Second and fourth Painlevé hierarchies and Jimbo-Miwa linear problems

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Abstract

The relations between the different linear problems for Painlevé equations is an intriguing open problem. Here we consider our second and fourth Painlevé hierarchies given in *Publ. Res. Inst. Math. Sci. (Kyoto)* **37** 327-347 (2001), and show that they could alternatively have been derived using the linear problems of Jimbo and Miwa. That is, we give a gauge transformation of our linear problems for these two hierarchies which maps those of the second and fourth Painlevé equations themselves onto those of Jimbo and Miwa.

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1 Introduction

The discovery of the Inverse Monodromy Transform [1]—[4] saw the inclusion of the six Painlevé equations amongst the panoply of integrable equations, i.e., of equations solvable using underlying linear problems. Recently, the question of obtaining linear problems for hierarchies of higher order analogues of the Painlevé equations has absorbed the attention of many authors. In particular, in a series of recent papers [5]–[8], non-isospectral scattering problems have been used as a means of deriving new completely integrable hierarchies of partial differential equations (PDEs) in $2+1$ and $1+1$ dimensions, and, by reduction, new hierarchies of ordinary differential equations (ODEs), all together with corresponding underlying linear problems.

In [8] we presented a generalized non-isospectral dispersive water wave hierarchy in $2+1$ dimensions; amongst the reductions to ODEs of this hierarchy we obtained a generalized $P_{IV}-P_{II}$ hierarchy which includes as special cases both a hierarchy of ODEs having the fourth Painlevé equation (P_{IV}) as first member, and a hierarchy having the second Painlevé equation (P_{II}) as first member. We note here that the P_{II} hierarchy of [8] is not equivalent to the standard P_{II} hierarchy given in [9, 1]; thus both the P_{II} and P_{IV} hierarchies of [8] were previously unknown.

The aim of the present paper is to explore the relationship between the linear problems for the P_{II} and P_{IV} hierarchies presented in [8], and other linear problems for these hierarchies. We give the important result that there exist gauge transformations which map the linear problems for these P_{II} and P_{IV} hierarchies onto two new sequences of linear problems, whose first members are the linear problems of P_{II} and P_{IV} given by Jimbo and Miwa [3].

2 A second Painlevé hierarchy

One of the hierarchies of ODEs obtained in [8], as a reduction of a $(2+1)$-dimensional non-isospectral hierarchy, can be expressed as

$$\mathcal{R}^n\mathbf{u}_x+\sum_{i=0}^{n-2}c_i\mathcal{R}^i\mathbf{u}_x+g_{n+1}\left(\begin{array}{c}1\\0\end{array}\right)=\left(\begin{array}{c}0\\0\end{array}\right),\qquad n\geq 1, \tag{1}$$

where $g_{n+1}(\neq 0)$ and each of the c_i are constants. Here $\mathbf{u}=(u(x),v(x))^T$ and $\mathcal{R}$ is the recursion operator of the dispersive water wave hierarchy as given in [10],

$$\mathcal{R}=\frac{1}{2}\left(\begin{array}{cc}\partial_x u\partial_x^{-1}-\partial_x & 2\\ 2v+v_x\partial_x^{-1} & u+\partial_x\end{array}\right). \tag{2}$$

In [8], we also gave the following matrix linear problem for the hierarchy (1):

$$\begin{aligned}\mathbf{\Psi}_x &= F\mathbf{\Psi}, &(3)\\ \left(\frac{1}{2}g_{n+1}\right)\mathbf{\Psi}_\lambda &= H_n\mathbf{\Psi}=\left[\left(\lambda^n+\sum_{i=0}^{n-2}c_i\lambda^i\right)F+G_n+\sum_{i=1}^{n-2}c_iG_i\right]\mathbf{\Psi}, &(4)\end{aligned}$$

where $\mathbf{\Psi}=(\psi_1,\psi_2)^T$ and the matrices F and G_i are given by

$$F=\left(\begin{array}{cc}-\frac{1}{2}(2\lambda-u) & 1\\ -v & \frac{1}{2}(2\lambda-u)\end{array}\right), \tag{5}$$

$$G_n = \left(\begin{array}{cc} -\frac{1}{4}\left((2\lambda - u)\, P_n + P_{n,x}\right) & \frac{1}{2} P_n \\ \frac{1}{2} g_{n+1} + \frac{1}{2}\lambda^n u_x - \frac{1}{2} M_n & \frac{1}{4}\left((2\lambda - u)\, P_n + P_{n,x}\right) \\ -\frac{1}{4}\left((2\lambda - u)\, P_n + P_{n,x}\right)_x - \frac{1}{2} v P_n & \end{array} \right), \quad (6)$$

and, for $i < n$,

$$G_i = \left(\begin{array}{cc} -\frac{1}{4}\left((2\lambda - u)\, P_i + P_{i,x}\right) & \frac{1}{2} P_i \\ \frac{1}{2}\lambda^i u_x - \frac{1}{2} M_i & \frac{1}{4}\left((2\lambda - u)\, P_i + P_{i,x}\right) \\ -\frac{1}{4}\left((2\lambda - u)\, P_i + P_{i,x}\right)_x - \frac{1}{2} v P_i & \end{array} \right). \quad (7)$$

