MAXIMAL SUBGROUPS OF FREE IDEMPOTENT-GENERATED SEMIGROUPS FACTORED OUT BY BIORDER RELATIONS

D. EASDOWN, S. GARDINER, AND B. MCELWEE

Abstract. Given any biordered set E, we may form the idempotent-generated semigroup F_E, which is generated by the set E, subject to the relations $ef = e * f$ whenever e and f are elements of E and $e * f$ is a basic product. Easdown proved in 1985 that the biordered set of F_E is biorder isomorphic to E, thus demonstrating that the biordered set axioms, introduced by Nambooripad in 1974, characterise certain partial algebras of idempotents of semigroups. Relatively little is known about the general structure of F_E, though it is known that every group can arise as a maximal subgroup of F_E for some E, and that, as a consequence, the word problem is unsolvable. In this article, presentations for maximal subgroups are studied using homomorphic images of fundamental groups of graphs associated with $\mathcal{D}$-classes of the biordered set E. To illustrate the technique, small biordered sets E are constructed where F_E contains maximal subgroups which are cyclic of order two and free abelian on two generators respectively, the second of which reconstructs an example due to Dolinka.

1. Introduction

It is a longstanding theme to understand roles played by idempotents in algebras. Howie [25] proved that every semigroup embeds in a semigroup generated by idempotents, at about the same time that Munn [28, 29] discovered a method for recovering information about inverse semigroups from partial symmetries of their semilattices of idempotents. This led to many different generalisations, particularly within the class of regular semigroups, and was the driving force behind Nambooripad's successful characterisation [32, 33] of systems of idempotents of regular semigroups, through his invention of biordered sets and and associated sandwich sets. Removing references to sandwich sets, one then obtains a general axiomatic definition of biordered sets, which was conjectured for a long time to characterise systems of idempotents of arbitrary (not necessarily regular) semigroups. That this characterisation turns out to be correct was proved in [16], using the semigroup F_E (also called $IG(E)$ in the literature) freely generated by an abstract biordered set E, subject to the relations of the biordered set. If one starts with an abstract biordered set E, then E is in a natural bijective correspondence with the idempotents of F_E, and this bijection respects the biorder relations between elements (made precise below using the notion of arrows and basic products in a biordered set). Attempting to understand the structure of F_E has intrigued many authors, and much progress has been made, including the result, by Gray and Ruskuc [20], that the general word problem for F_E is undecidable, the proof of which is a consequence of the fact that all groups (in particular, those with an unsolvable word problem) arise as maximal subgroups of F_E for some E. By contrast, many of the earlier investigations (see, for example, [30, 34–36]) had only fully worked out the details in certain classes of cases, in which the maximal subgroups turned out to be free.

The general difficulty, or intractability, of working with F_E is illustrated by the fact that the first example of a biordered set E for which it could be proved that the maximal subgroups are not free did not appear in the literature until 2009, when Brittenham, Meakin and Margolis [1] exhibited a semigroup with seventy-two elements with biordered set E, such that F_E contains

AMS Subject Classification (2010): 06A99, 20M10, 20M20.
Key Words: free idempotent-generated semigroups, biordered sets, maximal subgroups.

maximal subgroups that are free abelian of rank two. In fact, McElwee [31] had discovered, but not published, an example prior to this, with sixteen elements and maximal subgroups that are cyclic of order two, and this example appears below. Dolinka [10] subsequently found a semigroup, in fact a band with twenty elements, also with maximal subgroups that are free abelian of rank two, which is also reconstructed and explored below.

