Analysis and Applications of Autoregressive Moving Average Models with Stochastic Variance

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Abstract

It is known that volatility plays a central role in financial modelling problems. This paper studies, in detail, a class of discrete time stochastic volatility (SV) models driven by ARMA models with innovations having a stochastic variances. The autocorrelation function of this class of models is derived and methods of identification of such processes are described. An example is added to illustrate the development of the theory over the standard methods.

Keywords: GARCH models, Volatility, Stochastic variance, Innovations, Heteroscedasticity, Random, Conditional expectation, Autocorrelation, Estimation.

1 Introduction

The class of autoregressive moving average (ARMA) models has been used in many applications related to time series observations. This class of ARMA moedsl of order (p,q) or ARMA(p,q) is given by

$$X_t = C + \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \cdots + \alpha_p X_{t-p} + Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}, \quad (1.1)$$

where C is a constant, α_j's and β_j's are not all zeros for all j, $\{Z_t\}$ is a sequence of uncorrelated random variables (not necessarily independent) with mean zero and constant variance, σ^2 (known as white noise, $WN(0, \sigma^2)$).

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Under stationarity conditions (ie. the roots of $1 - \alpha_1\omega - \alpha_2\omega^2 - \cdots - \alpha_p\omega^p = 0$ lie out side the unit circle), one has the stationary solution to (1.1) given by

$$X_t = \mu + \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \tag{1.2}$$

where $\mu = E(X_t)$ is the unconditional mean of X_t and $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$ such that $\psi(B)\alpha(B) = \beta(B)$ ($\alpha(B) = I - \alpha_1 B - \alpha_2 B^2 - \cdots - \alpha_p B^p$; $\beta(B) = I + \beta_1 B + \beta_2 B^2 + \cdots + \beta_q B^q$, and B is the backshift operator satisfying $B^j X_t = X_{t-j}$ with $B^0 X_t = I X_t = X_t$).

Under stationary conditions, the unconditional mean of the process is

$$\mu = E(X_t) = \frac{C}{1 - \alpha_1 - \alpha_2 - \cdots - \alpha_p} \ ; \ 1 - \alpha_1 - \alpha_2 - \cdots - \alpha_p \neq 0 \tag{1.3}$$

and the corresponding constant variance is

$$Var(X_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j^2, \tag{1.4}$$

where $\sum_{j=0}^{\infty} \psi_j^2 < \infty$.

However, the conditional mean and variance of the process, respectively are

$$E_{t|t-1} = E(X_t|F_{t-1}) = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \cdots + \alpha_p X_{t-p} + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}$$

and $V_{t|t-1} = Var(X_t|F_{t-1}) = \sigma^2$, where F_{t-1} is the history of the process upto time $t-1$. Therefore, standard ARMA models are useful in modelling the conditional mean but not suitable for modelling the conditional variance.

In many practical problems in finance, it has been noticed that the models assuming constant conditional variance may not produce good forecast values. Further it has been recognised that many financial time series data are uncorrelated while the squared values are highly correlated. Engle (1982) exploited this idea and put forward a class of Autoregressive Conditional Heteroskedastic (ARCH) models to describe the conditional variance which proved to be extremely useful in many financial time series applications. Bollerslev (1986) generalised the class of ARCH models to incorporate the temporal dependence in conditional variances for skewness and excess kurtosis. The class of $(p,q)^{th}$ order generalised ARCH or GARCH is given by

$$\begin{aligned} X_t &= \sqrt{h_t}\zeta_t, \\ h_t &= \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}, \end{aligned} \tag{1.5}$$

where ζ_t is a sequence of independent and identically distributed (iid) random variables with zero mean and unit variance, $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$ and $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$.

Notes:

1. When $\beta_j = 0$, $j = 1, 2, \cdots, q$, equation (1.5) reduces to the ARCH(p) model. The conditions on $\alpha's$ and $\beta's$ ensure that $Var(X_t) > 0$, while $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$ is required for wide sense stationarity of $\{X_t\}$.

