, e_4 and send $e_5 \mapsto \pm e_5$ and $e_6 \mapsto \pm e_6$. Since every different isomorphism class of $B|_U$ can be realized by a two-dimensional $U \subseteq \operatorname{span}\{e_1, \ldots, e_4\}$, this completes the proof. $\square$

Up to a change of basis for V, the pencil of bilinear forms above is isomorphic to one of four possibilities

$$\begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & 0 & 0 \\ -y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & x & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & -y & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & x & y \\ 0 & 0 & -y & x \\ -x & y & 0 & 0 \\ -y & -x & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & 0 & y \\ -y & 0 & 0 & 0 \\ 0 & -y & 0 & 0 \end{pmatrix},$$

which have respective Pfaffians (up to sign) $q(x,y) = 0$, xy, $x^2 + y^2$, and x^2.

If $n = 2$ then $X := G \cap H_1 \cap H_2$ is homeomorphic to one of the following four respective types.

(i) If $q = 0$, then every linear combination $x\omega_1 + y\omega_2$ is degenerate. In this case X is the intersection of trains for $\ker \omega_1$ and $\ker \omega_2$, which intersect non-trivially. It follows that X is a union of $\mathbb{R}P^1 \times \mathbb{R}P^1$ with $\mathbb{R}P^2$ along a wedge sum $\mathbb{R}P^1 \vee \mathbb{R}P^1$.

(ii) If q has two distinct real roots, then X is the intersection of trains for a pair of two-dimensional subspaces $P_1, P_2 \subseteq V$ which intersect trivially. In this case $X = P(P_1) \times P(P_2) \cong \mathbb{R}P^1 \times \mathbb{R}P^1$ is a torus.

(iii) If q has one root with multiplicity two, then X can be identified with the intersection of the Lagrangian Grassmannian and the train of a Lagrangian subspace, for some symplectic form ω. Therefore, X is isomorphic to the Maslov cycle described by Arnol'd [3, §3]; it is homeomorphic to the one point compactification of $S^1 \times \mathbb{R}$.

(iv) If q has no real roots, then there exists a quaternionic structure I, J, K on V in which the pencil is spanned by symplectic forms ω_I and ω_J, and X can be identified with the intersection of their respective Lagrangian Grassmanians, $\Lambda_I \cap \Lambda_J$. Equivalently, $X \cong S^2$

is identified with the complex projective line $P(\mathbb{C}^2_K)$ with respect to the third complex structure K. In this case $G \setminus (H_1 \cap H_2)$ is not an MA space, because H_1 is not a train.

In Section 4 we construct a Maslov–Arnold space for the study of $n\times n$ systems of reaction–diffusion equations. When $n = 2$ it is of the type (iii) described above.

Proposition 3.12. *If $\mathcal{M}$ is one of the four cases above, then $H^1(\mathcal{M};\mathbb{Z}) \cong \mathbb{Z}$ and is generated by the Poincaré dual of $H_1 \cap \mathcal{M}$.*

Proof. By Poincaré duality $H^1(\mathcal{M};\mathbb{Z}) \cong H_3(G, X;\mathbb{Z})$. Consider the long exact sequence of the pair

$$H_3(G) \to H_3(G, X) \to H_2(X) \to H_2(G).$$

We know $H_3(G) = 0$ and $H_2(G) \cong \mathbb{Z}/2$ (see [13]) and that X is isomorphic to a two-dimensional cell complex, so $H_2(X;\mathbb{Z})$ is torsion free. Exactness therefore implies that $H_3(G, X;\mathbb{Z})$ is isomorphic to $H_2(X;\mathbb{Z})$. In all four cases above it is straightforward to check $H_2(X;\mathbb{Z}) \cong \mathbb{Z}$, so it follows that $H^1(\mathcal{M};\mathbb{Z}) \cong \mathbb{Z}$. In Theorem 3.8 we constructed a loop in $\mathcal{M}$ whose intersection number with $H_1 \cap \mathcal{M}$ is one, so it must generate $H_3(G, X;\mathbb{Z}) \cong H^1(\mathcal{M};\mathbb{Z})$. □

3.4. **The Fat Lagrangian Grassmannian.** In this section we prove Theorem 3.3, constructing a rank two Maslov–Arnold space $(\mathcal{F}, P)$ that extends the classical Maslov–Arnold space $(\Lambda(2), P)$ for any $P \in \Lambda(2)$.

As described above, $\mathcal{F}$ has the desirable property of being a large MA space that contains the entire Lagrangian Grassmannian, and the undesirable property of not being a smooth manifold. The lack of smoothness follows directly from the construction given below, but also from Theorem 3.4, which demonstrates that this problem is essential, and does not depend on the particular details of our construction.

Let $v_1, v_2, v_3, v_4 \in V \cong \mathbb{R}^4$ be a basis, with dual basis $v_1^*, v_2^*, v_3^*, v_4^* \in V^*$. Define symplectic forms

$$\omega_I := v_1^* \wedge v_3^* + v_2^* \wedge v_4^* \qquad\qquad \omega_J := v_1^* \wedge v_4^* - v_2^* \wedge v_3^*.$$

with corresponding Lagrangian Grassmannians

$$\Lambda_I := G \cap H_{\omega_I} \qquad\qquad \Lambda_J := G \cap H_{\omega_J}.$$

Observe that both $Q := [v_1 \wedge v_2]$ and $P := [v_3 \wedge v_4]$ lie in the intersection $\Lambda_I \cap \Lambda_J$.

Denote Plücker coordinates by $p_{ij} = v_i^* \wedge v_j^*$, regarded as linear functions $p_{ij} \colon \bigwedge^2(V) \to \mathbb{R}$. The image of the Plücker embedding, $G \subseteq P(\bigwedge^2(V))$, is defined by the homogeneous quadratic equation

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{24} = 0.$$

Consider the closed subset $S \subseteq P(\bigwedge^2(V))$ defined by the linear equation $p_{14} - p_{23} = 0$ and the inequality $p_{12}p_{34} \geq 0$. The inequality makes sense in $P(\bigwedge^2(V))$ because given $\xi \in \Lambda^2(V)$ and $c \in \mathbb{R}$, we have $p_{12}p_{34}(c\xi) = c^2 p_{12}p_{34}(\xi)$, so the sign of $p_{12}p_{34}$ is well-defined.

Lemma 3.13. *The intersection $\Lambda_I \cap S$ consists of the two points $P, Q \in G$.*

Proof. The intersection is determined by the system of inequalities

$$\begin{aligned} p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} &= 0 \\ p_{13} &= -p_{24} \\ p_{14} &= p_{23} \\ p_{12}p_{34} &\geq 0 \end{aligned}$$

where the first two equations determine Λ_I and the second two inequalities determine S. Substituting the first three equalities into the inequality yields $p_{13}^2 + p_{24}^2 \leq 0$, which is only possible if $p_{13} = p_{24} = 0$. So we are reduced to the equivalent equations

$$p_{12}p_{34} = p_{13} = p_{24} = p_{14} = p_{23} = 0,$$

which have only two solutions: $Q = [v_1 \wedge v_2]$ and $P = [v_3 \wedge v_4]$. □

Define $\mathcal{U} := G \setminus S$. This is an open, dense subset of G, hence it is a non-compact, orientable 4-manifold, so by Poincaré duality $H^1(\mathcal{U};\mathbb{Z})$ is naturally isomorphic to the relative homology group $H_3(G, S;\mathbb{Z})$ (alternatively, the Borel–Moore homology group $H_3^{BM}(\mathcal{U};\mathbb{Z})$). The train of P in $\mathcal{U}$ is the intersection $\mathcal{U} \cap H_{v_3^* \wedge v_4^*}$.

Lemma 3.14. *The train of $\mathcal{Z}_P \cap \mathcal{U}$ in $\mathcal{U}$ is a smooth, closed, co-orientable submanifold of $\mathcal{U}$.*

Proof. The intersection $G \cap H_{v_3^* \wedge v_4^*}$ is transverse except at $P = [v_3 \wedge v_4]$. By Lemma 3.13 we see $P \notin \mathcal{U}$, so the intersection $\mathcal{U} \cap H_{v_3^* \wedge v_4^*}$ is transverse, hence it is a smoothly embedded codimension one submanifold.

The intersection $\mathcal{U} \cap H_{v_3^* \wedge v_4^*}$ is determined in Plücker coordinates by

$$\begin{aligned} \mathcal{U} \cap H_{v_3^* \wedge v_4^*} &= G \cap (\{p_{14} - p_{23} = 0\} \cap \{p_{12}p_{34} \geq 0\})^c \cap \{p_{34} = 0\} \\ &= G \cap \{p_{34} = 0\} \cap \{p_{14} - p_{23} = 0\}^c \\ &= (G \setminus H_{\omega_J}) \cap H_{v_3^* \wedge v_4^*} \end{aligned}$$

where we have applied de Morgan's law and the fact that $\{p_{34} = 0\} \subseteq \{p_{12}p_{34} \geq 0\}$. Therefore, the normal bundle of $\mathcal{U} \cap H_{v_3^* \wedge v_4^*}$ in $\mathcal{U}$ is the pullback of the normal bundle of the affine space $\left(P(\bigwedge^2(V)) \setminus H_{\omega_J}\right) \cap H_{v_3^* \wedge v_4^*} \cong \mathbb{R}^4$ in the affine space $P(\bigwedge^2(V)) \setminus H_{\omega_J} \cong \mathbb{R}^5$. But this is clearly co-orientable, so we are done. □

Remark 3.15. *One might expect, based on the above argument, that since the linear inclusion $\mathbb{R}^4 \subseteq \mathbb{R}^5$ has a trivial Poincaré dual in $H^1(\mathbb{R}^5) \cong \{0\}$, the same must be true of $\mathcal{U} \cap H_{v_3^* \wedge v_4^*}$ in $\mathcal{U}$. However, since $\mathcal{U}$ is not a subset of $P(\bigwedge^2(V) \setminus H_{\omega_J} \cong \mathbb{R}^5$, there is no natural map in cohomology from $H^1(\mathbb{R}^5)$ to $H^1(\mathcal{U})$.*

Corollary 3.16. *The open set $\mathcal{U} \subseteq G$ is a Maslov–Arnold space with respect to P.*

Proof. Let $N := \Lambda_I \setminus \{P, Q\}$. By Lemma 3.13 we know $N = \mathcal{U} \cap \Lambda_I$. Since Λ_I is a 3-manifold and N is the complement of two isolated points in Λ_I, the inclusion determines an isomorphism $H^1(N;\mathbb{Z}) \cong H^1(\Lambda_I;\mathbb{Z}) \cong \mathbb{Z}$, which is generated in both cases by the Poincaré dual of the train of P (with a chosen co-orientation).