The results of Gray and Ruskuc [20] rely on finding a presentation for maximal subgroups of F_E, using a Reidemeister-Schreier method and general combinatorial semigroup machinery developed by Ruskuc [37]. By contrast, the methods of Brittenham, Meakin and Margolis [1] are topological in nature, related to graph-theoretical techniques of Graham [19] and Houghton [24]. In this paper, we provide presentations for the maximal subgroups of F_E, using homomorphic images of fundamental groups of graphs, which are particularly amenable for investigating small examples. These equivalent methods involve incorporating relators that arise from so-called singular squares in the biordered set. Gray and Ruskuc [20] renewed interest in the field, generating activity in finding simplifications or focusing on particular classes of semigroups. They proved [21] that symmetric groups arise as maximal subgroups of F_E when E is the biordered set of a transformation semigroup. Dolinka [11], and also with Gray [12], extended these results to partial transformation semigroups and full linear semigroups respectively. Dolinka and Rukuc [14] also showed that all groups can arise as maximal subgroups of F_E when the E comes from a band, and Dandan and Gould [6] also showed that all groups can arise, but using a simplified wreath product construction. Dandan and Gould [7, 8], and also with Dolinka [4, 5] and Quinn-Gregson [9], explore the word problem for F_E, and examine the structure of F_E beyond maximal subgroups, in a range of other settings. Easdown, Sapir and Volkov [17] adapt the word argument of [16] to show that all periodic elements in F_E belong to subgroups. Dolinka, Gray and Ruskuc [13] shift the emphasis further away from maximal subgroups of F_E, considering the interplay between regular and nonregular elements, and associated word problems.

In Section 2, we establish terminology and the arrow notation used to describe relationships between idempotents in a biordered set. We explain definitions related to singular squares in a biordered set E, and define F_E, the semigroup freely generated by elements of E, subject to relations arising from basic products in E. A method is introduced for creating ideal extensions of Rees matrix semigroups, using transformation and dual transformation semigroups and semigroups of strictly row and column monomial matrices. This is a special case, tailored for our purposes, of a general construction of Clifford [2], who investigated ideal extensions of completely simple semigroups. Several examples are given, which coalesce, later, to fully describe, up to isomorphism, the semigroup F_E for each of the biordered sets E constructed in the final two sections. Section 2 finishes by setting up the machinery for processing group presentations that arise as images of fundamental groups of graphs, using, as generators, edges that do not appear in a spanning tree for the graph. We explain how graphs naturally arise from $\mathcal{D}$-classes of a biordered set E, which are the connected components of E using paths consisting of alternating double arrows.

Section 3 provides a presentation of the maximal subgroups (all of which are isomorphic) associated with a given $\mathcal{D}$-class of E, based on singular squares located throughout the associated graph Γ. One first chooses a spanning tree T for Γ, then locates directed edges that do not lie in T, to be used as generators in the presentation, and then writes down traversals of singular squares in Γ, to be used as relators. In Sections 4 and 5, we find presentations of maximal subgroups associated with $\mathcal{D}$-classes of F_E in some special cases. In Section 4, we analyse McElwee's example [31], with cyclic maximal subgroups of order two. The underlying semigroup is a band with sixteen elements and F_E is finite and isomorphic to an ideal extension of a Rees matrix semigroup by a left-zero band. Section 5 analyses an example of Dolinka [10], with maximal subgroups that are free abelian of rank two. The underlying semigroup is a band with twenty elements and F_E is infinite and isomorphic to an ideal extension of one Rees matrix semigroup by another.

2. Preliminaries

Standard terminology and facts about semigroups and Green's relations $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{D}$ as given in say [3], [23] or [26], will be assumed. Denote the set of idempotents of a semigroup S by $E(S)$. Denote the full transformation semigroup on a set X by $\mathcal{T}_X$, and its dual by $\mathcal{T}_X^*$. Elements of $\mathcal{T}_X$ are composed from left to right, whereas elements of $\mathcal{T}_X^*$ are composed from right to left.