2. It is clear that the unconditional variance of X_t is greater than the conditional variance of X_t provided the past realizations of $X_t^2 > \frac{\alpha_0}{1-\sum_{i=1}^{p}\alpha_i - \sum_{j=1}^{q}\beta_j}$.

Remark:

When the original time series has significant serial correlations, first we fit an ARMA model and and then investigate the residuals for any heteroscedasticity.

A shortcoming of the GARCH model in (1.5) is that the sign of X_t does not influence the conditional variance, which may contradict the observed dynamics of the process. Therefore, the main concern of this paper is to further develop the theory which may turn out to be useful in modelling the statistical properties of financial data. With that view in mind, the next section considers the class of ARMA models with a stochastic variance specification.

2 ARMA Models with Stochastic Variance

Suppose that $\{X_t\}$ in (1.1) represents the mean corrected return of a stock given by $X_t = Y_t - \mu$, where $\mu = E(Y_t)$. The word volatility in finance literature is frequently associated with the $Var(X_t)$ and the changes of volatility occur in almost all classes of assets and have been extensively studied and reported in the past. This paper considers the changes of volatility by allowing the variance of the noise process in (1.1) to be σ_t^2. For example, set Z_t such that $Z_t \sim NID(0, \sigma_t^2)$. The assumption of normality of Z_t is not necessary, but can be used to simplify many related results. There are two interesting cases arise in practice, where σ_t^2 is

(i) a deterministic function of time and

(ii) a stochastic process.

These two cases can be analysed to emphasize different related practical issues. However, in each case we need additional assumptions relating to σ_t^2 such as

(a) $0 < m < \sigma_t^2 < M < \infty$ in (i) and

(b) stationarity in (ii)

for stable solutions. Case (i) has been considered by many authors (see, for example, Peiris and Singh (1987), Peiris (1991) and Singh and Peiris (1997) and references there in for details).

This paper considers the case (ii) with the following specification for $h_t = log\ \sigma_t^2$ to describe a class of stochastic volatility (SV) models. Suppose that $\{h_t\}$ follows an $ARMA(p', q')$ model satisfying

$$h_t = C' + \eta_1 h_{t-1} + \eta_2 h_{t-2} + \cdots + \eta_{p'} h_{t-p'} + \nu_0 V_t + \nu_1 V_{t-1} + \cdots + \nu_{q'} V_{t-q'}, \quad (2.1)$$

where C' is a positive constant, η_j's and ν_j's are (constants) not all zeros for all j and $\{V_t\}$ is assumed to be a sequence of serially uncorrelated random variables which are mutually uncorrelated with $\{Z_t\}$. Further assume that the mean and the variance of $\{V_t\}$ are zero and σ_V^2 respectively(ie.$\{V_t\} \sim WN(0, \sigma_V^2)$). Without loss of generality (w.l.o.g.) we may take $\sigma_V^2 = 1$.

It is known that the volatility logarithm given in (2.1) follows a stationary ARMA(p', q') process when the zeros of $\eta(\omega) = 1 - \eta_1\omega - \eta_2\omega^2 - \cdots - \eta_{p'}\omega^{p'}$ are outside the unit circle. In this case $C' = (1 - \sum_{i=1}^{p'} \eta_i)\ \delta$, where $\delta = E(h_t)$ is the unconditional mean of h_t.

Notes:

1. The class of SV models driven by (1.1) with $Z_t \sim NID(0, \sigma_t^2)$ and $h_t = ln(\sigma_t^2)$, where h_t follows (2.1) is called the class of $ARMA - ARMASV(p, q, p', q')$.

2. It is convenient to set $Z_t = \sigma_t U_t$, where σ_t and U_t are two independent variables and $\{U_t\} \sim NID(0, 1)$. In this case, it is clear that

$$E(Z_t^j) = M_j E(\sigma_t^j), \tag{2.2}$$

where $M_j = E(U_t^j) = \frac{d^j}{dt^j}(exp(t^2/2))_{|t=0}$ and $exp(t^2/2)$ is the moment generating function (mgf) of an $N(0, 1)$ distribution. For example, $M_1 = 0, M_2 = 1, M_3 = 0$ and $M_4 = 3$.