In the above, M_i and P_i are given respectively by

$$\begin{aligned} \left(\begin{array}{c} M_n \\ N_n \end{array} \right) &= \mathcal{R}^n \mathbf{u}_x + g_{n+1} \left(\begin{array}{c} 1 \\ 0 \end{array} \right), & (8) \\ \left(\begin{array}{c} M_i \\ N_i \end{array} \right) &= \mathcal{R}^i \mathbf{u}_x, \qquad \text{for } i < n, & (9) \end{aligned}$$

and

$$P_i = \partial_x^{-1} \sum_{j=0}^{i-1} \lambda^{i-1-j} M_j. \quad (10)$$

The compatibility condition of the matrix linear problem (3), (4) is equation (1).

Since each member of the dispersive water wave hierarchy is in conservation form, each component of our hierarchy (1) integrates immediately to give

$$\left(\begin{array}{c} \tilde{M}_n \\ \tilde{N}_n \end{array} \right) \equiv \left(\begin{array}{cc} \partial_x^{-1} & 0 \\ 0 & \partial_x^{-1} \end{array} \right) \left[\mathcal{R}^n \mathbf{u}_x + \sum_{i=0}^{n-2} c_i \mathcal{R}^i \mathbf{u}_x \right] + \left(\begin{array}{c} g_{n+1} x \\ -\delta_n \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right), \quad (11)$$

where δ_n is one of the constants of integration; our assumption that $g_{n+1} \neq 0$ allows us to set the second constant of integration to zero without loss of generality. It is this hierarchy that is our P_{II} hierarchy; as we shall see, in the case $n = 1$, this system of equations yields P_{II} itself. We note, however, that higher order members of this hierarchy are different from those of the P_{II} hierarchy presented in [9, 1].

It is a simple matter to write down a linear problem whose compatibility condition gives the hierarchy (11) directly; this is given by the pair of equations

$$\begin{aligned} \mathbf{\Psi}_x &= F \mathbf{\Psi}, & (12) \\ \mathbf{\Psi}_\lambda &= K_n \mathbf{\Psi}, & (13) \end{aligned}$$

where

$$K_n = \frac{1}{g_{n+1}} \left[2H_n + \left(\begin{array}{cc} -\tilde{M}_n & 0 \\ 2\tilde{N}_n & \tilde{M}_n \end{array} \right) \right] = \left(\begin{array}{cc} (K_n)_{11} & (K_n)_{12} \\ (K_n)_{21} & (K_n)_{22} \end{array} \right). \quad (14)$$

Note that the second matrix term in the square brackets of (14) is identically zero when the equations of the hierarchy are satisfied. The addition of this matrix to $2H_n$ is equivalent to substituting higher order derivatives in H_n in order to obtain as compatibility condition the integrated hierarchy (11) rather than the original hierarchy (1).

2.1 The relationship with the Jimbo-Miwa linear problem for the second Painlevé equation

Let us consider the linear problem for the hierarchy (11) given by the system of equations (12), (13), where F and K_n are as above. We now consider the gauge transformation

$$\mathbf{\Psi} = M\mathbf{\Phi}, \tag{15}$$

where the matrix M is given by

$$M = \begin{pmatrix} e^{\frac{1}{2}s(x)} & 0 \\ 0 & e^{-\frac{1}{2}s(x)} \end{pmatrix}, \tag{16}$$

and where $s_x = u$. This maps the linear system (12), (13) onto

$$\begin{aligned} \mathbf{\Phi}_x &= A\mathbf{\Phi}, & (17) \\ \mathbf{\Phi}_\lambda &= B_n\mathbf{\Phi}, & (18) \end{aligned}$$

where (noting that $M_\lambda = 0$)

$$A = M^{-1}FM - M^{-1}M_x = \begin{pmatrix} -\lambda & \frac{w}{2} \\ -2\frac{v}{w} & \lambda \end{pmatrix}, \tag{19}$$

$$B_n = M^{-1}K_nM = \begin{pmatrix} (K_n)_{11} & \frac{w}{2}(K_n)_{12} \\ \frac{2}{w}(K_n)_{21} & (K_n)_{22} \end{pmatrix}, \tag{20}$$

and where we have introduced the auxiliary function $w = w(x)$ defined by $w = 2e^{-s}$, and which therefore satisfies the relation $\frac{w_x}{w} = -u$.

Thus we obtain a different sequence of linear problems (17), (18) for the P_{II} hierarchy (11): the first of these is, up to a trivial change of variables (see next section), the linear problem for P_{II} given by Jimbo and Miwa [3]. We thus obtain the result that the P_{II} hierarchy obtained in [8] could also have been obtained by expanding in powers of λ (or $\mu = -2\lambda$; again, see next section) in the Jimbo-Miwa linear problem for P_{II}, i.e., by using an Ablowitz-Kaup-Newell-Segur (AKNS) type approach [11] therein. We now illustrate this remark with some examples.