It follows from Lemma 3.14 that the train $\mathcal{Z}_P \cap \mathcal{U}$, equipped with a chosen co-orientation, represents a well-defined cohomology class in $H^1(\mathcal{U};\mathbb{Z}) \cong H_3^{BM}(\mathcal{U};\mathbb{Z})$. This cohomology class must have infinite order, because it is sent to a generator of $H^1(N;\mathbb{Z})$ under restriction to $N \subseteq \mathcal{U}$. □

We now define the *Fat Lagrangian Grassmannian*

$$\mathcal{F} := \mathcal{U} \cup \Lambda(2) = \mathcal{U} \cup \{P, Q\}. \tag{16}$$

Note that $\mathcal{F}$ is not a manifold. However, it is a semialgebraic set, since $\mathcal{U}$ is defined by polynomial inequalities.

Consider the coordinate neighbourhood of $P \in G$ by

$$U_P = \{gr(A) : A \in \mathrm{Hom}(P, Q)\}$$

consisting of all 2-planes that intersect Q trivially, and hence can be realized as graphs of linear maps from P to Q. Denote by $J \colon P \to Q$ the complex structure with $J(v_3) = -v_2$ and $J(v_4) = v_1$. As in the proof of Theorem 3.4, we have

$$U_P = \{gr(JA) : A \in \mathrm{Hom}(P, P)\} \cong \mathrm{Hom}(P, P).$$

Using the matrix representation with respect to the basis $\{v_3, v_4\}$ of P determines a coordinate chart

$$U_P \cong \mathbb{R}^4 = \left\{ A = \begin{pmatrix} x & y \\ z & w \end{pmatrix} : x, y, z, w \in \mathbb{R} \right\}.$$

Under this identification

$$\begin{aligned} \Lambda_J \cap U_P &= \{gr(JA) : A = A^T\}, \\ H_{v_3^* \wedge v_4^*} \cap U_P &= \{gr(JA) : \det(A) = 0\}. \end{aligned}$$

Similarly, we have a coordinate neighbourhood of $Q \in G$,

$$U_Q = \{gr(A) : A \in \mathrm{Hom}(Q, P)\} = \{gr(AJ^{-1}) : A \in \mathrm{Hom}(P, P)\} \cong \mathbb{R}^4.$$

Lemma 3.17. *The spaces $U_Q \setminus S$ and $U_P \setminus S$ are both homeomorphic to $\mathbb{R} \times (\mathbb{R}^3 \setminus \{0\})$, and are therefore homotopy equivalent to S^2.*

Proof. Under the identification $U_P \cong \mathrm{Hom}(P, P) \cong \mathbb{R}^4$, the intersection $S \cap U_P$ is defined by the equations $\mathrm{tr}(A) = x + w = 0$ and $\det(A) = xw - yz \geq 0$. These describe a solid, closed double cone in the three-dimensional subspace $\{x + w = 0\}$. The complement $U_P \setminus S$ is therefore invariant under multiplication by the positive scalar $\mathbb{R}_+$ and intersects the unit sphere S^3 in the complement of two closed 2-disks, which is diffeomorphic to $\mathbb{R}^3 \setminus \{0\}$. The case $U_Q \setminus S$ is similar. □

Proposition 3.18. *The inclusion $\mathcal{U} \subseteq \mathcal{F}$ defines an isomorphism $H^1(\mathcal{F};\mathbb{Z}) \cong H^1(\mathcal{U};\mathbb{Z})$. Consequently, $\mathcal{F}$ is an MA space that extends Λ_I and is dense in G.*

Proof. By definition $\mathcal{U} = \mathcal{F} \setminus \{P, Q\}$. Let $\mathcal{U}'$ be the union of two small open balls around P and Q in U_P and U_Q, respectively, intersected with $\mathcal{F}$. From the local picture described in the proof of Lemma 3.17, it is clear that $\mathcal{U}'$ deformation retracts onto the two point set $\{Q, P\}$ and that $\mathcal{U} \cap \mathcal{U}'$

deformation retracts onto $S^2 \coprod S^2$. The isomorphism follows from the Mayer–Vietoris long exact sequence

$$H^0(\mathcal{U}) \oplus H^0(\mathcal{U}') \twoheadrightarrow H^0(\mathcal{U} \cap \mathcal{U}') \to H^1(\mathcal{F}) \to H^1(\mathcal{U}) \oplus H^1(\mathcal{U}') \to H^1(\mathcal{U} \cap \mathcal{U}')$$

since $H^1(\mathcal{U}') \cong H^1(\mathcal{U} \cap \mathcal{U}') \cong \{0\}$ and $H^0(\mathcal{U}) \oplus H^0(\mathcal{U}') \twoheadrightarrow H^0(\mathcal{U} \cap \mathcal{U}')$ is surjective.

Any sufficiently generic loop $\gamma \colon S^1 \to \mathcal{F}$ is contained in $\mathcal{U}$, so $\mathcal{F}$ is an MA space extending $\mathcal{U}$. Finally, following the proof of Corollary 3.16, subspace inclusions determine a commuting diagram of isomorphisms

$$\begin{array}{ccc} H^1(\mathcal{F};\mathbb{Z}) & \xrightarrow{\cong} & H^1(\Lambda_I,\mathbb{Z}) \\ \downarrow \cong & & \downarrow \cong \\ H^1(\mathcal{U};\mathbb{Z}) & \xrightarrow{\cong} & H^1(N;\mathbb{Z}) \end{array}$$

so $\mathcal{F}$ also extends Λ_I. □

4. Counting unstable eigenvalues

We now explain how the theory of Maslov–Arnold spaces relates to the eigenvalue problem $\mathcal{L}v = \lambda v$ for the operator $\mathcal{L}$ defined in (2). In this section we give the general framework and some preliminary results, and construct an MA space that has desirable monotonicity properties for reaction–diffusion systems, allowing us to relate real unstable eigenvalues to conjugate points. Specific examples will be explored in the following section.

We consider a coupled system of eigenvalue equations on a bounded interval $(0, L)$, with separated boundary conditions given by subspaces $P_0, P_1 \in Gr_n(\mathbb{R}^{2n})$. That is, we seek solutions to the first-order system

$$\frac{d}{dx}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & D^{-1} \\ \lambda I - \nabla F(\bar{u}) & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} \tag{17}$$

satisfying the boundary conditions

$$\begin{pmatrix} v(0) \\ w(0) \end{pmatrix} \in P_0, \qquad \begin{pmatrix} v(L) \\ w(L) \end{pmatrix} \in P_1. \tag{18}$$

For instance, Dirichlet and Neumann boundary conditions correspond to the subspaces $P_D = \{(0, p) : p \in \mathbb{R}^n\}$ and $P_N = \{(q, 0) : q \in \mathbb{R}^n\}$, respectively. The Robin boundary condition $Du_x = \Theta u$, where Θ is a real $n \times n$ matrix, corresponds to $P_R = \{(q, \Theta q) : q \in \mathbb{R}^n\}$. Note that P_R is Lagrangian if and only if Θ is symmetric, and $\Theta = 0$ gives Neumann boundary conditions.

For each $x \in [0, L]$ and $\lambda \geq 0$ we define the subspace

$$W(x, \lambda) = \left\{ \begin{pmatrix} v(x) \\ w(x) \end{pmatrix} : \begin{pmatrix} v \\ w \end{pmatrix} \text{ satisfies (17) and } \begin{pmatrix} v(0) \\ w(0) \end{pmatrix} \in P_0 \right\}, \tag{19}$$

so that λ is an eigenvalue of $\mathcal{L}$ if and only $W(L, \lambda) \cap P_1 \neq \{0\}$.

Our goal is to obtain a generalized Morse index theorem, relating unstable eigenvalues of $\mathcal{L}$ to conjugate points, where x_* is said to be a conjugate point if $W(x_*, 0) \cap P_1 \neq \{0\}$.

Theorem 4.1. *Assume that* $P_1 = P_D$ *and* P_0 *is either* P_D *or* P_R *for some* $\Theta \in M_n(\mathbb{R})$*. For* $W(x,\lambda)$ *defined by* (19)*, there exists* $\lambda_\infty > 0$ *such that* $W(x,\lambda) \cap P_1 = \{0\}$ *for all* $0 < x \le L$ *and* $\lambda \ge \lambda_\infty$*. Let* H_1 *denote the hyperplane corresponding to* P_1*. If* $H_2 \ne H_1$ *is a hyperplane such that* $W(x,\lambda) \in G \setminus (H_1 \cap H_2)$ *for all* $(x,\lambda) \in (0,L] \times [0,\lambda_\infty]$*, then the generalized Maslov index of* W *is defined, and*

$$\begin{aligned}\#\big\{\textit{nonnegative eigenvalues of } \mathcal{L}\big\} &\ge -\operatorname{Mas}\Big(W(L,\lambda)\big|_{\lambda\in[0,\lambda_\infty]}; P_1\Big) \\ &= \operatorname{Mas}\Big(W(x,0)\big|_{x\in[\delta,L]}; P_1\Big)\end{aligned} \tag{20}$$

for $0 < \delta \ll 1$*. Moreover, there is a particular hyperplane* H_2 *with the property that if* $W(x,\lambda) \in G \setminus (H_1 \cap H_2)$ *for all* $(x,\lambda) \in (0,L] \times [0,\lambda_\infty]$*, then*

$$\operatorname{Mas}\Big(W(x,0)\big|_{x\in[\delta,L]}; P_1\Big) = \#\big\{\textit{conjugate points in } (0,L]\big\} \tag{21}$$

for $0 < \delta \ll 1$*, and hence*

$$\#\big\{\textit{nonnegative eigenvalues of } \mathcal{L}\big\} \ge \#\big\{\textit{conjugate points in } (0,L]\big\}. \tag{22}$$

Since $\lambda = 0$ is an eigenvalue of $\mathcal{L}$ if and only if $x = L$ is a conjugate point (with the same multiplicity), we immediately obtain the following.

Corollary 4.2. *Assuming the hypotheses of Theorem 4.1, we have*

$$\#\big\{\textit{positive eigenvalues of } \mathcal{L}\big\} \ge \#\big\{\textit{conjugate points in } (0,L)\big\}.$$

The generalized Maslov index only detects real eigenvalues, whereas $\mathcal{L}$ can have complex eigenvalues, since it is not assumed to be selfadjoint. However, it is immediate that

$$\#\big\{\text{unstable eigenvalues of } \mathcal{L}\big\} \ge \#\big\{\text{positive eigenvalues of } \mathcal{L}\big\},$$

where an eigenvalue is said to be unstable its real part is positive, and so the existence of an interior conjugate point is a sufficient condition for instability.

The main restriction in Theorem 4.1 is the condition that $W(x,\lambda) \in G\setminus(H_1\cap H_2)$, which means that the relevant curve of subspaces stays in the MA space. In the Hamiltonian case, this is guaranteed by the invariance of the Lagrangian Grassmannian under the associated flow. The idea here is that this can be applied on a case-by-case basis. It also allows us to discuss different mechanisms of instability, namely those that can be "counted" by a Maslov index and those associated with a breakdown of this Hamiltonian-like behavior due to a failure of the invariance condition on $W(x,\lambda)$, see the discussion below on the Turing problem in Section 5.3.