3. When σ_t and U_t are not independent, we may consider a more general case such that the pairs (V_t, U_t) are iid and $corr(V_t, U_t) \neq 0$. For example, (V_t, U_t) are iid and bivariate normal with the correlation matrix $\begin{pmatrix} 1 & \delta' \\ \delta' & 1 \end{pmatrix}$.

Let $\nu(\omega) = \nu_0 + \nu_1\omega + \nu_2\omega^2 + \cdots + \nu_{q'}\omega^{q'}$ and let the sequence of square summable constants $\{\psi_i'\}$ is obtained by $\Psi'(B)\eta(B) = \nu(B)$, where $\Psi'(B) = \sum_{j=0}^{\infty} \psi_j' B^j$. In this case

$$h_t = \delta + \sum_{j=0}^{\infty} \psi_j' V_{t-j} \tag{2.3}$$

is a valid solution to (2.1). The assumption on the zeros of $\eta(\omega)$ ensures that the process $\{h_t\}$ is weakly stationary with mean δ and variance, $\xi^2 = \sum_{j=0}^{\infty} \psi_j'^2$. Clearly, with the assumption of normality, the unconditional distribution of X_t is a lognormal mixture of normal distributions. See, for instance, Taylor (1994).

In many applications of time series with the specification in (2.1), one needs the moments of $\{X_t\}$. The next section derives the moments , $E(X_t^j)$ for $j,\ 1 \leq j \leq 4$.

3 Moments and Kurtosis

When $\{V_t\} \sim NID(0,1)$, the unconditional distribution of σ_t^2 is log normal with mean δ and variance ξ^2. Since $h_t \sim N(\delta, \xi^2)$, we have

$$E[(\sigma_t^2)^j] = exp(j\delta + j^2\xi^2/2).$$

Now the mean and variance of σ_t^2 respectively are $exp(\delta + \xi^2/2)$ and $exp(2\delta + \xi^2)(exp\xi^2 - 1)$. From (1.2) it is obvious that $E[(X_t - \mu)^r] < \infty$ for all $r > 0$ where the mean return is $E(X_t) = \mu$ and its variance is $E[(X_t - \mu)^2] = \sum_{j=0}^{\infty} \psi_j^2 E(\sigma_t^2) = exp(\delta + \xi^2/2) \sum_{j=0}^{\infty} \psi_j^2$. Further, we obtain a lower bound for $E[(X_t - \mu)^4]$ is obtained as follows:

$$\begin{aligned} E[(X_t - \mu)^4] &= E[(\sum_{j=0}^{\infty} \psi_j Z_{t-j})^4] && (3.1) \\ &= E[(\sum_{j=0}^{\infty} \psi_j Z_{t-j})^2 (\sum_{j=0}^{\infty} \psi_j Z_{t-j})^2] && (3.2) \\ &\geq (\sum_{j=0}^{\infty} \psi_j^2)^2 E(\sigma_t^4) && (3.3) \\ &= 3exp(2\delta + 2\xi^2)(\sum_{j=0}^{\infty} \psi_j^2)^2. \end{aligned}$$

Therefore, the kurtosis of X_t is greater than $3exp(\xi^2)$ and clearly, the distribution of X_t is leptokurtic.

Example 1

Consider an $ARMA - ARMASV(1,0,1,0)$ given by the following two equations:

$$X_t = C + \alpha X_{t-1} + Z_t,\ \{Z_t\} \sim WN(0, \sigma_t^2) \tag{3.4}$$

$$h_t = C' + \eta h_{t-1} + V_t,\ \{V_t\} \sim WN(0,1), \tag{3.5}$$

where $|\alpha|,\ |\eta| < 1$.