2.2 Examples

2.2.1 Example n = 1

For the case $n = 1$ of the hierarchy (11) we have the system of equations

$$\begin{aligned} v + \frac{1}{2}(u^2 - u_x) + g_2 x &= 0, & (21) \\ uv + \frac{1}{2}v_x - \delta_1 &= 0. & (22) \end{aligned}$$

This system is equivalent to the second order ODE

$$u_{xx} = 2u^3 + 4g_2xu + 2(g_2 + 2\delta_1), \tag{23}$$

which for $g_2 \neq 0$ is just the second Painlevé equation P_{II}. We have the corresponding linear problem given by equations (12), (13), where

$$F = \begin{pmatrix} -\lambda + \frac{1}{2}u & 1 \\ -v & \lambda - \frac{1}{2}u \end{pmatrix}, \tag{24}$$

$$K_1 = \frac{1}{g_2}\sum_{j=0}^{2} K_{1,j}\lambda^j, \tag{25}$$

and where the matrices $K_{1,j}$ are given by

$$K_{1,2} = 2\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_{1,1} = 2\begin{pmatrix} 0 & 1 \\ -v & 0 \end{pmatrix}, \quad K_{1,0} = \begin{pmatrix} -v - g_2 x & u \\ uv - 2\delta_1 & v + g_2 x \end{pmatrix}. \tag{26}$$

After the gauge transformation (15) we obtain the linear problem (17), (18) with

$$A = \begin{pmatrix} -\lambda & \frac{w}{2} \\ -2\frac{v}{w} & \lambda \end{pmatrix}, \tag{27}$$

$$B_1 = \frac{1}{g_2}\sum_{j=0}^{2} B_{1,j}\lambda^j, \tag{28}$$

where each $B_{1,j} = M^{-1}K_{1,j}M$, and where w satisfies $\frac{w_x}{w} = -u$. This transformation leads to the introduction of factors $(w/2)$ and $(2/w)$ in the off-diagonal elements of the matrix B_1, as in (20).

If in the linear system (17), (18), with A and B_1 given as above, we set $\lambda = -(\mu/2)$, $v = z/2$, $g_2 = 1/4$, $u = y$ and $4\delta_1 = \alpha - \frac{1}{2}$, then we obtain the linear system for P_{II} as given by Jimbo and Miwa [3], i.e.

$$\mathbf{\Phi}_x = \tilde{A}\mathbf{\Phi} \tag{29}$$

$$\mathbf{\Phi}_\mu = \tilde{B}_1\mathbf{\Phi} \tag{30}$$

with

$$\tilde{A} = \frac{1}{2}\begin{pmatrix} \mu & w \\ -2\frac{z}{w} & -\mu \end{pmatrix}, \tag{31}$$

$$\tilde{B}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\mu^2 + \begin{pmatrix} 0 & w \\ -2\frac{z}{w} & 0 \end{pmatrix}\mu + \begin{pmatrix} z + \frac{1}{2}x & -wy \\ \frac{2\alpha - 1 - 2zy}{w} & -z - \frac{1}{2}x \end{pmatrix}. \tag{32}$$

The compatibility condition of this last gives $w_x = -yw$ and, after elimination of w, the system of equations

$$z_x = -2yz + \alpha - \frac{1}{2}, \tag{33}$$

$$y_x = y^2 + z + \frac{x}{2}. \tag{34}$$

These two equations imply that y satisfies P_{II}:

$$y_{xx} = 2y^3 + xy + \alpha. \tag{35}$$

2.2.2 Example $\mathbf{n = 2}$

In the case $n = 2$ of the hierarchy (11) we have the system of equations

$$\begin{aligned}
\frac{1}{4}\left(u_{xx} - 3uu_x + u^3 + 6uv\right) + c_0 u + g_3 x &= 0, &(36)\\
\frac{1}{4}\left(v_{xx} + 3v^2 + 3uv_x + 3u^2 v\right) + c_0 v - \delta_2 &= 0. &(37)
\end{aligned}$$

This system arises as the compatibility condition of the linear problem (12), (13) where F is given by (24) and K_2 by

$$K_2 = \frac{1}{g_3}\sum_{j=0}^{3} K_{2,j}\lambda^j, \tag{38}$$

where the matrices $K_{2,j}$ are as follow:

$$\begin{aligned}
K_{2,3} &= 2\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, &(39)\\
K_{2,2} &= 2\begin{pmatrix} 0 & 1 \\ -v & 0 \end{pmatrix}, &(40)\\
K_{2,1} &= \begin{pmatrix} -v - 2c_0 & u \\ -v_x - uv & v + 2c_0 \end{pmatrix}, &(41)\\
K_{2,0} &= \begin{pmatrix} -\frac{1}{2}v_x - uv - g_3 x & \frac{1}{2}\left(u^2 - u_x\right) + v + 2c_0 \\ \frac{1}{2}\left(v^2 + uv_x - u_x v\right) + u^2 v - 2\delta_2 & \frac{1}{2}v_x + uv + g_3 x \end{pmatrix}. &(42)
\end{aligned}$$

After the gauge transformation (15) we obtain the linear problem (17), (18) with

$$\begin{aligned}
A &= \begin{pmatrix} -\lambda & \frac{w}{2} \\ -2\frac{v}{w} & \lambda \end{pmatrix}, &(43)\\
B_2 &= \frac{1}{g_3}\sum_{j=0}^{3} B_{2,j}\lambda^j, &(44)
\end{aligned}$$

where as before each $B_{2,j} = M^{-1}K_{2,j}M$. We note that this particular member of our P_{II} hierarchy was also obtained by Kitaev [12] (see Appendix).