The rest of this section is devoted to the proof of Theorem 4.1. In Section 4.1 we give some preliminary calculations that will be of use here, and also in the applications in Section 5. In Section 4.2 we construct the promised MA space, and in Section 4.3 we complete the proof by computing the relevant Maslov indices.

4.1. **Preliminary calculations.** We start by considering the more general system

$$\frac{du}{dx} = A(x)u, \tag{23}$$

where $u \in \mathbb{R}^{2n}$ and $A(\cdot)$ is a continuous family of real $2n \times 2n$ matrices.

For an oriented n-plane $\widetilde{W} \subseteq \mathbb{R}^{2n}$ we define

$$\psi_\omega(\widetilde{W}) := \frac{\omega(f_1, \dots, f_n)}{|f_1 \wedge \dots \wedge f_n|}, \tag{24}$$

where $(f_1, \dots, f_n)$ is any positively oriented basis. The denominator can be computed as $\sqrt{\det G}$, where G denotes the *Gram matrix*, with entries $G_{ij} = \langle f_i, f_j \rangle$. For a positive orthonormal basis we have $G_{ij} = \delta_{ij}$ and hence $\psi_\omega(\widetilde{W}) = \omega(f_1, \dots, f_n)$.

Since ω is skew symmetric, we have $\psi_\omega(-\widetilde{W}) = -\psi_\omega(\widetilde{W})$, where $-\widetilde{W}$ is the negatively oriented version of $\widetilde{W}$. For an unoriented subspace W, $\psi_\omega(W)$ is only defined up to a sign, but the product and quotient $\psi_i(W)\psi_j(W)$ and $\psi_i(W)/\psi_j(W)$ are both well defined, where we have abbreviated $\psi_i = \psi_{\omega_i}$.

Lemma 4.3. *Let $W(x)$ be an integral curve of* (23)*. If $(f_1, \dots, f_n)$ is a positive orthonormal basis for $W(x_0)$, then*

$$\left.\frac{d\psi_\omega(\widetilde{W})}{dx}\right|_{x=x_0} = \sum_{j=1}^{n} \omega(f_1, \dots, A(x_0)f_j, \dots, f_n) - \psi_\omega(\widetilde{W}) \sum_{j=1}^{n} \langle A(x_0)f_j, f_j \rangle . \tag{25}$$

Proof. Write $\psi_\omega(\widetilde{W}) = n/d$ where n and d are the numerator and denominator of the expression (24). Then

$$\left.\frac{d\psi_\omega(\widetilde{W})}{dx}\right|_{x=x_0} = \left.\frac{dn' - nd'}{d^2}\right|_{x=x_0} = n'(x_0) - \psi_\omega(\widetilde{W})d'(x_0),$$

where we have substituted $d(x_0) = 1$ and $n(x_0) = \psi_\omega(\widetilde{W})$. Using the fact that $W(x)$ is an integral curve, one easily calculates

$$n'(x_0) = \sum_{j=1}^{n} \omega(f_1, \dots, A(x_0)f_j, \dots, f_n).$$

Moreover, since $d(x) = \sqrt{\det G(x)}$ and $G(x_0)$ is the identity matrix, Jacobi's formula for the derivative of the determinant yields

$$d'(x_0) = \frac{1}{2} \operatorname{tr} \left(\left.\frac{dG}{dx}\right|_{x=x_0} \right) = \sum_{j=1}^{n} \langle A(x_0)f_j, f_j \rangle$$

which completes the proof. □

As defined above, the index of a curve $W(t)$ in $\mathcal{M}$ is equal to the winding number, through the point $[0:1]$, of the curve $\phi \circ \gamma$ in $\mathbb{R}P^1$, where ϕ is defined in (11). Here we write W as a function of t to emphasize that it can be any continuous curve in $\mathcal{M}$, not necessarily an integral curve of the system (23).

We thus need to understand the motion of the curve $t \mapsto [\omega_1(v_1, \dots, v_n) : \omega_2(v_1, \dots, v_n)]$ in $\mathbb{R}P^1$, where $\{v_1, \dots, v_n\}$ is a basis for $W(t)$.

Suppose $\omega_1(v_1,\dots,v_n)=0$ for some t_0. This implies $\omega_2(v_1,\dots,v_n)\neq 0$ for $|t-t_0|\ll 1$, hence

$$\frac{\omega_1(v_1,\dots,v_n)}{\omega_2(v_1,\dots,v_n)}=\frac{\psi_1(t)}{\psi_2(t)},$$

where we have abbreviated $\psi_i(t)=\psi_i(\widetilde{W}(t))$. Since $\psi_1(t_0)=0$, we find that

$$\frac{d}{dt}\frac{\omega_1(v_1,\dots,v_n)}{\omega_2(v_1,\dots,v_n)}\bigg|_{t=t_0}=\frac{\psi_1'(t_0)\psi_2(t_0)-\psi_1(t_0)\psi_2'(t_0)}{\psi_2(t_0)^2}=\frac{\psi_1'(t_0)}{\psi_2(t_0)}. \tag{26}$$

In the next section we will use this formula, in combination with (25), to obtain monotonicity results for integral curves of (23).

4.2. **Choosing hyperplanes for a reaction–diffusion system.** We now return to the eigenvalue problem (3), letting

$$A(x,\lambda)=\begin{pmatrix}0 & D^{-1}\\ B(x,\lambda) & 0\end{pmatrix} \tag{27}$$

in (23), where $D=\operatorname{diag}(d_1,\dots,d_n)$.

For Dirichlet boundary conditions it is natural to let $H_1\cap G$ be the train of the Dirichlet subspace. We thus choose H_1 to be the hyperplane corresponding to the degenerate n-form

$$\omega_1=e_1^*\wedge\cdots\wedge e_n^*, \tag{28}$$

where $e_1,\dots,e_{2n}$ denotes the standard orthonormal basis for $\mathbb{R}^{2n}$. Since the resulting index equals the geometric intersection number with $H_1\cap G$, it will count solutions to the Dirichlet problem, which are (by definition) conjugate points. When $n=2$, the two-form ω_1 corresponds to the matrix

$$\Omega_1=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

in the sense that $\omega_1(v,w)=v^T\Omega_1 w$ for any $v,w\in\mathbb{R}^4$.

The choice of H_2 is less obvious. Motivated by the calculation to follow in Lemma 4.4, we let

$$\omega_2=\sum_{j=1}^n\frac{1}{d_j}e_1^*\wedge\cdots\wedge\widehat{e_j^*}\wedge e_{j+n}^*\wedge\cdots\wedge e_n^*, \tag{29}$$

i.e. the jth summand is proportional to ω_1 with e_j^* replaced by e_{j+n}^*. When $n=2$ this is

$$\omega_2=\frac{1}{d_1}e_3^*\wedge e_2^*+\frac{1}{d_2}e_1^*\wedge e_4^*,$$

corresponding to the matrix

$$\Omega_2=\begin{pmatrix}0&0&0&1/d_2\\0&0&-1/d_1&0\\0&1/d_1&0&0\\-1/d_2&0&0&0\end{pmatrix}. \tag{30}$$

This choice allows us to obtain a monotonicity result that is central to the proof of Theorem 4.1. Moreover, it will play a prominent role in Section 5, where we prove a long-time invariance result for reaction–diffusion systems with large diffusivities.

We now apply Lemma 4.3 to these symplectic forms. To state the result, we additionally define

$$\omega_3 = \sum_{\substack{j,k=1 \\ j<k}}^{n} \frac{2}{d_j d_k} e_1^* \wedge \cdots \wedge e_{j+n}^* \wedge \cdots \wedge e_{k+n}^* \wedge \cdots \wedge e_n^*. \tag{31}$$

That is, the j-k summand is obtained from ω_1 by replacing e_j^* and e_k^* by e_{j+n}^* and e_{k+n}^*, respectively. For $n = 2$ we have

$$\omega_3 = \frac{2}{d_1 d_2} e_3^* \wedge e_4^*,$$

which is a degenerate two-form corresponding to the train of the Neumann subspace.

Lemma 4.4. *Let $W(x,\lambda)$ be an integral curve of $\frac{du}{dx} = A(x,\lambda)u$, with $A(x,\lambda)$ given by* (27)*, and define ω_1, ω_2 and ω_3 by* (28)*,* (29) *and* (31)*, respectively. Then*

$$\frac{d\psi_1}{dx} = \psi_2 - \gamma\psi_1 \tag{32}$$

and

$$\frac{d\psi_2}{dx} = \left(\frac{b_{11}}{d_1} + \cdots + \frac{b_{nn}}{d_n}\right)\psi_1 + \psi_3 - \gamma\psi_2 \tag{33}$$

where $\gamma = \sum_{j=1}^n \langle Af_j, f_j\rangle$ and b_{ij} is the i-j component of the matrix B. Moreover, if $W(x_0,\lambda) = P_D$, then

$$\psi_1(x_0) = \psi_1'(x_0) = \cdots = \psi_1^{(n-1)}(x_0) = 0 \tag{34}$$

and

$$\psi_1^{(n)}(x_0) = \frac{n!}{d_1 \cdots d_n} \neq 0. \tag{35}$$

Proof. From Lemma 4.3 we have

$$\frac{d\psi_i}{dx} = \sum_{j=1}^{n} \omega_i(f_1, \ldots, Af_j, \ldots, f_n) - \gamma\psi_i$$

for $i \in \{1,2\}$. For ω_1 we observe that

$$\omega_1(f_1, \ldots, Af_j, \ldots, f_n) = \big(e_1^* \wedge \cdots \wedge e_j^* A \wedge \cdots \wedge e_n^*\big)(f_1, \ldots, f_n).$$

The composition $e_j^* A \colon V \to \mathbb{R}$ is given by $e_j^* A = \sum_{k=1}^{2n} A_{jk} e_k^*$, hence

$$e_j^* A = \frac{1}{d_j} e_{j+n}^*, \qquad e_{j+n}^* A = \sum_{k=1}^{n} b_{jk} e_k^*,$$

for any $1 \leq j \leq n$. It follows that

$$e_1^* \wedge \cdots \wedge e_j^* A \wedge \cdots \wedge e_n^* = \frac{1}{d_j} e_1^* \wedge \cdots \wedge e_{j+n}^* \wedge \cdots \wedge e_n^*,$$

which is precisely the jth summand in the definition of ω_2. This implies

$$\sum_{j=1}^{n} \omega_1(f_1, \ldots, Af_j, \ldots, f_n) = \omega_2(f_1, \ldots, f_n), \tag{36}$$

and completes the proof of (32).