Let I and J be nonempty indexing sets and G be a group. Let P be a $J \times I$ sandwich matrix with entries from G, and recall that we may form the Rees matrix semigroup

$$\mathcal{M} \;=\; \mathcal{M}(G, I \times J, P) \;=\; \{(i,g,j) \mid i \in I,\;\; j \in J\,,\; g \in G\}$$

with associative multiplication

$$(i,g,j)(k,h,\ell) \;=\; (i, gP_{jk}h, \ell)\,,$$

for all $i,k \in I$, $j,\ell \in J$ and $g \in G$. The idempotents of $\mathcal{M}$ have the form (i, P_{ji}^{-1}, j) as i and j range over I and J respectively, where P_{ji}^{-1} denotes the inverse of P_{ij} in G. If all elements of P are the group identity element 1 (which occurs automatically if G is trivial) then the rule for multiplication trivialises in the middle coordinate, and by making the identification

$$(i,j) \;\equiv\; (i,1,j)\,,$$

we have rectangular band multiplication:

$$(i,j)(k,\ell) \;=\; (i,\ell)\,.$$

If, further, the indexing set I [J] has exactly one element, then $\mathcal{M}$ becomes a right-zero [left-zero] band. Put $G^0 = G \dot{\cup} \{0\}$, where 0 behaves as a zero. Recall (see, for example, [3]) that we may form the following semigroups using formal matrix multiplication:

$$M(J) \;=\; \{\; J \times J \;\text{ strictly row monomial matrices over } G^0 \,\}$$

and

$$M^*(I) \;=\; \{\; I \times I \;\text{ strictly column monomial matrices over } G^0 \,\}\,.$$

Note that $M^*(I)$ is anti-isomorphic to $M(I)$ under transposition.

Example 2.1. Put $M = \{e,f,g,h\}$ where

$$e \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\;\; f \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 0 \end{bmatrix},\;\; g \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 0 \end{bmatrix},\;\; h \;=\; \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

where G is any nontrivial group and a and b are elements of G not equal to the identity element. Then M is a left-zero band and a subsemigroup of $M(J)$ where $J = \{1,2,3\}$. In an application below, we take $a = b$ and G to be a cyclic group of order 2, where a is the generator of G.

Example 2.2. Put $M = \{e,f,g,h\}$ where

$$e = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\; f = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},\; g = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},\; h = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then M is also a four element left-zero band (isomorphic to the band of the previous example) and a subsemigroup of $M^*(I)$ where $I = \{1,2,3,4\}$ and G is any group.

Example 2.3. Let G be an infinite cyclic group generated by b. Put

$$M \ = \ \{\alpha_n, \beta_n, \gamma_n, \delta_n \mid n \in \mathbb{Z}\}$$

where, for each integer n,

$$\alpha_n = \begin{bmatrix} 0 & b^n & 0 & 0 \\ 0 & b^n & 0 & 0 \\ 0 & 0 & b^n & 0 \\ 0 & 0 & b^n & 0 \end{bmatrix}, \quad \beta_n = \begin{bmatrix} b^n & 0 & 0 & 0 \\ b^n & 0 & 0 & 0 \\ 0 & 0 & 0 & b^n \\ 0 & 0 & 0 & b^n \end{bmatrix},$$

$$\delta_n = \begin{bmatrix} 0 & b^{n-1} & 0 & 0 \\ 0 & b^n & 0 & 0 \\ 0 & 0 & b^n & 0 \\ 0 & 0 & b^{n-1} & 0 \end{bmatrix}, \quad \gamma_n = \begin{bmatrix} b^n & 0 & 0 & 0 \\ b^{n+1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b^{n+1} \\ 0 & 0 & 0 & b^n \end{bmatrix}.$$