2.2.3 Example $\mathbf{n = 3}$

In the case $n = 3$ the system of equations (11) is:

$$\begin{aligned}
&\frac{1}{8}\left(-u_{xxx} + 6v^2 + 2v_{xx} + 4uu_{xx} - 6vu_x - 6u^2u_x + 3u_x^2 + 12u^2v + u^4\right)\\
&\qquad + c_1\left(v + \frac{1}{2}u^2 - \frac{1}{2}u_x\right) + c_0 u + g_4 x = 0, \qquad (45)\\
&\frac{1}{8}\left(v_{xxx} + 6vv_x + 2vu_{xx} + 2u_x v_x + 12v^2u + 4u^3v + 6u^2v_x + 4uv_{xx}\right)\\
&\qquad + c_1\left(uv + \frac{1}{2}v_x\right) + c_0 v - \delta_3 = 0. \qquad (46)
\end{aligned}$$

This system arises as the compatibility condition of the linear problem (12), (13), where F is given by (24) and K_3 by

$$K_3 = \frac{1}{g_4}\sum_{j=0}^{4} K_{3,j}\lambda^j, \tag{47}$$

and where the matrices $K_{3,j}$ are:

$$K_{3,4} = 2\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{48}$$

$$K_{3,3} = 2\begin{pmatrix} 0 & 1 \\ -v & 0 \end{pmatrix}, \tag{49}$$

$$K_{3,2} = \begin{pmatrix} -v-2c_1 & u \\ -v_x - uv & v+2c_1 \end{pmatrix}, \tag{50}$$

$$K_{3,1} = \begin{pmatrix} -\frac{1}{2}v_x - uv - 2c_0 & \frac{1}{2}\left(u^2 - u_x\right) + v + 2c_1 \\ -\frac{1}{2}\left(v_{xx} + 2v^2 + u^2v + 2uv_x + u_xv\right) & \frac{1}{2}v_x + uv + 2c_0 \\ -2c_1v & \end{pmatrix}, \tag{51}$$

$$K_{3,0} = \begin{pmatrix} -\frac{1}{4}\left(v_{xx} + 3v^2 + 3u^2v + 3uv_x\right) & \frac{1}{4}\left(u_{xx} - 3uu_x + 6uv + u^3\right) \\ -c1v - g_4x & +c_1u + 2c_0 \\ \frac{1}{4}\left(vu_{xx} + uv_{xx} + 3u^2v_x - 3uvu_x\right. & \frac{1}{4}\left(v_{xx} + 3v^2 + 3u^2v + 3uv_x\right) \\ \left.-u_xv_x + 3u^3v + 6uv^2\right) & +c1v + g_4x \\ +c_1uv - 2\delta_3 & \end{pmatrix}. \tag{52}$$

Again, after the gauge transformation (15) we obtain the linear problem (17), (18) with

$$A = \begin{pmatrix} -\lambda & \frac{w}{2} \\ -2\frac{v}{w} & \lambda \end{pmatrix}, \tag{53}$$

$$B_3 = \frac{1}{g_4}\sum_{j=0}^{4} B_{2,j}\lambda^j, \tag{54}$$

and where as before $B_{3,j} = M^{-1}K_{3,j}M$.

3 A fourth Painlevé hierarchy

We now consider the P_{IV} hierarchy which, together with its linear problem, we introduced in [8]. Without any loss of generality we assume that our P_{IV} hierarchy is of the form

$$\mathcal{R}^n\mathbf{u}_x + \sum_{i=1}^{n-1} c_i\mathcal{R}^i\mathbf{u}_x + g_n\mathcal{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad n \geq 1, \tag{55}$$

where again $g_n(\neq 0)$ and c_i are constants, and where $\mathcal{R}$ is the recursion operator of the dispersive water wave hierarchy (2). Bäcklund transformations for this hierarchy have been given in [13]. We also note that a detailed singularity analysis of the case $n = 2$, which presented certain difficulties, was undertaken in [14]. Here we concentrate on the linear problem for (55), whose matrix form is [8]

$$\begin{aligned} \mathbf{\Psi}_x &= F\mathbf{\Psi}, &(56)\\ \left(\frac{1}{2}\lambda g_n\right)\mathbf{\Psi}_\lambda &= H_n\mathbf{\Psi} = \left[\left(\lambda^n + \sum_{i=1}^{n-1} c_i\lambda^i\right)F + G_n + \sum_{i=1}^{n-1} c_i G_i\right]\mathbf{\Psi}, &(57)\end{aligned}$$

where $\mathbf{\Psi} = (\psi_1, \psi_2)^T$ and the matrices F, G_n, and G_i for $i < n$, are given by:

$$F = \begin{pmatrix} -\frac{1}{2}(2\lambda - u) & 1 \\ -v & \frac{1}{2}(2\lambda - u) \end{pmatrix}, \qquad (58)$$

$$G_n = \begin{pmatrix} -\frac{1}{4}\left((2\lambda - u)P_n + P_{n,x}\right) & \frac{1}{2}P_n \\ \begin{array}{c} \frac{1}{2}\lambda g_n + \frac{1}{2}\lambda^n u_x - \frac{1}{2}M_n \\ -\frac{1}{4}\left((2\lambda - u)P_n + P_{n,x}\right)_x - \frac{1}{2}vP_n \end{array} & \frac{1}{4}\left((2\lambda - u)P_n + P_{n,x}\right) \end{pmatrix}, \qquad (59)$$

where

$$\begin{aligned} \begin{pmatrix} M_n \\ N_n \end{pmatrix} &= \mathcal{R}^n\mathbf{u}_x + g_n\mathcal{R}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, &(60)\\ \begin{pmatrix} M_{n-1} \\ N_{n-1} \end{pmatrix} &= \mathcal{R}^{n-1}\mathbf{u}_x + g_n\begin{pmatrix} 1 \\ 0 \end{pmatrix}, &(61)\\ \begin{pmatrix} M_j \\ N_j \end{pmatrix} &= \mathcal{R}^j\mathbf{u}_x, \qquad \text{for } j < n-1, &(62)\end{aligned}$$

and

$$P_n = \partial_x^{-1}\sum_{j=0}^{n-1}\lambda^{n-1-j}M_j; \qquad (63)$$

and (for $i < n$)

$$G_i = \begin{pmatrix} -\frac{1}{4}\left((2\lambda - u)P_i + P_{i,x}\right) & \frac{1}{2}P_i \\ \begin{array}{c} \frac{1}{2}\lambda^i u_x - \frac{1}{2}\widehat{M}_i \\ -\frac{1}{4}\left((2\lambda - u)P_i + P_{i,x}\right)_x - \frac{1}{2}vP_i \end{array} & \frac{1}{4}\left((2\lambda - u)P_i + P_{i,x}\right) \end{pmatrix}, \qquad (64)$$

where

$$\begin{pmatrix} \widehat{M}_i \\ \widehat{N}_i \end{pmatrix} = \mathcal{R}^i\mathbf{u}_x \qquad (65)$$

and now

$$P_i = \partial_x^{-1}\sum_{j=0}^{i-1}\lambda^{i-1-j}\widehat{M}_j. \qquad (66)$$

The compatibility condition of the matrix linear problem (56), (57) is equation (55).

Again, as in the case of the P_{II} hierarchy discussed in Section 2, this hierarchy (55) can be integrated. The integrated version of this hierarchy was presented in [8] and its derivation was given in [15]. A general statement of this integration process can be found in [16]. We briefly sumarize the results here.

The hierarchy (55) can be a written in the alternative form

$$B_2\mathbf{K}_n[\mathbf{u}] = 0, \tag{67}$$

where

$$\mathbf{K}_n[\mathbf{u}] = \mathbf{L}_n[\mathbf{u}] + \sum_{i=1}^{n-1} c_i\mathbf{L}_i[\mathbf{u}] + g_n \begin{pmatrix} 0 \\ x \end{pmatrix}, \tag{68}$$

B_2 is one of the three Hamiltonian operators of the dispersive water wave hierarchy,

$$B_2 = \frac{1}{2}\begin{pmatrix} 2\partial_x & \partial_x u - \partial_x^2 \\ u\partial_x + \partial_x^2 & v\partial_x + \partial_x v \end{pmatrix}, \tag{69}$$

and each $\mathbf{L}_i[\mathbf{u}]$ is the variational derivative of the Hamiltonian density corresponding to the operator B_2 for the t_i-flow of the dispersive water wave hierarchy, $\mathbf{u}_{t_i} = \mathcal{R}^i\mathbf{u}_x = B_2\mathbf{L}_i[\mathbf{u}]$ (for further details, see [10]).

Here we have used the fact that $\mathcal{R} = B_2B_1^{-1}$, where

$$B_1 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \tag{70}$$

is another of the Hamiltonian operators of the dispersive water wave hierarchy. We note also that we have the recursion relation $B_1\mathbf{L}_{i+1}[\mathbf{u}] = B_2\mathbf{L}_i[\mathbf{u}]$, and that $\mathbf{L}_1[\mathbf{u}] = (v,u)^T$.

The integrated form of (55) is

$$\tilde{M}_n \equiv L_{n,x} - 2K_n - uL_n - (g_n - 2\alpha_n) = 0, \tag{71}$$

$$\tilde{N}_n \equiv K_{n,x} - \frac{(K_n + \frac{1}{2}g_n - \alpha_n)^2 - \frac{1}{4}\beta_n^2}{L_n} + vL_n = 0, \tag{72}$$

where $\mathbf{K}_n = (K_n, L_n)^T$, and where α_n and β_n^2 are the two arbitrary constants of integration.