For (33) we need to compute

$$\sum_{j=1}^{n} \omega_2(f_1, \dots, Af_j, \dots, f_n) = \sum_{j,k=1}^{n} \omega_2^k(f_1, \dots, Af_j, \dots, f_n),$$

where $\omega_2^k := d_k^{-1} e_1^* \wedge \cdots \wedge e_{k+n}^* \wedge \cdots \wedge e_n^*$ denotes the kth summand in the definition of ω_2. For summands with $j = k$ we have

$$\begin{aligned}\frac{1}{d_j} e_1^* \wedge \cdots \wedge e_{j+n}^* A \wedge \cdots \wedge e_n^* &= \frac{1}{d_j} e_1^* \wedge \cdots \wedge \left(\sum_{l=1}^{n} b_{jl} e_l^* \right) \wedge \cdots \wedge e_n^* \\ &= \frac{b_{jj}}{d_j} \omega_1.\end{aligned}$$

For summands with $j \neq k$ we have

$$\frac{1}{d_k} e_1^* \wedge \cdots \wedge e_j^* A \wedge \cdots \wedge e_{k+n}^* \wedge \cdots \wedge e_n^* = \frac{1}{d_k} e_1^* \wedge \cdots \wedge \left(\frac{1}{d_j} e_{j+n}^* \right) \wedge \cdots \wedge e_{k+n}^* \wedge \cdots \wedge e_n^*,$$

which is precisely the j,k term in the definition of ω_3, so the proof of (33) is complete.

To prove the final statement, we recall that $P_D = \operatorname{span}\{e_{n+1}, \dots, e_{2n}\}$, so an n-form $\omega = e_{j_1}^* \wedge \cdots \wedge e_{j_n}^*$ will vanish on P_D unless $\{j_1, \dots, j_n\} = \{n+1, \dots, 2n\}$. In general, suppose m of the indices $j_1, \dots, j_n$ are contained in $\{n+1, \dots, 2n\}$, with the remaining $n - m$ in $\{1, \dots, n\}$. Then, as in the calculations above, the derivative of ψ_ω will have terms with $m - 1$, m and $m + 1$ indices in $\{n+1, \dots, 2n\}$. To find the first nonvanishing derivative of ψ_ω on P_D, we therefore only need to keep track of the $m + 1$ term. We thus compute

$$\begin{aligned}
\frac{d\psi_1}{dx} &= \psi_2 + \cdots, \\
\frac{d\psi_2}{dx} &= \psi_3 + \cdots, \\
\frac{d\psi_3}{dx} &= \psi_4 + \cdots, \qquad \omega_4 := \sum_{\substack{j,k,l=1 \\ j<k<l}}^{n} \frac{3!}{d_j d_k} e_1^* \wedge \cdots \wedge e_{j+n}^* \wedge \cdots \wedge e_{k+n}^* \wedge \cdots \wedge e_{l+n}^* \wedge \cdots \wedge e_n^* \\
&\vdots \\
\frac{d\psi_n}{dx} &= \psi_{n+1} + \cdots, \qquad \omega_{n+1} := \frac{n!}{d_1 \cdots d_n} e_{n+1}^* \wedge \cdots \wedge e_{2n}^*,
\end{aligned}$$

and the result follows. □

Remark 4.5. *The form ω_2 was chosen to make the equality* (36) *hold. The fact that we can do this, and end up with a form that does not depend on x or λ (even though A does) is a consequence of the block structure of A and the fact that ω_1 only depends on the first n coordinates.*

4.3. **Positive eigenvalues and conjugate points.** We are now ready to begin the proof of Theorem 4.1. We start with the existence of λ_∞.

Lemma 4.6. *Assuming the hypotheses of Theorem 4.1, there exists $\lambda_\infty > 0$ such that $W(x, \lambda) \cap P_1 = \{0\}$ for all $0 < x \leq L$ and $\lambda \geq \lambda_\infty$. Moreover, every eigenvalue $\lambda \in \sigma(\mathcal{L})$ has $\operatorname{Re} \lambda \leq \lambda_\infty$.*

Note that the property $W(x,\lambda) \cap P_1 = \{0\}$ is only guaranteed for $0 < x \leq L$. It is possible for $W(0,\lambda)$ to intersect P_1 nontrivially, for instance if $P_0 = P_1$.

Proof. Suppose there is a (possibly complex-valued) solution v to

$$Dv_{xx} + \nabla F(\bar{u})v = \lambda v$$

on $[0, x_*]$, satisfying the boundary conditions

$$\begin{pmatrix} v \\ Dv_x \end{pmatrix}\bigg|_{x=0} \in P_0, \qquad \begin{pmatrix} v \\ Dv_x \end{pmatrix}\bigg|_{x=x_*} \in P_1.$$

Since $P_1 = P_D$, this means $v(x_*) = 0$. Similarly, at $x = 0$ we have either $v(0) = 0$ or $Dv_x(0) = \Theta v(0)$, depending on the choice of P_0.

Multiplying the eigenvalue equation by the conjugate of v and integrating by parts, using $v(x_*) = 0$, we find that

$$\lambda \int_0^{x_*} |v|^2\, dx = -\langle Dv_x(0), v(0)\rangle + \int_0^{x_*} \big(\langle \nabla F(\bar{u})v, v\rangle - \langle Dv_x, v_x\rangle \big)\, dx, \tag{37}$$

where $\langle\cdot,\cdot\rangle$ denotes the $\mathbb{C}^n$ inner product. Defining constants

$$d = \min\{d_1, \dots, d_n\}, \qquad K = \sup_{x\in[0,L]} \big\|\nabla F(\bar{u}(x))\big\|,$$

we obtain

$$\mathrm{Re} \int_0^{x_*} \big(\langle \nabla F(\bar{u})v, v\rangle - \langle Dv_x, v_x\rangle \big)\, dx \leq K \int_0^{x_*} |v|^2\, dx - d \int_0^{x_*} |v_x|^2\, dx. \tag{38}$$

To deal with the boundary term in (37), we treat the Dirichlet and Robin cases separately. If $P_0 = P_D$, then the boundary term vanishes, so we find that

$$\mathrm{Re}\, \lambda \leq K$$

and hence it suffices to choose any $\lambda_\infty > K$. On the other hand, if $P_0 = P_R$, the boundary term satisfies $|\langle Dv_x(0), v(0)\rangle| = |\langle \Theta v(0), v(0)\rangle| \leq C|v(0)|^2$ for some positive constant C. Moreover, since $v(x_*) = 0$, we have

$$\begin{aligned} |v(0)|^2 &= \left| \int_0^{x_*} \frac{d}{dx} |v(x)|^2 dx \right| \\ &\leq 2 \int_0^{x_*} |v||v_x|\, dx \\ &\leq \epsilon^{-1} \int_0^{x_*} |v|^2\, dx + \epsilon \int_0^{x_*} |v_x|^2\, dx \end{aligned}$$

for any $\epsilon > 0$. Choosing $\epsilon = d/C$, and combining the above inequality with (37) and (38), we obtain

$$\mathrm{Re}\, \lambda \leq K + \frac{C^2}{d},$$

which completes the proof. ∎

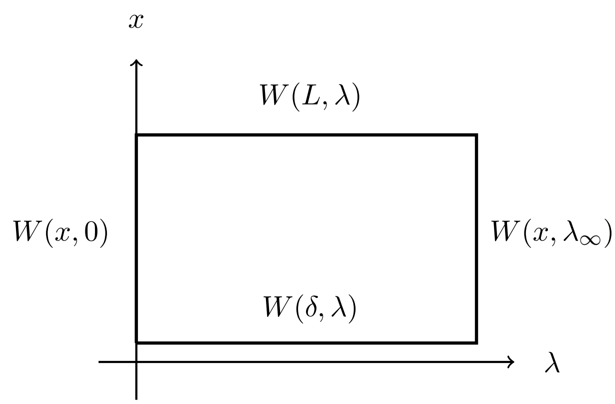

FIGURE 2. The homotopy argument in (39)

Now define the Maslov–Arnold space $\mathcal{M} = G \setminus (H_1 \cap H_2)$, with H_1 as in Section 4.2 and any $H_2 \neq H_1$, and consider the path $W(x,\lambda)$ in $Gr_n(\mathbb{R}^{2n})$ defined by (19).

By assumption, $W(x,\lambda) \in \mathcal{M}$ for all $(x,\lambda) \in [\delta, L] \times [0,\lambda_\infty]$. Therefore, the image under W of the boundary of $[\delta, L] \times [0,\lambda_\infty]$ is null-homotopic, and hence has zero index. Adding the four sides with appropriate orientation (see Figure 2), we obtain

$$\begin{aligned} &\operatorname{Mas}\Big(W(x,0)\big|_{x\in[\delta,L]}; P_1\Big) + \operatorname{Mas}\Big(W(L,\lambda)\big|_{\lambda\in[0,\lambda_\infty]}; P_1\Big) \\ &\quad = \operatorname{Mas}\Big(W(\delta,\lambda)\big|_{\lambda\in[0,\lambda_\infty]}; P_1\Big) + \operatorname{Mas}\Big(W(x,\lambda_\infty)\big|_{x\in[\delta,L]}; P_1\Big). \end{aligned} \tag{39}$$

We will prove the theorem by evaluating each of these terms. We start by showing that

$$\operatorname{Mas}\Big(W(x,\lambda_\infty)\big|_{x\in[\delta,L]}; P_1\Big) = 0, \tag{40}$$

$$\Big|\operatorname{Mas}\Big(W(L,\lambda)\big|_{\lambda\in[0,\lambda_\infty]}; P_1\Big)\Big| \leq \#\big\{\text{nonnegative eigenvalues of } \mathcal{L}\big\}, \tag{41}$$

$$\operatorname{Mas}\Big(W(\delta,\lambda)\big|_{\lambda\in[0,\lambda_\infty]}; P_1\Big) = 0, \tag{42}$$

These inequalities, combined with (39), immediately yield (20).

Lemma 4.6 implies that

$$\operatorname{Mas}\Big(W(x,\lambda_\infty)\big|_{x\in[\delta,L]}; P_1\Big) = 0,$$

for any $\delta \in (0,1)$, so (40) is verified. Moreover, since the index counts *signed* intersections between $W(x,\lambda)$ and P_1, we have

$$\begin{aligned} \Big|\operatorname{Mas}\Big(W(L,\lambda)\big|_{\lambda\in[0,\lambda_\infty]}; P_1\Big)\Big| &\leq \#\big\{\lambda \in [0,\lambda_\infty] : W(1,\lambda) \cap P_1 \neq \{0\}\big\} \\ &= \#\big\{\text{eigenvalues of } \mathcal{L} \text{ in } [0,\lambda_\infty]\big\} \\ &= \#\big\{\text{nonnegative eigenvalues of } \mathcal{L}\big\}, \end{aligned}$$

where the last equality follows from Lemma 4.6. This confirms (41).