Then the following rules for multiplication hold, for any $m, n \in \mathbb{Z}$:

$$\alpha_m\alpha_n \ = \ \alpha_m\delta_n \ = \ \beta_m\alpha_n \ = \ \alpha_{m+n}\,, \qquad \beta_m\beta_n \ = \ \beta_m\gamma_n \ = \ \alpha_m\beta_n \ = \ \beta_{m+n}\,,$$

$$\gamma_m\gamma_n \ = \ \gamma_m\beta_n \ = \ \delta_m\gamma_n \ = \ \gamma_{m+n}\,, \qquad \delta_m\delta_n \ = \ \delta_m\alpha_n \ = \ \gamma_m\delta_n \ = \ \delta_{m+n}\,,$$

$$\alpha_m\gamma_n = \beta_{m+n+1}\,, \quad \gamma_m\alpha_n = \delta_{m+n+1}\,, \qquad \beta_m\delta_n = \alpha_{m+n-1}\,, \quad \delta_m\beta_n = \gamma_{m+n-1}\,,$$

from which it follows that M is a subsemigroup of $M(I)$ where $I = \{1, 2, 3, 4\}$. Further, one may check that the Rees matrix semigroup

$$\mathcal{M} \ = \ \mathcal{M}\left(G, \{1,2\} \times \{1,2\}, \begin{bmatrix} 1 & 1 \\ 1 & b \end{bmatrix}\right) \tag{1}$$

is isomorphic to M using the following mapping μ from $\mathcal{M}$ to M defined by the following rule:

$$\mu : \ (i, b^n, j) \ \mapsto \begin{cases} \alpha_{-n} & \text{if } (i,j) = (1,1) \\ \beta_{-n} & \text{if } (i,j) = (1,2) \\ \delta_{-n} & \text{if } (i,j) = (2,1) \\ \gamma_{-n-1} & \text{if } (i,j) = (2,2)\,. \end{cases} \tag{2}$$

Example 2.4. Again let G be an infinite cyclic group generated by b and put

$$M \ = \ \{\alpha_n, \beta_n, \gamma_n, \delta_n \mid n \in \mathbb{Z}\}$$

where, for each integer n, by contrast with the previous example,

$$\alpha_n = \begin{bmatrix} b^n & 0 & 0 & b^{n+1} \\ 0 & b^n & b^{n+1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \beta_n = \begin{bmatrix} b^n & 0 & 0 & b^n \\ 0 & b^n & b^n & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\delta_n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b^{n-1} & b^n & 0 \\ b^{n-1} & 0 & 0 & b^n \end{bmatrix}, \quad \gamma_n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b^n & b^n & 0 \\ b^n & 0 & 0 & b^n \end{bmatrix}.$$

One may check that the same rules for multiplication hold as in the previous example, but now M is a subsemigroup of $M^*(I)$ where $I = \{1, 2, 3, 4\}$. Again M is isomorphic to the Rees matrix semigroup $\mathcal{M}$ defined by (1) and the mapping μ defined by (2) becomes an isomorphism.

Consider $M(J)$, where J is any nonempty set, and let $A \in M(J)$. For each $j \in J$ there exists a unique $k \in J$ such that A_{jk} is nonzero, inducing a mapping $\rho_A : J \to J$ such that

$$(\forall j \in J) \quad A_{j,j\rho_A} \neq 0 \ .$$

For all $A, B, C \in M(J)$, the equation $AB = C$ implies that

$$(\forall j \in J) \quad j\rho_A\rho_B \ = \ j\rho_C \quad \text{and} \quad A_{j,j\rho_A}B_{j\rho_A,j\rho_A\rho_B} \ = \ C_{j,j\rho_C} \ .$$

In particular, $\rho_A\rho_B = \rho_{AB}$ for all $A, B \in M(J)$, so that $\rho : M(J) \to \mathcal{T}(J)$ is a semigroup homomorphism. Moreover, ρ is always onto, and an isomorphism when G is trivial.