A matrix linear problem whose compatibility condition is precisely the integrated hierarchy (71), (72) can easily be given. This can be done, in the same way as for the P_{II} hierarchy, by adding to H_n in the linear equation (57) a matrix whose entries (being the integrated equations themselves) are identically zero. This matrix is obtained by taking into account the dependence of H_n on the higher order derivatives $u_{n,x}$ and $v_{n,x}$, as well as that of $\tilde{M}_n$ and $\tilde{N}_n$ on these derivatives: $\tilde{M}_n \sim \left(-\frac{1}{2}\right)^{n-1} u_{n,x}$, $\tilde{N}_n \sim \left(\frac{1}{2}\right)^{n-1} v_{n,x}$. We thus obtain the linear problem

$$\mathbf{\Psi}_x = F\mathbf{\Psi}, \tag{73}$$

$$\mathbf{\Psi}_\lambda = K_n\mathbf{\Psi}, \tag{74}$$

where the matrix K_n is given by

$$K_n = \frac{1}{\lambda g_n}\left[2H_n + \begin{pmatrix} \frac{1}{2}\tilde{M}_n & 0 \\ \tilde{N}_n & -\frac{1}{2}\tilde{M}_n \end{pmatrix}\right] = \begin{pmatrix} (K_n)_{11} & (K_n)_{12} \\ (K_n)_{21} & (K_n)_{22} \end{pmatrix}. \tag{75}$$

The compatibility condition of the linear problem (73), (74) is the integrated hierarchy (71), (72).

3.1 The relationship with the Jimbo-Miwa linear problem for the fourth Painlevé equation

As for the P_{II} hierarchy discussed in Section 2, we now consider a gauge transformation of the linear problem (73), (74): we set

$$\mathbf{\Psi} = M\mathbf{\Phi}, \tag{76}$$

where as before

$$M = \begin{pmatrix} e^{\frac{1}{2}s(x)} & 0 \\ 0 & e^{-\frac{1}{2}s(x)} \end{pmatrix} \tag{77}$$

and $s_x = u$. The linear system (73), (74) is then mapped to

$$\begin{aligned} \mathbf{\Phi}_x &= A\mathbf{\Phi}, &(78)\\ \mathbf{\Phi}_\lambda &= B_n\mathbf{\Phi}, &(79) \end{aligned}$$

where A and B_n are now

$$\begin{aligned} A &= M^{-1}FM - M^{-1}M_x = \begin{pmatrix} -\lambda & w \\ -\frac{v}{w} & \lambda \end{pmatrix}, &(80)\\ B_n &= M^{-1}K_nM = \begin{pmatrix} (K_n)_{11} & w(K_n)_{12} \\ \frac{1}{w}(K_n)_{21} & (K_n)_{22} \end{pmatrix}, &(81) \end{aligned}$$

and where we have introduced the auxiliary function $w = w(x)$ defined — differently from in Section 2 — by $w = e^{-s}$, and which therefore satisfies the relation $\frac{w_x}{w} = -u$.

We now consider some examples. In particular we will see how, using the above gauge transformation, the linear problem for the first nontrivial flow of our hierarchy is mapped onto the linear problem for P_{IV} given by Jimbo and Miwa [3]. This then means that our P_{IV} hierarchy (71), (72) could alternatively have been obtained using an AKNS type approach in the Jimbo-Miwa linear problem for P_{IV}.

3.2 Examples

3.2.1 Example n = 1

The first non-trivial flow of the hierarchy (71), (72) consists of the pair of equations

$$\begin{aligned} u_x &= 2v + u^2 + g_1xu - 2\alpha_1, &(82)\\ v_x &= \frac{\left[v - \alpha_1 + \frac{1}{2}g_1\right]^2 - \frac{1}{4}\beta_1^2}{(u + g_1x)} - v(u + g_1x), &(83) \end{aligned}$$

where α_1 and β_1 are two independent constants of integration. Eliminating v between these equations and performing the change of variables

$$u = y - g_1x, \tag{84}$$

we obtain

$$y_{xx} = \frac{1}{2}\frac{y_x^2}{y} + \frac{3}{2}y^3 - 2g_1xy^2 + 2[(g_1^2x^2/4) - \alpha_1]y - \frac{1}{2}\frac{\beta_1^2}{y}. \tag{85}$$

Setting $g_1 = -2$, which can be done without loss of generality for $g_1 \neq 0$, gives

$$y_{xx} = \frac{1}{2}\frac{y_x^2}{y} + \frac{3}{2}y^3 + 4xy^2 + 2(x^2 - \alpha_1)y - \frac{1}{2}\frac{\beta_1^2}{y}, \tag{86}$$

i.e., P_{IV}. Corresponding to the system (82), (83) we have the linear problem given by equations (73), (74) where

$$F = \begin{pmatrix} -\lambda + \frac{1}{2}u & 1 \\ -v & \lambda - \frac{1}{2}u \end{pmatrix}, \tag{87}$$

$$K_1 = \frac{1}{g_1}\sum_{j=-1}^{1} K_{1,j}\lambda^j, \tag{88}$$

with $K_{1,j}$ given by

$$K_{1,1} = 2\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{89}$$

$$K_{1,0} = \begin{pmatrix} -g_1 x & 2 \\ -2v & g_1 x \end{pmatrix}, \tag{90}$$

$$K_{1,-1} = \begin{pmatrix} -v - \frac{1}{2}g_1 + \alpha_1 & u + g_1 x \\ -\frac{\left[v - \alpha_1 + \frac{1}{2}g_1\right]^2 - \frac{1}{4}\beta_1^2}{(u + g_1 x)} & v + \frac{1}{2}g_1 - \alpha_1 \end{pmatrix}. \tag{91}$$