We next deal with (42).

Lemma 4.7. *There exists $\delta > 0$ such that $W(x,\lambda) \cap P_1 = \{0\}$ for all $\lambda \in [0,\lambda_\infty]$ and $x \in (0,\delta)$.*

Proof. There are two cases to consider. If $P_0 = P_R$, then $P_0 \cap P_1 = \{0\}$, and so $W(0,\lambda) \cap P_1 = \{0\}$ for all λ, because $W(0,\lambda) = P_0$. Since $W(x,\lambda)$ in continuous in x and λ, and $[0,\lambda_\infty]$ is compact, there exists $\delta > 0$ such that $W(x,\lambda) \cap P_0 = \{0\}$ for all $\lambda \in [0,\lambda_\infty]$ and $x \in [0,\delta)$. (Note that $x = 0$ is allowed in this case.)

The other case is when $P_0 = P_D$, so $W(0,\lambda) \cap P_1 \neq \{0\}$. Defining $\eta(x,\lambda) = \psi_1(x,\lambda)^2$, we have

$$\eta(0,\lambda) = \cdots = \eta^{(2n-1)}(0,\lambda) = 0 \qquad \text{and} \qquad \eta^{(2n)}(0,\lambda) > 0$$

from Lemma 4.4. Therefore, for fixed λ we have $\eta(x,\lambda) > 0$ for sufficiently small $x > 0$, and so by compactness there exists $\delta > 0$ such that $\eta(x,\lambda) > 0$ for all $\lambda \in [0,\lambda_\infty]$ and $x \in (0,\delta)$. This completes the proof, since $\eta(x,\lambda) > 0$ implies $\psi_1(x,\lambda) \neq 0$ and hence $W(x,\lambda) \cap P_1 = \{0\}$. □

This completes the proof of (20). The following lemma verifies (22), and hence completes the proof of Theorem 4.1. Note that up to this point H_2 has been an arbitrary hyperplane different from H_1, and did not appear explicitly in any of the preceeding calculations.

Lemma 4.8. *For H_2 as defined in Section 4.2 we have*

$$\operatorname{Mas}\Big(W(x,0)\big|_{x\in[\delta,L]}; P_1\Big) = \#\{\textit{conjugate points in } (0,L]\}$$

for $0 < \delta \ll 1$.

The Maslov index on the left-hand side is a *signed* count of the $x_* \in [\delta, L]$ for which $W(x_*,0) \cap P_1 \neq \{0\}$. These are conjugate points (by definition) so to prove the lemma we just need to show that they all contribute to the Maslov index with the same sign. This is where the choice of H_2 becomes crucial.

Proof. Suppose $x_* \in [\delta, L]$ is a conjugate point, so $\psi_1(x_*) = \psi_1(W(x_*,0)) = 0$. Then (32) implies $\psi_1'(x_0) = \psi_2(x_0)$. Substituting this in (26), we obtain

$$\frac{d}{dx}\frac{\omega_1(v_1,\ldots,v_n)}{\omega_2(v_1,\ldots,v_n)}\bigg|_{x=x_*} = 1 > 0.$$

Recalling Definition 3.10, and in particular (14), this says that the Maslov index equals the number of conjugate points in $(\delta, L]$, and hence the number of conjugate points in $(0,L]$ if δ is sufficiently small. □

5. Applications

5.1. **Systems with large diffusion.** Here we give an example where the curve $W(x,\lambda)$ is guaranteed to remain in the Arnold–Maslov space $\mathcal{M}$ constructed above, hence Theorem 4.1 can be applied.

Specifically, we consider the eigenvalue problem with mixed boundary conditions

$$D\frac{d^2u}{dx^2} + Vu = \lambda u, \quad u'(0) = u(L) = 0, \tag{43}$$

recalling that $D = \operatorname{diag}(d_1,\ldots,d_n)$. The corresponding boundary subspaces are

$$P_0 = \{(q,0) : q \in \mathbb{R}^n\}, \quad P_1 = \{(0,p) : p \in \mathbb{R}^n\},$$

and so x_* is a conjugate point if there exists a nontrivial solution to the boundary value problem

$$D\frac{d^2u}{dx^2} + Vu = 0, \quad u'(0) = u(x_*) = 0.$$

Our main result is that Theorem 4.1 applies to the above system as long as none of the d_j are too small, and all of the products $d_j d_k$ with $j \neq k$ are sufficiently large.

Theorem 5.1. *Fix L and $d_* > 0$, and suppose $V \in C[0,L]$. There exists a constant $\Delta > 0$ such that if $d_j \geq d_*$ for all j and $d_j d_k \geq \Delta$ for $j \neq k$, then the hypotheses of Theorem 4.1 are satisfied, and hence*

$$\#\big\{\textit{positive real eigenvalues of } (43)\big\} \geq \#\big\{\textit{conjugate points in } (0,L)\big\}. \tag{44}$$

The constant Δ appearing in the theorem depends on L, d_* and V, and can be estimated from the proof if desired.

Proof. From Lemma 4.6 we see that λ_∞ can be any number satisfying

$$\lambda_\infty > \sup_{x\in[0,L]} \big\|V(x)\big\|.$$

In particular, it can be chosen independent of D.

We now use Lemma 4.4, with $B(x,\lambda) = \lambda I - V(x)$. Define

$$\rho = \frac{1}{2}(\psi_1^2 + \psi_2^2), \tag{45}$$

so that $\rho(0,\lambda) = 1/2$. It follows that

$$\frac{d\rho}{dx} = -2\gamma\rho + \left(1 + \frac{b_{11}}{d_1} + \cdots + \frac{b_{nn}}{d_n}\right)\psi_1\psi_2 + \psi_2\psi_3.$$

From the definition of γ (in Lemma 4.4) we obtain

$$|\gamma(x,\lambda)| \leq n\|A(x,\lambda)\| \leq n\left(\|B(x,\lambda)\| + \|D^{-1}\|\right) \leq n\left(\max\|B(x,\lambda)\| + \frac{1}{d_*}\right) =: C_1$$

where the maximum is taken over $(x,\lambda) \in [0,L]\times[0,\lambda_\infty]$. We similarly have

$$\left|\left(1 + \frac{b_{11}}{d_1} + \cdots + \frac{b_{nn}}{d_n}\right)\psi_1\psi_2\right| \leq \underbrace{\left(1 + \frac{\max|b_{11}(x,\lambda)|}{d_*} + \cdots + \frac{\max|b_{nn}(x,\lambda)|}{d_*}\right)}_{C_2}\rho.$$

Moreover, using

$$|\psi_3| \leq \sum_{\substack{j,k=1 \\ j<k}}^{n} \frac{2}{d_j d_k} \leq \frac{n(n-1)}{\Delta},$$

we obtain

$$|\psi_2\psi_3| \leq |\psi_2|\frac{n(n-1)}{\Delta} \leq \underbrace{\frac{1}{2}\left(\frac{n(n-1)}{d_*}\right)^2}_{C_3}\rho + \frac{1}{\Delta},$$

and hence $\frac{d\rho}{dx} \geq -C\rho - \Delta^{-1}$, where $C = 2C_1 + C_2 + C_3$ depends only on d_* and V. This is equivalent to

$$\frac{d}{dx}\left(e^{Cx}\rho(x)\right) \geq -\frac{e^{Cx}}{\Delta}$$

so we have

$$e^{Cx}\rho(x) - \frac{1}{2} \geq -\frac{1}{\Delta}\int_0^x e^{Ct}dt = \frac{1-e^{Cx}}{C\Delta}.$$

Therefore, we will have $\rho(x,\lambda) > 0$ for $\lambda \in [0,\lambda_\infty]$ provided

$$e^{Cx} < 1 + \frac{C\Delta}{2}.$$

This equality will hold for all $(x,\lambda) \in [0,L] \times [0,\lambda_\infty]$ if it holds when $x = L$. Therefore, we need $e^{CL} < 1 + \frac{C\Delta}{2}$. This is satisfied for a sufficiently large choice of Δ, depending only on L and C (i.e. on L, d_* and V). □

5.2. **Stability of homogeneous equilibria.** If the steady state $\bar{u}$ is homogeneous (constant in x), then the linearized operator (2) has the form

$$\mathcal{L} = \frac{d^2}{dx^2} + V$$

where $V = \nabla F(\bar{u}) \in M_2(\mathbb{R})$ is a constant real matrix. Consider the Dirichlet problem on $(0,L)$,

$$\mathcal{L}v = \lambda v, \quad v(0) = v(1) = 0 \in \mathbb{R}^2. \tag{46}$$

Assume for simplicity that V is diagonalizable, with eigenvalues ν_1 and ν_2. Then $\mathcal{L}$ is similar to the decoupled operator

$$\widetilde{\mathcal{L}} = \begin{pmatrix} \frac{d^2}{dx^2} + \nu_1 & 0 \\ 0 & \frac{d^2}{dx^2} + \nu_2 \end{pmatrix}$$

and hence has spectrum

$$\sigma(\mathcal{L}) = \left\{\nu_1 - (n\pi/L)^2 : n \in \mathbb{N}\right\} \cup \left\{\nu_2 - (n\pi/L)^2 : n \in \mathbb{N}\right\}. \tag{47}$$

It follows that $\mathcal{L}$ has a positive eigenvalue if and only if both ν_1 and ν_2 are real and at least one of them is greater than $(\pi/L)^2$. More generally, when both eigenvalues are real we obtain

$$\#\{\text{positive eigenvalues of } \mathcal{L}\} = \#\{\text{conjugate points in } (0,L)\}, \tag{48}$$

since the eigenvalue equation $\widetilde{\mathcal{L}}v = \lambda v$ consists of two decoupled Sturm–Liouville problems. This equality also holds when V is not diagonalizable, provided the eigenvalues of $\mathcal{L}$ are counted with geometric multiplicity.

On the other hand, if ν_1 and ν_2 have nonzero imaginary part, then $\mathcal{L}$ has no real eigenvalues, and hence no positive eigenvalues (though it will have eigenvalues with *positive real part* if $\operatorname{Re}\nu_1 = \operatorname{Re}\nu_2 > (\pi/L)^2$). Moreover, it is easy to see that when ν is complex, the equation

$$v''(x) + \nu v(x) = 0, \quad v(0) = v(x_*) = 0$$

does not admit nontrivial solutions for any $x_* > 0$, and so there are no conjugate points. Therefore, the equality (48) holds trivially in this case.

We have thus verified (48) for any constant potential $V \in M_2(\mathbb{R})$. We now reconsider this problem using the machinery developed in the previous section, to see whether or not the same conclusion can be obtained from our generalized Maslov index. Our result is the following.