Consider $M^*(I)$, where I is any nonempty set. Dually, for each $A \in M^*(I)$, there is an induced mapping $\lambda_A : I \to I$ such that

$$(\forall i \in I) \quad A_{\lambda_A i,i} \neq 0 \ .$$

Note here that the action of λ_A is on the left (by contrast to the mapping ρ_A above, where the action is on the right). Further, we have, for all $A, B, C \in M^*(I)$, that $AB = C$ implies that

$$(\forall i \in I) \quad \lambda_A\lambda_B i \ = \ \lambda_C i \quad \text{and} \quad A_{\lambda_A\lambda_B i,\lambda_B i}B_{\lambda_B i,i} \ = \ C_{\lambda_C i,i} \ .$$

In particular, dual to the above, $\lambda_A\lambda_B = \lambda_{AB}$ for all $A, B \in M(I)$, so that $\lambda : M^*(I) \to \mathcal{T}^*(I)$ is a semigroup homomorphism. As before, λ is always onto, and an isomorphism when G is trivial.

Consider the Rees matrix semigroup $\mathcal{M} = \mathcal{M}(G, I \times J, P)$, and suppose that we have a semigroup $\mathcal{S}$ and two homomorphisms

$$L : \mathcal{S} \to M^*(I) \,, \qquad \text{and} \qquad R : \mathcal{S} \to M(J) \ .$$

We wish to combine $\mathcal{S}$ and $\mathcal{M}$ in a natural way, using disjoint union, and the transformation and dual transformations actions that arise by applying L and R to $\mathcal{S}$, to form a semigroup $\mathcal{S}\dot{\cup}\mathcal{M}$, which will turn out to be an ideal extension of $\mathcal{M}$. We first decongest the notation. Consider $\alpha \in \mathcal{S}$ so that

$$L\alpha \in M^*(I) \qquad \text{and} \qquad \alpha R \in M(J) \ ,$$

inducing the dual transformation $\lambda_{L\alpha} \in \mathcal{T}^*(I)$ and transformation $\rho_{\alpha R} \in \mathcal{T}(J)$ respectively. We write

$$\alpha i \ \equiv \ \lambda_{L\alpha} i \quad (\forall i \in I) \qquad \text{and} \qquad j\alpha \ \equiv \ j\rho_{\alpha R} \quad (\forall j \in J) \ ,$$

noting actions on the left and right respectively. We also wish to impose the following condition, referred to as the *adjoint property*, describing a type of conjugation relation:

$$(\forall \alpha \in \mathcal{S}) \qquad (\alpha R)P \ = \ P(L\alpha) \ . \tag{3}$$

As a consequence of (3), using our notation, $(\alpha R)_{j,j\alpha} \neq 0$ and $(L\alpha)_{\alpha k,k} \neq 0$ for all $\alpha \in S$, $j \in J$ and $k \in I$, and

$$(\alpha R)_{j,j\alpha}P_{j\alpha,k} \ = \ P_{j,\alpha k}(L\alpha)_{\alpha k,k} \ . \tag{4}$$

Note that (3) simplifies to the following condition, in the case that the images of L and R contain only matrices with zeros and ones:

$$P_{j\alpha,k} \ = \ P_{j,\alpha k} \tag{5}$$

for all $\alpha \in \mathcal{S}$, $j \in J$ and $k \in I$. This is the case if G happens to be trivial, in which case (3) holds trivially anyway, as all entries of the sandwich matrix P would be 1. We have the following general result, providing a natural semigroup multiplication on $\mathcal{S}\dot{\cup}\mathcal{M}$, which then becomes an ideal extension of $\mathcal{M}$. This may also be deduced from a general construction of Clifford, describing ideal extensions of completely simple semigroups [2].

Proposition 2.5. *Suppose that $\mathcal{M} = \mathcal{M}(G, I \times J, P)$ is a Rees matrix semigroup and $\mathcal{S}$ is a semigroup for which there are homomorphisms $L : \mathcal{S} \to M^*(I)$ and $R : \mathcal{S} \to M(J)$ such that the adjoint property* (3) *holds. Then the set $\mathcal{S}\dot{\cup}\mathcal{M}$ becomes a semigroup with multiplication, extending the multiplications of $\mathcal{S}$ and $\mathcal{M}$, given by, for all $\alpha \in \mathcal{S}$, $i \in I$, $j \in J$ and $g \in G$,*

$$\alpha(i,g,j) \;=\; \big(\alpha i, (L\alpha)_{\alpha i, i}\, g, j\big) \quad \textit{and} \quad (i,g,j)\alpha \;=\; \big(i, g(\alpha R)_{j, j\alpha}, j\alpha\big)\ .$$