Our gauge transformation then yields the alternative linear problem (78), (79) with

$$A = \begin{pmatrix} -\lambda & w \\ -\frac{v}{w} & \lambda \end{pmatrix}, \tag{92}$$

$$B_1 = \frac{1}{g_1}\sum_{j=-1}^{1} B_{1,j}\lambda^j, \tag{93}$$

where w satisfies $\frac{w_x}{w} = -u$ and each $B_{1,j} = M^{-1}K_{1,j}M$. This transformation leads to the introduction of factors w and $(1/w)$ in the off-diagonal elements of the matrix B_1, as in (81).

If in the linear system (78), (79), with A and B_1 given as above, we set $\lambda = -\mu$ and choose $g_1 = -2$, make the transformation

$$u = y + 2x, \tag{94}$$

$$v = -2z + 2(\theta_0 + \theta_\infty), \tag{95}$$

and redefine the parameters α_1 and β_1 as

$$\alpha_1 = 2\theta_\infty - 1, \tag{96}$$

$$\beta_1^2 = 16\theta_0^2, \tag{97}$$

we obtain the linear problem

$$\mathbf{\Phi}_x = \tilde{A}\mathbf{\Phi}, \tag{98}$$

$$\mathbf{\Phi}_\mu = \tilde{B}_1\mathbf{\Phi}, \tag{99}$$

where

$$\tilde{A} = \begin{pmatrix} \mu & w \\ \frac{2(z-\theta_0-\theta_\infty)}{w} & -\mu \end{pmatrix}, \tag{100}$$

$$\tilde{B}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mu + \begin{pmatrix} x & w \\ \frac{2(z-\theta_0-\theta_\infty)}{w} & -x \end{pmatrix} + \begin{pmatrix} -z+\theta_0 & -\frac{1}{2}wy \\ \frac{2z(z-2\theta_0)}{wy} & z-\theta_0 \end{pmatrix} \frac{1}{\mu}, \tag{101}$$

and where we now have the relation $\frac{w_x}{w} = -y - 2x$. The compatibility condition of this linear problem gives this last relation and, after elimination of w, the system of equations

$$z_x = -\frac{2}{y}z^2 - yz + \frac{4\theta_0}{y}z + (\theta_0 + \theta_\infty)\, y, \tag{102}$$

$$y_x = -4z + y^2 + 2xy + 4\theta_0, \tag{103}$$

which is of course equivalent to P_{IV}. The above linear problem is the linear problem for P_{IV} given by Jimbo and Miwa in [3].

3.2.2 Example $\mathbf{n = 2}$

As a further example we give here the results for the second member of our P_{IV} hierarchy (71), (72),

$$u_{xx} = 3uu_x - u^3 - 6uv - 2g_2xu + 2c_1\left(u_x - 2v - u^2\right) + 4\alpha_2, \tag{104}$$

$$\begin{aligned} v_{xx} &= 2\left(\frac{\left[uv + \frac{1}{2}v_x + c_1v - \alpha_2 + \frac{1}{2}g_2\right]^2 - \frac{1}{4}\beta_2^2}{v + \frac{1}{2}u^2 - \frac{1}{2}u_x + g_2x + c_1u}\right) - 2(uv)_x \\ &\quad -2v\left(v + \frac{1}{2}u^2 - \frac{1}{2}u_x + g_2x\right) - 2c_1\left(v_x + uv\right), \end{aligned} \tag{105}$$

where α_2 and β_2 are two independent constants of integration.

Corresponding to the system (104), (105) we have the linear problem given by equations (73), (74) where

$$F = \begin{pmatrix} -\lambda + \frac{1}{2}u & 1 \\ -v & \lambda - \frac{1}{2}u \end{pmatrix}, \tag{106}$$

$$K_2 = \frac{1}{g_2}\sum_{j=-1}^{2} K_{2,j}\lambda^j, \tag{107}$$

with $K_{2,j}$ given by

$$K_{2,2} = 2\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{108}$$

$$K_{2,1} = \begin{pmatrix} -2c_1 & 2 \\ -2v & 2c_1 \end{pmatrix}, \tag{109}$$

$$K_{2,0} = \begin{pmatrix} -v - g_2 x & u + 2c_1 \\ & \\ -v_x - uv - 2c_1 v & v + g_2 x \end{pmatrix}, \tag{110}$$

$$K_{2,-1} = \begin{pmatrix} \begin{array}{c} -\frac{1}{2}(2uv + v_x) \\ -c_1 v - \frac{1}{2}g_2 + \alpha_2 \end{array} & \begin{array}{c} \frac{1}{2}(2v + u^2 - u_x) \\ +c_1 u + g_2 x \end{array} \\ & \\ -\frac{\left[uv + \frac{1}{2}v_x + c_1 v - \alpha_2 + \frac{1}{2}g_2\right]^2 - \frac{1}{4}\beta_2^2}{v + \frac{1}{2}u^2 - \frac{1}{2}u_x + g_2 x + c_1 u} & \begin{array}{c} \frac{1}{2}(2uv + v_x) \\ +c_1 v + \frac{1}{2}g_2 - \alpha_2 \end{array} \end{pmatrix}. \tag{111}$$