Theorem 5.2. *If the eigenvalues of V satisfy one of the following conditions*

(i) ν_1 *and* ν_2 *are not real*
(ii) ν_1 *and* ν_2 *are real and* $\min\{\nu_1, \nu_2\} < (\pi/L)^2$

then $W(x,\lambda) \in \mathcal{M}$ for all $0 < x \le L$ and $\lambda \ge 0$, so the generalized Maslov index of W is defined, and for $0 < \delta \ll 1$ we have

$$\begin{aligned}\#\big\{\textit{nonnegative eigenvalues of } \mathcal{L}\big\} &= -\operatorname{Mas}\Big(W(L,\cdot)\big|_{\lambda\in[0,\lambda_\infty]}; P_1\Big)\\ &= -\operatorname{Mas}\Big(W(\cdot,0)\big|_{x\in[\delta,L]}; P_1\Big)\\ &= \#\big\{\textit{conjugate points in } (0,L]\big\}\end{aligned}$$

and hence

$$\#\big\{\textit{positive eigenvalues of } \mathcal{L}\big\} = \#\big\{\textit{conjugate points in } (0,L)\big\}.$$

The hypothesis on the eigenvalues only fails when both ν_1 and ν_2 are real and greater than or equal to $(\pi/L)^2$. As seen above, $\mathcal{L}$ has no positive eigenvalues if ν_1 and ν_2 are complex, or if $\nu_1, \nu_2 \le (\pi/L)^2$. Therefore, the result is most interesting, in the sense that the Maslov index is nonzero, when V has precisely one eigenvalue in the interval $\big((\pi/L)^2, \infty\big)$.

We start by writing the eigenvalue problem in the general form

$$\frac{d}{dx}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & I \\ B(\lambda) & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}, \tag{49}$$

where $B(\lambda) \in M_2(\mathbb{R})$ does not depend on x. Later we will set $B(\lambda) = \lambda I - V$.

This is of the form considered in Section 4, with $d_1 = d_2 = 1$, so we choose ω_1 and ω_2 corresponding to the matrices

$$\Omega_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{50}$$

Let H_1 and H_2 denote the corresponding hyperplanes, and $\mathcal{M} = G \setminus (H_1 \cap H_2)$ the resulting Maslov–Arnold space.

As above, we define a family of two-dimensional subspaces

$$W(x,\lambda) = \left\{ \begin{pmatrix} v(x) \\ w(x) \end{pmatrix} : \begin{pmatrix} v \\ w \end{pmatrix} \text{ satisfies (49) and } v(0) = 0 \right\} \subseteq \mathbb{R}^4 \tag{51}$$

for $x \ge 0$. The proof of Theorem 5.2 consists of two steps. First, we show that $W(x,\lambda)$ is contained in $\mathcal{M}$ for all $(x,\lambda) \in (0,L] \times [0,\lambda_\infty]$, and hence Theorem 4.1 applies. Then, we show that $W(x,\lambda)$ is monotone in λ, which implies

$$\#\big\{\text{nonnegative eigenvalues of } \mathcal{L}\big\} = \operatorname{Mas}\Big(W(L,\lambda)\big|_{\lambda\in[0,\lambda_\infty]}; P_1\Big)$$

and thus completes the proof.

We start with the invariance result that guarantees the index of $W(x,\lambda)$ is defined.

Proposition 5.3. *Let $W(x,\lambda)$ be defined by (51). Then $W(x_*,\lambda) \in H_1 \cap H_2$ for some $x_* > 0$ if and only if the eigenvalues β_1, β_2 of $B(\lambda)$ are real and negative and satisfy*

$$\sin\sqrt{-\beta_1}x_* = \sin\sqrt{-\beta_2}x_* = 0.$$

Proof. We first compute a frame for $W(x,\lambda)$. A frame for a two-dimensional subspace W is (by definition) a 4×2 matrix whose columns span W. Writing this as

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix},$$

and denoting the columns by u_1 and u_2, we compute

$$\begin{aligned} \omega_1(u_1,u_2) &= x_{11}x_{22} - x_{12}x_{21} \\ &= \det X \end{aligned}$$

and

$$\begin{aligned} \omega_2(u_1,u_2) &= x_{11}y_{22} - x_{21}y_{12} + y_{11}x_{22} - y_{21}x_{12} \\ &= \det(X+Y) - \det X - \det Y. \end{aligned}$$

It follows that

$$W \in H_1 \iff \det X = 0$$

and

$$W \in H_2 \iff \det(X+Y) = \det X + \det Y.$$

Note that $W(x,\lambda)$ is spanned by the last two columns of the fundamental solution matrix e^{Ax}, where $A = \left(\begin{smallmatrix} 0 & I \\ B & 0 \end{smallmatrix}\right)$. We thus compute

$$e^{Ax} = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \begin{pmatrix} B^m x^{2m} & 0 \\ 0 & B^m x^{2m} \end{pmatrix} + \sum_{m=0}^{\infty} \frac{1}{(2m+1)!} \begin{pmatrix} 0 & B^m x^{2m+1} \\ B^{m+1} x^{2m+1} & 0 \end{pmatrix}$$

to conclude that a frame for $W(x,\lambda)$ is given by

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \sum_{m=0}^{\infty} \begin{pmatrix} \frac{B^m x^{2m+1}}{(2m+1)!} \\ \frac{B^m x^{2m}}{(2m)!} \end{pmatrix} = \begin{pmatrix} B^{-1/2}\sinh(\sqrt{B}x) \\ \cosh(\sqrt{B}x) \end{pmatrix}. \tag{52}$$

The functions on the right-hand side are defined by their power series, which converge for all numbers x and matrices B.

Letting β_1 and β_2 denote the eigenvalues of $B(\lambda)$, it follows that $W(x,\lambda) \in H_1 \cap H_2$ if and only if

$$\det X = \frac{\sinh(\sqrt{\beta}_1 x)}{\sqrt{\beta}_1} \frac{\sinh(\sqrt{\beta}_2 x)}{\sqrt{\beta}_2} = 0 \tag{53}$$

and

$$\det(X+Y) - \det X - \det Y = \frac{\sinh(\sqrt{\beta_1}x)}{\sqrt{\beta_1}}\cosh(\sqrt{\beta_2}x) + \frac{\sinh(\sqrt{\beta_2}x)}{\sqrt{\beta_2}}\cosh(\sqrt{\beta_1}x) = 0. \quad (54)$$

As in (52), the functions $\beta^{-1/2}\sinh(\sqrt{\beta}x)$ and $\cosh(\sqrt{\beta}x)$ are defined by power series which converge for all values of x and β. In particular, when $\beta = 0$ we obtain $\beta^{-1/2}\sinh(\sqrt{\beta}x) = x$, and when $x = 0$ we obtain $\beta^{-1/2}\sinh(\sqrt{\beta}x) = 0$ for any value of β.

Now suppose that $W(x_*, \lambda) \in H_1 \cap H_2$ for some $x_* > 0$, so both (53) and (54) are satisfied. If $\beta_1^{-1/2}\sinh(\sqrt{\beta_1}x_*) = 0$, then $\beta_1 \neq 0$, hence $\sinh(\sqrt{\beta_1}x_*) = 0$ and so $\cosh(\sqrt{\beta_1}x_*) \neq 0$. Then (54) implies $\beta_2^{-1/2}\sinh(\sqrt{\beta_2}x_*) = 0$, hence $\beta_2 \neq 0$ and $\sinh(\sqrt{\beta_2}x_*) = 0$. Therefore, $W(x_*, \lambda) \in H_1 \cap H_2$ if and only if $\sinh(\sqrt{\beta_1}x_*) = \sinh(\sqrt{\beta_2}x_*) = 0$, which is possible if and only if β_1 and β_2 are both real and negative and satisfy $\sin(\sqrt{-\beta_1}x_*) = \sin(\sqrt{-\beta_2}x_*) = 0$. □

Remark 5.4. *In terms of the frame computed above, $x_* \in (0, L]$ is a conjugate point if and only if at least one of the eigenvalues of X is zero. On the other hand, $W(x_*, \lambda) \in H_1 \cap H_2$ if and only if both eigenvalues of X are zero, so there are three possibilities:*

(i) *For some $x_* \in (0, L]$ both eigenvalues of X vanish, so the index is not defined.*
(ii) *For some $x_* \in (0, L]$ exactly one eigenvalue of X vanishes, so the index is nonzero.*
(iii) *The eigenvalues of X do not vanish for any $x_* \in (0, L]$, so the index is zero.*

Therefore, the most interesting case is when there are points where one, but not both, eigenvalues of X vanish.

Remark 5.5. *The curve $W(x, \lambda)$ in Proposition 5.3 satisfies Dirichlet boundary conditions at $x = 0$, i.e. $W(0, \lambda) = P_D$. For the path starting at the Neumann subspace,*

$$W(x, \lambda) = \left\{ \begin{pmatrix} v(x) \\ w(x) \end{pmatrix} : \begin{pmatrix} v \\ w \end{pmatrix} \ \textit{satisfies (49) and } w(0) = 0 \right\} \subseteq \mathbb{R}^4,$$

a similar computation shows that $W(x_*, \lambda) \in H_1 \cap H_2$ if and only if β_1 and β_2 are real and negative and satisfy

$$\cos\sqrt{-\beta_1}x_* = \cos\sqrt{-\beta_2}x_* = 0.$$

Proposition 5.3 immediately implies the following.

Corollary 5.6. *There exists $x_* \in (0, L]$ such that $W(x_*, \lambda) \in H_1 \cap H_2$ if and only if $\beta_1, \beta_2 < 0$, and*

$$\frac{\beta_1}{\beta_2} = \left(\frac{m}{n}\right)^2 \quad (55)$$

for integers m and n satisfying

$$1 \leq m \leq \frac{\sqrt{-\beta_1}L}{\pi}, \quad 1 \leq n \leq \frac{\sqrt{-\beta_2}L}{\pi}. \quad (56)$$

This result can be visualized as in Figure 3. The condition (55) is satisfied if and only if the line through $(0, 0)$ and $(\sqrt{-\beta_1}, \sqrt{-\beta_2})$ intersects one of the indicated lattice points. If $\min\{-\beta_1, -\beta_2\} < (\pi/L)^2$, then no such lattice points exist, and so $W(x, \lambda) \in \mathcal{M} = G \setminus (H_1 \cap H_2)$ for all $x \in (0, L]$. It is easy to see that the set of β_1 and β_2 for which (55) is *not* satisfied is open and dense.