In this case we have

$$X_t = \mu + \sum_{j=0}^{\infty} \alpha^j Z_{t-j}$$

$$h_t = \delta + \sum_{j=0}^{\infty} \eta^j V_{t-j},$$

where $\mu = \frac{C}{1-\alpha}$ and $\delta = \frac{C'}{1-\eta}$.

Under the assumption of normality

$$Var(X_t) = E[(X_t - \mu)^2] = \frac{exp(\delta + \xi^2/2)}{1-\alpha^2},$$

where $\xi^2 = var(h_t) = \sum_{j=0}^{\infty} \eta^{2j} = \frac{1}{1-\eta^2}$.

The corresponding kurtosis $K_1 > 3exp(1/(1-\eta^2))$.

Example 2

Consider an $ARMA - ARMASV(1,0,1,1)$ model given by (3.4) and the following volatility equation:

$$h_t = C' + \eta h_{t-1} + V_t + \nu V_{t-1},\ \{V_t\} \sim WN(0,1). \tag{3.6}$$

In this case we have

$$h_t = \delta + \sum_{j=0}^{\infty} \psi'_j V_{t-j},$$

where $\psi'_0 = 1$ and $\psi'_j = \eta^{j-1}(\eta + \nu)$ for $j \geq 1$.

Under the assumption of normality

$$Var(X_t) = E[(X_t - \mu)^2] = \frac{exp(\delta + \xi'^2/2)}{1-\alpha^2},$$

where $\xi^2 = var(h_t) = \sum_{j=0}^{\infty} \psi'^{2j} = \frac{1+2\eta\nu+\nu^2}{1-\eta^2}$.

The corresponding kurtosis $K_2 > 3exp(\frac{1+2\eta\nu+\nu^2}{1-\eta^2})$.

Next section considers the modelling of excess equity returns in the US(Jan 1971 to Sept 2004) as an application of ARMA models with stochastic variance (SV).

4 An application of SV modelling

We obtained the data for equity index from Morgan Stanley Capital International and the short-term interest rate data from the International Financial Statistics of the International Monetary Fund. Excess return is calculated by subtracting the interest rate from the return computed from the equity index. These are all annualised figures. The time series plot, acf and the pacf of data show that there is no significant serial correlation (see, Appendix 1). The analysis shows that the squared values are autocorrelated (see Appendix 2) and therefore we fit AR(1) and ARMA(1,1) models to these squared values. The following tables report the corresponding results:

Table 1: Results for Excess Equity Return (USA)

Estimated Model Parameters				
Model	α_0	α_1	θ_1	σ_v^2
AR	-0.1099 (0.0475)	0.9207 (0.0313)		0.2130 (0.0577)
ARMA	-0.1110 (0.0504)	0.9199 (0.0331)	0.5016 (0.0044)	0.0204 (0.0105)

Model parameters are estimated by the quasi-maximum likelihood method using numerical maximization algorithm in Gauss^{T}M. The robust standard errors (se) are given in parentheses.

Entries are p-values for the respective statistics. These diagnostics are computed from the recursive residual (standardised) of the measurement equation. The null hypothesis in the portmanteau test is that the residuals are serially uncorrelated. The ARCH test checks for no serial correlations in the squared residual up to lag 26. This is applicable to recursive residuals as explained in Wells (1996, p 27). MNR is

Table 2: Residual Diagnostics and Model Adequacy Tests

Estimated Model Parameters				
Model	Portmanteau	ARCH	MNR	Recursice T
AR	0.443	0.320	0.019	0.946
ARMA	0.469	0.642	0.116	0.933

the modified Von Neumann ratio test using a recursive residual for model adequacy (see, Harvey, 1990, Chapter 5). Similarly, if the model is correctly specified then the recursive T has a Student's t-distribution (see, Harvey, 1990, p 157).