Our gauge transformation then yields the alternative linear problem (78), (79), with

$$A = \begin{pmatrix} -\lambda & w \\ -\frac{v}{w} & \lambda \end{pmatrix}, \tag{112}$$

$$B_2 = \frac{1}{g_2} \sum_{j=-1}^{2} B_{2,j} \lambda^j, \tag{113}$$

where w satisfies $\frac{w_x}{w} = -u$ and where each $B_{2,j} = M^{-1} K_{2,j} M$.

4 Conclusions

We have studied in detail the P_{II} and P_{IV} hierarchies derived in [8]. We have shown that the corresponding linear problems can be mapped on to alternative linear problems such that those for the first members of our hierarchies (i.e. for P_{II} and P_{IV} themselves) are precisely the linear problems given by Jimbo and Miwa [3]. This then means that our hierarchies could alternatively have been obtained by using an AKNS type approach, i.e. expanding in powers of λ, in the Jimbo-Miwa linear problems for P_{II} and P_{IV}. Our work here then raises the interesting problem of whether other Jimbo-Miwa linear problems can be used to obtain hierarchies based on Painlevé equations. This is a topic that we will pursue in subsequent studies.

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Appendix

In this Appendix we briefly consider the linear problems for P_{II} given in [1] and [3],

$$\mathbf{\Phi}_t = \mathcal{U}\mathbf{\Phi}, \qquad \mathbf{\Phi}_\xi = \mathcal{V}\mathbf{\Phi}, \tag{114}$$

which we take here in the forms presented in [17], i.e. respectively in the form of [1] with

$$\mathcal{U} = \begin{pmatrix} -i\xi & ip \\ -ip & i\xi \end{pmatrix}, \tag{115}$$

$$\mathcal{V} = \begin{pmatrix} -i(4\xi^2+t+2p^2) & i(4p\xi-\beta/\xi)-2p_t \\ -i(4p\xi-\beta/\xi)-2p_t & i(4\xi^2+t+2p^2) \end{pmatrix}, \tag{116}$$

or, corresponding to [3], with

$$\mathcal{U} = \begin{pmatrix} -i\xi & iq \\ -ir & i\xi \end{pmatrix}, \tag{117}$$

$$\mathcal{V} = \begin{pmatrix} -i(4\xi^2+t+2qr) & 4iq\xi-2q_t \\ -4ir\xi-2r_t & i(4\xi^2+t+2qr) \end{pmatrix}. \tag{118}$$

The compatibility condition of each of these linear problems yields P_{II} (in the case of (117), (118) after integrating twice). The linear problem (114) with (117), (118) is put into standard Jimbo-Miwa form (29)—(32) by making the change of variables

$$q = -\frac{1}{2}i\gamma w, \quad r = -i\gamma\frac{z}{w}, \quad t = \frac{x}{\gamma}, \quad \xi = \frac{1}{2}i\gamma\mu, \tag{119}$$

and using the relations

$$q_t = \frac{1}{2}i\gamma^2 wy, \qquad r_t = -i\gamma^2\left(\frac{\alpha-(1/2)-zy}{w}\right), \tag{120}$$

where $\gamma^3 = -2$. This identification means that our P_{II} hierarchy decribed in Section 2 could also have been derived beginning with an AKNS type linear problem, which then explains why Kitaev obtained the second member of our hierarchy when seeking to isolate examples of higher order ODEs related to such a linear problem [12].

It is argued in [17] that there is no elementary relation between the linear problems given by (114) with (115), (116), and (114) with (117), (118); for example, there is no gauge transformation mapping one into the other. However, whilst there is no known gauge transformation, we give a general linear problem which encapsulates both Lax pairs. Consider the linear problem (114) with

$$\mathcal{U} = \begin{pmatrix} -i\xi & iq \\ -ir & i\xi \end{pmatrix}, \tag{121}$$

$$\mathcal{V} = \begin{pmatrix} -i(4\xi^2+t+2qr) & i(4q\xi-\beta/\xi)-2q_t \\ -i(4r\xi-\beta/\xi)-2r_t & i(4\xi^2+t+2qr) \end{pmatrix}. \tag{122}$$

The compatibility condition of this linear problem yields the three equations

$$q_{tt} = 2q^2r+tq+\beta, \tag{123}$$

$$r_{tt} = 2qr^2+tr+\beta, \tag{124}$$

$$\beta(q-r) = 0, \tag{125}$$

the third of which tells us that we must either have $q = r$ or $\beta = 0$, corresponding to the choices (115), (116) and (117), (118) respectively. That is, the linear problems with (115), (116) and (117), (118) are in fact both special cases of (121), (122).

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