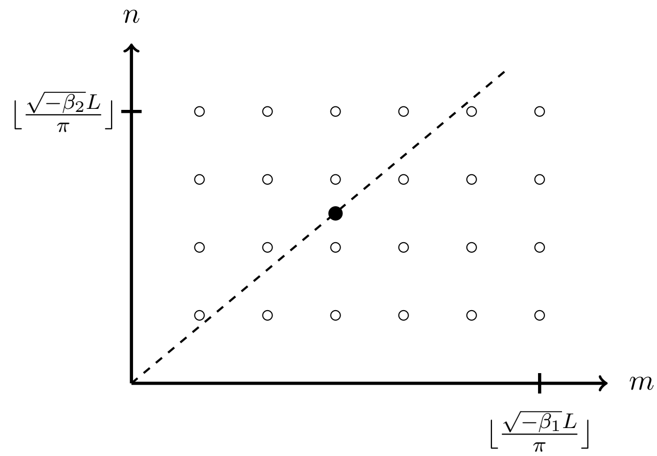

FIGURE 3. Illustrating the result of Corollary 5.6: (55) is satisfied if and only if the line through $(0,0)$ and $(\sqrt{-\beta_1}, \sqrt{-\beta_2})$ intersects a lattice point (m,n) with m and n as in (56).

We now let $B(\lambda) = \lambda I - V$, with eigenvalues $\beta_j(\lambda) = \lambda - \nu_j$. If $\beta_1(0) = -\nu_1$ and $\beta_2(0) = -\nu_2$ do not satisfy (55), then $\beta_1(\lambda)$ and $\beta_2(\lambda)$ will not satisfy (55) for λ close to zero. In particular, if λ_∞ is sufficiently small, we can conclude that $W(x,\lambda) \in \mathcal{M}$ for all $(x,\lambda) \in (0,L] \times [0,\lambda_\infty]$. Rather than precisely quantify the notion of smallness in order to obtain the most general result, we will make the simple observation that under the hypotheses of Theorem 5.2 we have $\min\{-\beta_1(\lambda), -\beta_2(\lambda)\} = \min\{\nu_1,\nu_2\} - \lambda < (\pi/L)^2$ for all $\lambda \geq 0$ (or else β_1 and β_2 are complex).

It follows that $W(x,\lambda) \in \mathcal{M}$ for all $(x,\lambda) \in (0,L] \times [0,\infty)$. We now apply Theorem 4.1 to obtain

$$\begin{aligned}\#\big\{\text{nonnegative eigenvalues of } \mathcal{L}\big\} &\geq -\operatorname{Mas}\Big(W(L,\cdot)\big|_{\lambda\in[0,\lambda_\infty]}; P_1\Big) \\ &= \#\big\{\text{conjugate points in } (0,L]\big\}.\end{aligned}$$

The proof of Theorem 5.2 is completed by the following lemma, which shows that the inequality above is in fact an equality.

Lemma 5.7. *Assuming the hypotheses of Theorem 5.2, we have*

$$\#\big\{\textit{nonnegative eigenvalues of } \mathcal{L}\big\} = -\operatorname{Mas}\Big(W(L,\cdot)\big|_{\lambda\in[0,\lambda_\infty]}; P_1\Big)$$

for sufficiently large λ_∞.

Proof. It is enough to show that the curve $\lambda \mapsto W(L,\lambda)$ is negative, i.e. its image in $\mathbb{R}P^1$ under the map ϕ defined in (11) always passes though the point $[0:1]$ in the negative (clockwise) direction). Using (15), this will imply

$$\begin{aligned}\operatorname{Mas}\Big(W(L,\lambda)\big|_{\lambda\in[0,\lambda_\infty]}; P_1\Big) &= -\#\big\{\lambda \in [0,\lambda_\infty) : W(L,\lambda) \cap P_1 \neq \{0\}\big\} \\ &= -\#\big\{\text{eigenvalues of } \mathcal{L} \text{ in } [0,\lambda_\infty)\big\} \\ &= -\#\big\{\text{nonnegative eigenvalues of } \mathcal{L}\big\}\end{aligned}$$

and hence complete the proof.

We prove monotonicity using (26). For convenience we abbreviate $\psi_i(W(L,\lambda)) = \psi_i(\lambda)$. From the computations in Proposition 5.3 we have

$$\frac{\psi_1(\lambda)}{\psi_2(\lambda)} = \frac{\det X}{\det(X+Y) - \det X - \det Y}$$

and so

$$\frac{\psi_1'(\lambda_*)}{\psi_2(\lambda_*)} = \frac{\frac{d}{d\lambda}\det X}{\det(X+Y) - \det Y}$$

at any point λ_* where $\det X = 0$.

To differentiate $\det X$, as given by (53), we first observe that

$$\frac{d}{d\lambda}\frac{\sinh(\sqrt{\lambda-\nu}x)}{\sqrt{\lambda-\nu}} = \frac{1}{2(\lambda-\nu)}\Big(x\cosh(\sqrt{\lambda-\nu}x) - \sinh(\sqrt{\lambda-\nu}x)\Big).$$

If $\sinh(\sqrt{\lambda_*-\nu_1}L) = 0$, then

$$\frac{d}{d\lambda}\det X\Big|_{\lambda=\lambda_*} = \frac{L}{2(\lambda_*-\nu_1)}\cosh(\sqrt{\lambda_*-\nu_1}L)\frac{\sinh(\sqrt{\lambda_*-\nu_2}L)}{\sqrt{\lambda_*-\nu_2}}.$$

Similarly, using (54) we obtain

$$\Big(\det(X+Y) - \det Y\Big)\Big|_{\lambda=\lambda_*} = \frac{\sinh(\sqrt{\lambda_*-\nu_2}L)}{\sqrt{\lambda_*-\nu_2}}\cosh(\sqrt{\lambda_*-\nu_1}L)$$

and hence

$$\frac{\psi_1'(\lambda_*)}{\psi_2(\lambda_*)} = \frac{L}{2(\lambda_*-\nu_1)} < 0$$

where $\lambda_* - \nu_1 < 0$ because $\sinh(\sqrt{\lambda-\nu_1}L) = 0$. The case $\sinh(\sqrt{\lambda_*-\nu_2}L) = 0$ is identical. □

5.3. **The Turing instability.** In this section, we seek insight into what it means when the conditions of Theorem 4.1 *do not* hold, so that the generalized Maslov index cannot be used directly to prove (in)stability of a steady state. The setting is a two-component reaction-diffusion system (1) with a so-called *Turing instability*. This phenomenon — first discovered by A.M. Turing [22] — refers to a stable, homogeneous equilibrium of a chemical reaction that is counter-intuitively destabilized in the presence of diffusion.

Explicitly, assume that there exists $\bar{u} \in \mathbb{R}^2$ such that $F(\bar{u}) = 0$, and the eigenvalues ν_1, ν_2 of $\nabla F(\bar{u})$ have negative real part. In other words, $\bar{u}$ is a stable equilibrium of the dynamical system

$$u_t = F(u). \tag{57}$$

Setting

$$\nabla F(\bar{u}) := A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{58}$$

we thus have

$$\det A > 0, \qquad \operatorname{tr} A < 0. \tag{59}$$

We further assume that $\bar{u}$ undergoes a Turing bifurcation, which is to say that $D = \operatorname{diag}(d_1, d_2)$ is chosen so that (2) has a positive eigenvalue (and hence $\bar{u}$ is unstable). It is well known (see, for instance, [20, §2.3]) that a Turing instability exists in this setting if and only if

$$d_1 a_{22} + d_2 a_{11} > 2\sqrt{d_1 d_2 \det A}. \tag{60}$$

It is worth noting that a necessary condition for a Turing instability is that $a_{12}a_{21} < 0$, so in particular $F(u)$ cannot be a gradient. Moreover, (59) and (60) together imply that $d_1 \neq d_2$, so Theorem 5.2 does not apply.

As mentioned earlier, the critical ingredient needed to apply the machinery of this paper is that W maps the rectangle $[0, L] \times [0, \lambda_\infty]$ into the MA space $G \setminus (H_1 \cap H_2)$, with λ_∞ chosen to bound the spectrum of L from above. The Turing instability condition (60) is actually derived for bounded perturbations on all of $\mathbb{R}$, so ideally this inclusion would hold for any $L > 0$. Indeed, it is clear from Theorem 5.2 that L can be taken as large as desired in the case where $D = I$ and the equilibrium of the reaction term is stable.

The following result shows that not only does the required inclusion fail for $\lambda \in [0, \lambda_\infty]$, but it actually fails for λ in *any* interval $[0, \epsilon]$ with $\epsilon > 0$. Moreover, when one views the diffusion coefficients d_i as parameters, this rather spectacular violation of the conditions of Theorem 4.1 occurs exactly at the onset of the Turing instability. In other words, for a fixed $F(u)$ and equilibrium $\bar{u}$ satisfying the conditions of this section, the generalized Maslov index can be used to study the stability of $\bar{u}$ as a solution of (1) if and only if D is such that there is no Turing instability.

Proposition 5.8. *Suppose A and D satisfy* (59) *and* (60)*. Then for any $\epsilon > 0$, there exists a point $(x_*, \lambda_*) \in (0, \infty) \times [0, \epsilon]$ for which $W(x_*, \lambda_*) \in H_1 \cap H_2$.*

In other words, the image under W of the rectangle $[\delta, \infty) \times [0, \epsilon]$ is not contained in the Maslov–Arnold space $\mathcal{M}$ for any $\epsilon, \delta > 0$.

Proof. To apply the results of Section 5.2, we write the eigenvalue equation in the form (49), i.e.

$$\frac{d}{dx}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & I \\ D^{-1}(\lambda I - A) & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}, \tag{61}$$

and set $B(\lambda) = D^{-1}(\lambda I - A)$. According to Proposition 5.3 (cf. Corollary 5.6), it suffices to find λ_* such that the eigenvalues $\beta_i(\lambda)$ of $B(\lambda)$ satisfy $\beta_1(\lambda_*)/\beta_2(\lambda_*) = (m/n)^2$ for some $m, n \in \mathbb{N}$. We thus define

$$\rho(\lambda) = \frac{\beta_1(\lambda)}{\beta_2(\lambda)}. \tag{62}$$

We claim that for any $\epsilon > 0$ the set $\{\rho(\lambda) : 0 \le \lambda \le \epsilon\}$ has nonempty interior. The immediately gives the result, since every non-degenerate interval contains a number of the form $(m/n)^2$.

Suppose the claim is false. Since ρ is continuous, this is only possible if ρ is constant on $[0, \epsilon]$. We will show that this is not consistent with the hypotheses on A and D, making use of the fact that β_1 and β_2 depend continuously on λ, and are analytic whenever they are distinct.