Figure 1
Estimated Variance from the SV Model (AR Structure)

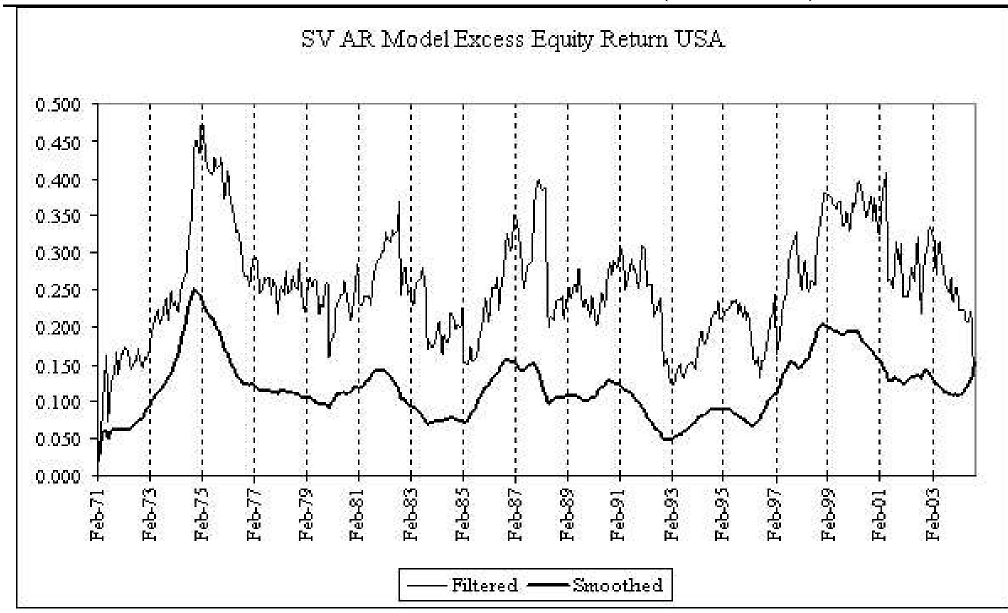
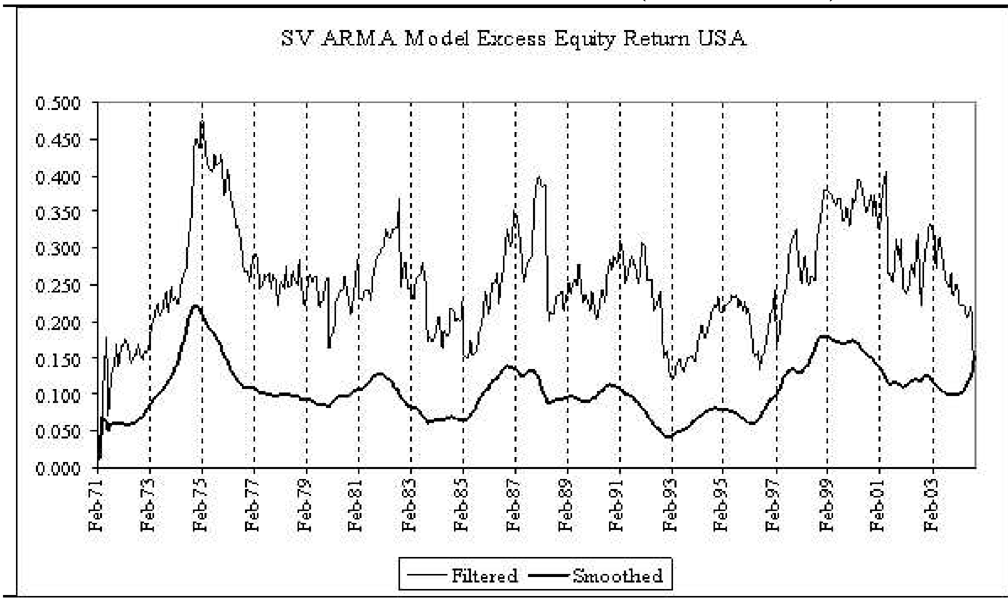


Figure 2
Estimated Variance from the SV Model (ARMA Structure)