We first compute

$$\det B(\lambda) = \frac{1}{d_1 d_2}\big(\lambda^2 - \lambda \operatorname{tr} A + \det A\big) \tag{63}$$

$$\operatorname{tr} B(\lambda) = \frac{1}{d_1 d_2}\big(\lambda(d_1 + d_2) - (d_1 a_{22} + d_2 a_{11})\big). \tag{64}$$

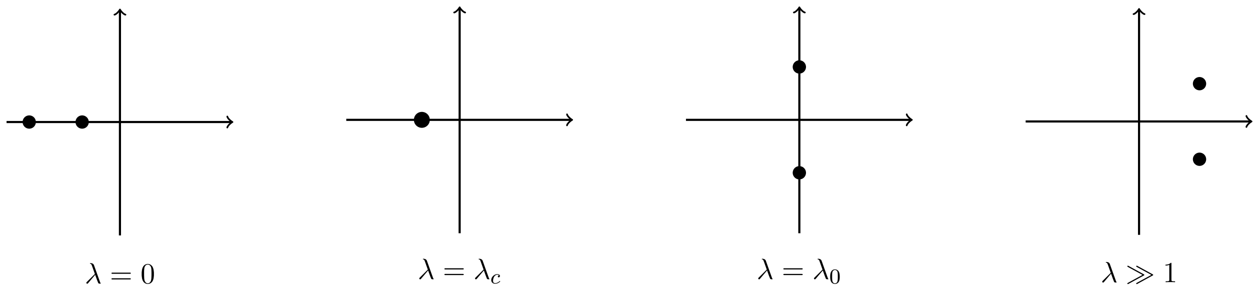

FIGURE 4. The eigenvalues of $B(\lambda)$ are negative and distinct when $\lambda = 0$. At $\lambda = \lambda_c$ they collide and leave the real axis as a complex conjugate pair, crossing the imaginary axis into the right-half plane at $\lambda = \lambda_0$.

It follows from (59) that $\det B(\lambda) > 0$ for all $\lambda \geq 0$. Therefore, the eigenvalues of $B(\lambda)$ are either complex conjugates or real numbers of the same sign. In particular, they are never zero.

From (60) and (64) we see that $\operatorname{tr} B(0) < 0$. Moreover, using (60), we find that the discriminant

$$\begin{aligned} \Delta B(0) &= \big(\operatorname{tr} B(0)\big)^2 - 4\det B(0) \\ &= \frac{1}{(d_1 d_2)^2}\big((d_1 a_{22} + d_2 a_{11})^2 - 4 d_1 d_2 (a_{11}a_{22} - a_{12}a_{21})\big) \end{aligned} \tag{65}$$

is negative, so the eigenvalues of $B(0)$ are negative, real, and distinct. In particular, $\rho(0) \neq 1$. On the other hand, (64) implies $\operatorname{tr} B(\lambda) > 0$ for λ sufficiently large, in which case $\beta_1(\lambda)$ and $\beta_2(\lambda)$ both have positive real part. Therefore, as λ increases, both eigenvalues cross the imaginary axis. Since $\det B(\lambda) \neq 0$, they must cross through $i\mathbb{R} \setminus \{0\}$ as a conjugate pair at some $\lambda = \lambda_0$, which means there exists $\lambda_c \in (0, \lambda_0)$ at which $\beta_1(\lambda_c) = \beta_2(\lambda_c) < 0$, hence $\rho(\lambda_c) = 1$; see Figure 4. This implies $\rho(\lambda)$ is not constant on $[0, \lambda_c]$.

However, the eigenvalues of $B(\lambda)$ are distinct for $\lambda < \lambda_c$, and hence depend analytically on λ, so $\rho(\lambda)$ is analytic on $[0, \lambda_c)$. Therefore, if it is constant on $[0, \epsilon]$, it will be constant on $[0, \lambda_c]$. This contradiction finishes the proof. □

5.4. **Comparing (non)invariance results.** We now compare the results given in Sections 5.1 and 5.3, namely Theorem 5.1 and Proposition 5.8. To compare these directly, there are two issues that must be addressed.

The first is that the two sections assume different boundary conditions. The large diffusion result in Theorem 5.1 requires Neumann boundary conditions at $x = 0$, whereas the analysis of the Turing problem in the previous section relies on Proposition 5.3, which assumes Dirichlet boundary conditions at $x = 0$. This does not pose a serious difficulty — if we impose Neumann boundary conditions at $x = 0$ in the Turing problem, the same conclusion is easily seen to hold, i.e. there exists $x_* > 0$ and λ_* arbitrarily close to zero for which $W(x_*, \lambda_*) \in H_1 \cap H_2$; cf. Remark 5.5.

The second issue is that the two results in question involve writing the eigenvalue problem as a first-order system in two different ways, see (17) vs (49). As a result, the solution spaces $W(x, \lambda)$ are different, as are the resulting Maslov–Arnold spaces (see (30) and (50)), so they cannot be compared directly. To clarify this, we define

$$W(x,\lambda) = \left\{ \begin{pmatrix} v(x) \\ w(x) \end{pmatrix} : \frac{d}{dx}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & D^{-1} \\ \lambda I - A & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} \text{ and } w(0) = 0 \right\}$$

and

$$\widehat{W}(x,\lambda) = \left\{ \begin{pmatrix} v(x) \\ w(x) \end{pmatrix} : \frac{d}{dx}\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} 0 & I \\ D^{-1}(\lambda I - A) & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix} \text{ and } w(0) = 0 \right\}.$$

We also define two-forms ω_1 and ω_2 corresponding to the matrices

$$\Omega_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \Omega_2 = \begin{pmatrix} 0 & 0 & 0 & 1/d_2 \\ 0 & 0 & -1/d_1 & 0 \\ 0 & 1/d_1 & 0 & 0 \\ -1/d_2 & 0 & 0 & 0 \end{pmatrix},$$

and $\widehat{\omega}_2$ corresponding to

$$\widehat{\Omega}_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Finally, we define hyperplanes H_1, H_2 and $\widehat{H}_2$, and the corresponding Maslov–Arnold spaces $\mathcal{M} = G \setminus (H_1 \cap H_2)$ and $\widehat{\mathcal{M}} = G \setminus (H_1 \cap \widehat{H}_2)$. In terms of the functions ψ_1, ψ_2 and $\widehat{\psi}_2$ (defined in Section 4.1) we have

$$W \in \mathcal{M} \iff \psi_1(W) = 0 \text{ and } \psi_2(W) = 0$$

and

$$\widehat{W} \in \widehat{\mathcal{M}} \iff \psi_1(\widehat{W}) = 0 \text{ and } \widehat{\psi}_2(\widehat{W}) = 0.$$

To calculate the ψ_j (up to a nonzero factor) it suffices to compute frames for W and $\widehat{W}$. Let

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$$

be a frame for $W(x,\lambda)$, with columns denoted u_1 and u_2. Computing as in the proof of Proposition 5.3, we find

$$\omega_1(u_1,u_2) = \det X$$

and

$$\begin{aligned} \omega_2(u_1,u_2) &= \frac{x_{11}y_{22}}{d_2} - \frac{x_{21}y_{12}}{d_1} + \frac{y_{11}x_{22}}{d_1} - \frac{y_{21}x_{12}}{d_2} \\ &= \det(X + D^{-1}Y) - \det X - \det(D^{-1}Y). \end{aligned}$$

On the other hand, it is easy to see that

$$\begin{pmatrix} DX \\ Y \end{pmatrix}$$

is a frame for $\widehat{W}(x,\lambda)$. Denoting the columns of by $\widehat{u}_1$ and $\widehat{u}_2$, we find

$$\omega_1(\widehat{u}_1,\widehat{u}_2) = \det(DX) = (\det D)\,\omega_1(u_1,u_2)$$

and

$$\widehat{\omega}_2(\widehat{u}_1,\widehat{u}_2) = \det(DX+Y) - \det(DX) - \det(Y) = (\det D)\,\omega_2(u_1,u_2),$$

and hence conclude that

$$W(x,\lambda) \in \mathcal{M} \quad \Longleftrightarrow \quad \widehat{W}(x,\lambda) \in \widehat{\mathcal{M}}. \tag{66}$$

Proposition 5.8 guarantees that there exists λ_* arbitrarily close to 0 such that $\widehat{W}(x,\lambda_*)$ leaves the Maslov–Arnold space $\widehat{M}$ for some $x_* > 0$, while Theorem 5.1 guarantees $W(x,\lambda)$ remains in $\mathcal{M}$ for all $(x,\lambda) \in [0,L] \times [0,\lambda_\infty]$, provided $d_1 d_2$ is large. Comparing these results, and making use of (66), we conclude that for any $\lambda_* \in [0,\lambda_\infty]$, the point x_* in Proposition 5.8 must be greater than L. In other words, while the path $W(x,\lambda)$ must leave $\mathcal{M}$ for some $x_* > 0$ and $\lambda_* \in [0,\lambda_\infty]$, we can ensure $x_* > L$ if $d_1 d_2$ is sufficiently large.

5.5. **Numerical prospects.** The classical Maslov index has seen many successful numerical treatments; see for instance [4, 6, 7]. In closing, we mention that the theory developed in this paper is also expected to be very amenable to numerical applications.

To explain this, we go back to Theorem 4.1, where it was shown that

$$\#\big\{\text{nonnegative eigenvalues of } \mathcal{L}\big\} \geq \operatorname{Mas}\Big(W(x,0)\big|_{x\in[\delta,L]};P_1\Big) \tag{67}$$

as long as $W(x,\lambda) \in \mathcal{M} = G \setminus (H_1 \cap H_2)$ for all $(x,\lambda) \in (0,L] \times [0,\lambda_\infty]$, where H_1 is the hyperplane corresponding to the train of the Dirichlet subspace and H_2 is arbitrary.

The particular choice of H_2 in the second half of Theorem 4.1 guaranteed monotonicity in x, but this is not important if the index is to be computed numerically — for any choice of H_2 the Maslov index computation simply becomes a winding number calculation in $\mathbb{R}P^1$. This is numerically robust, due to the homotopy invariance of the index. For instance, the curves

$$\eta(t) = \begin{cases} e^{it}, & -\pi/2 \leq t \leq 0 \\ e^{-it}, & 0 \leq t \leq \pi/2 \end{cases}, \qquad \eta_+(t) = e^{i\epsilon}\eta(t), \qquad \eta_-(t) = e^{-i\epsilon}\eta(t)$$

are ϵ-close, pass through the point $1 \in S^1$ one, two and zero times, respectively, and all have zero winding number. That is, the signed count of conjugate points (i.e. the generalized Maslov index) is stable under small perturbations, while the unsigned count is not.

Therefore, a small approximation error in the calculation of the path $W(x,\lambda)$ (i.e. in the numerical solution of an initial-value problem) will not change the numerically computed winding number. The only possible complication is the presence of a conjugate point near the endpoint $x = L$. If there is a conjugate point near (but not exactly at) the endpoint, it will be possible to determine so with sufficiently accurate numerics. Indeed, this can be established rigorously using validated numerics; see [23] for an overview of rigorous numerical methods applied to dynamical systems.

The case of a conjugate point at $x = L$ is more subtle, since it cannot be distinguished from a conjugate point that is very close (within some numerical tolerance) to $x = L$. Generically the endpoint is not a conjugate point, and when it is, this is usually a consequence of an underlying symmetry of the system. If we know a priori that $x = L$ *is* a conjugate point, then we can (rigorously) find a neighbourhood around it containing no other conjugate points, and hence the discussion in the previous paragraph applies.

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