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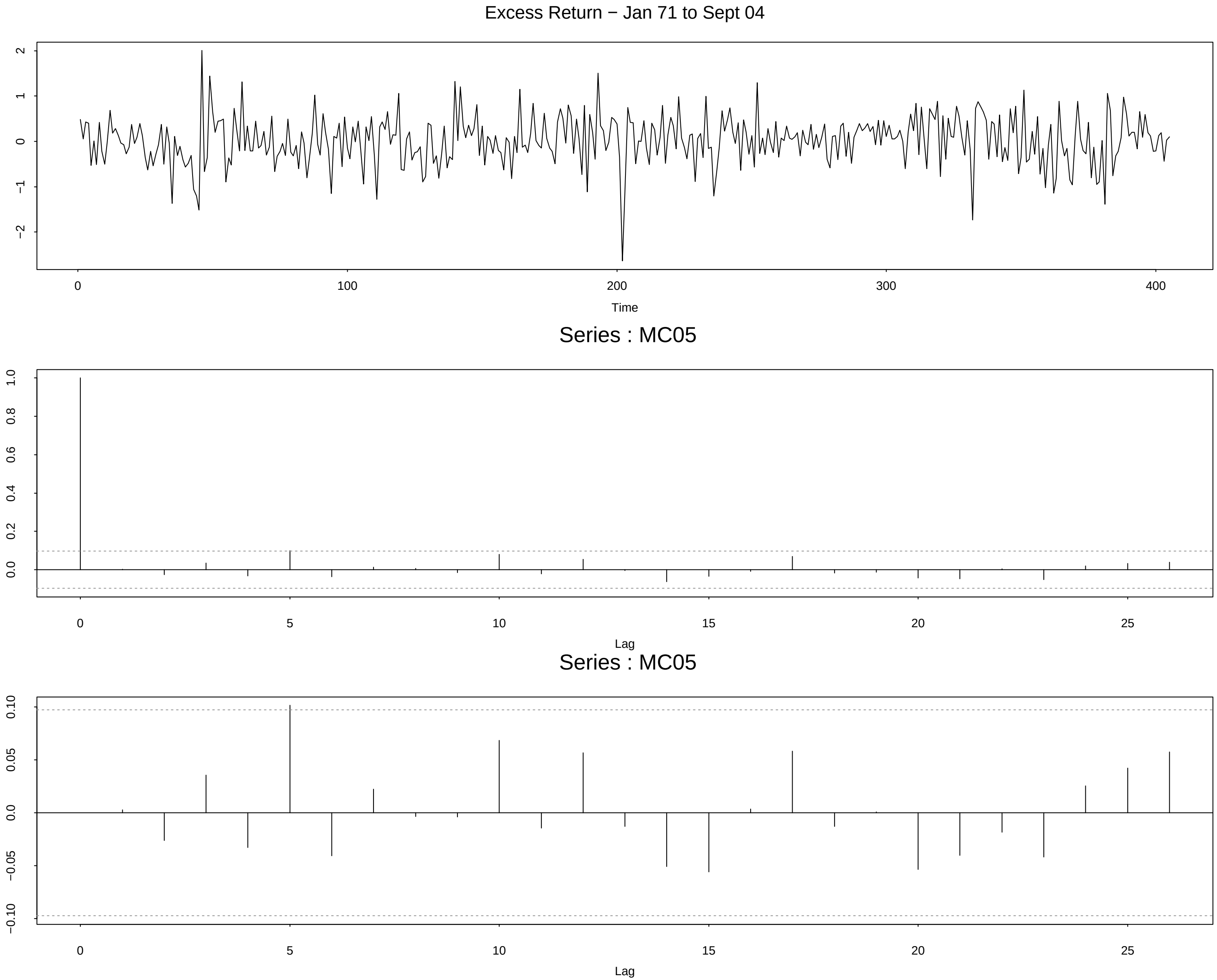
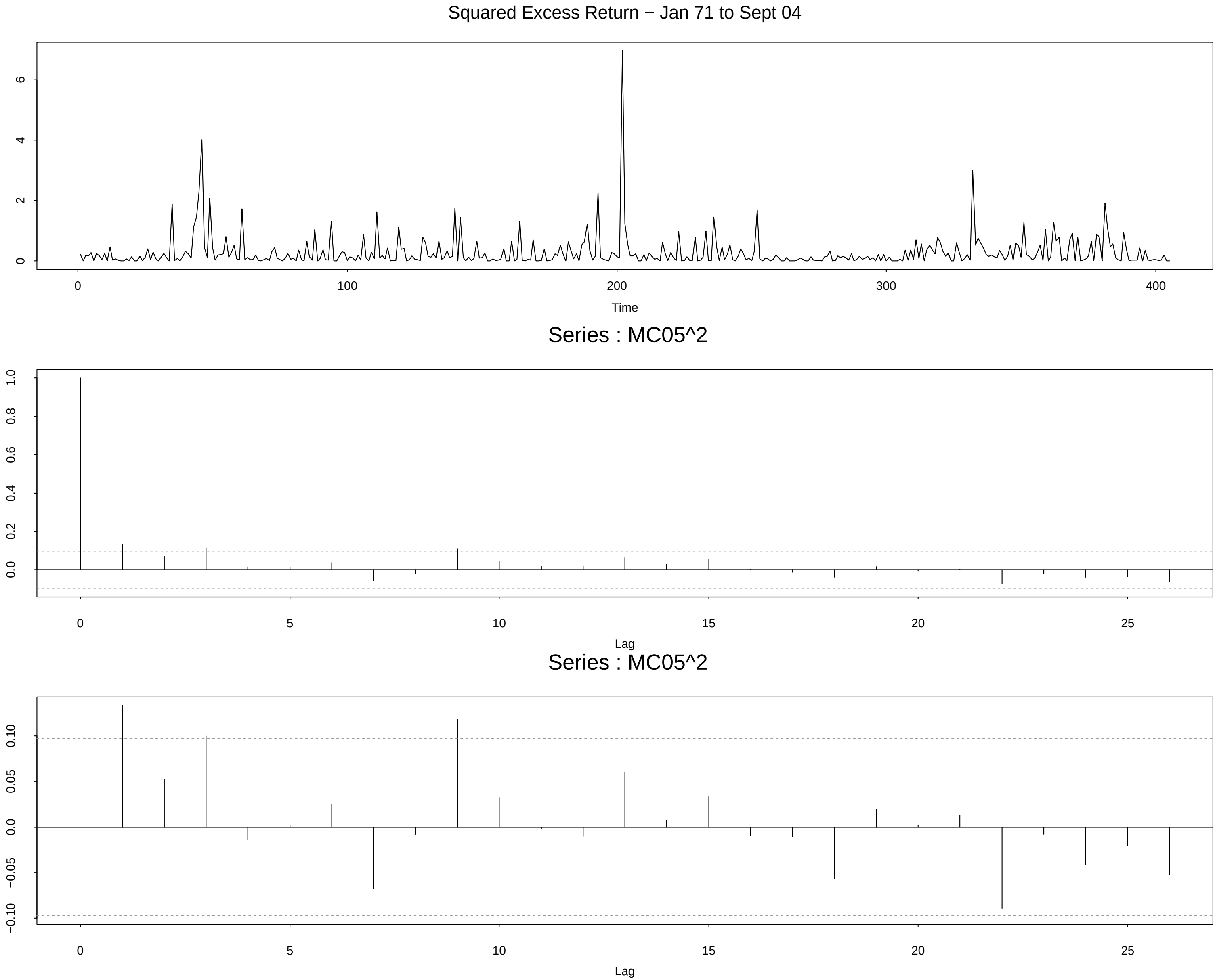
Appendix 1


Excess Return − Jan 71 to Sept 04
Time
Series : MC05
Lag
Series : MC05
Lag

Appendix 2


Squared Excess Return − Jan 71 to Sept 04
Time
Series : MC05^2
Lag
Series : MC05^2
Lag