Proof. The verification of associativity is routine, and we mention only the following case, which relies on the adjoint property: for $\alpha \in \mathcal{S}$ and $(i,g,j),(k,h,\ell) \in \mathcal{R}$, we have, by (4),

$$\begin{aligned}\big((i,g,j)\alpha\big)(k,h,\ell) &= \big(i, g(\alpha R)_{j,j\alpha}, j\alpha\big)(k,h,\ell) = \big(i, g(\alpha R)_{j,j\alpha} P_{j\alpha,k}\, h, \ell\big) \\ &= \big(i, gP_{j,\alpha k}(L\alpha)_{\alpha k,k} h, \ell\big) = (i,g,j)\big(\alpha k, (L\alpha)_{\alpha k,k} h, \ell\big) = (i,g,j)\big(\alpha(k,h,\ell)\big)\ .\end{aligned}$$

□

The following result is routine to prove directly, and also follows from Proposition 2.5 when G is trivial, noting that $M(J)$ and $M^*(I)$ become isomorphic to $\mathcal{T}(J)$ and $\mathcal{T}^*(I)$ respectively, and that (3) then holds automatically:

Corollary 2.6. *Suppose that $\mathcal{R}$ is a rectangular band and $\mathcal{S}$ is a subsemigroup of $\mathcal{T}^*(I) \times \mathcal{T}(J)$. The set $\mathcal{S}\dot{\cup}\mathcal{R}$ becomes a semigroup with multiplication, extending the multiplications of $\mathcal{S}$ and $\mathcal{R}$, given by, for all $\alpha \in \mathcal{S}$, $i \in I$ and $j \in J$,*

$$\alpha(i,j) \;=\; (\alpha i, j) \qquad \textit{and} \qquad (i,j)\alpha \;=\; (i, j\alpha)\ .$$

In particular, if $\mathcal{S}$ is a band then $\mathcal{S}\dot{\cup}\mathcal{R}$ is a band, which is an ideal extension of $\mathcal{R}$.

Example 2.7. We now coalesce Examples 2.1 and 2.2 to create a novel ideal extension of the Rees matrix semigroup $\mathcal{M} = \mathcal{M}(G, I \times J, P)$, where $G = \langle a \rangle = \{1, a\}$ is a cyclic group of order two, $I = \{1,2,3,4\}$, $J = \{1,2,3\}$ and

$$P \;=\; \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & a & a & 1 \\ 1 & 1 & a & a \end{bmatrix}\ .$$

Let $\mathcal{S} = \{e,f,g,h\}$ be a four element left-zero semigroup and define isomorphisms $R : \mathcal{S} \to M(J)$ and $L : \mathcal{S} \to M^*(I)$ of left-zero semigroup as follows:

$$eR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},\quad fR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 0 \end{bmatrix},\quad gR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & 0 & 0 \end{bmatrix},\quad hR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$Le = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ Lf = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},\ Lg = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},\ Lh = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is then routine to check, by direct calculation, that the adjoint property (3) holds everywhere. For example,

$$(hR)P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & a & a & 1 \\ 1 & a & a & 1 \end{bmatrix} = P(Lh)\,.$$

By Proposition 2.5, $\mathcal{S}\dot{\cup}\mathcal{M}$ is a semigroup, and an ideal extension of $\mathcal{M}$ by $\mathcal{S}$.

Example 2.8. We now coalesce Examples 2.3 and 2.4 to create a novel ideal extension of the Rees matrix semigroup $\mathcal{M} = \mathcal{M}(G, I \times I, P)$, where $G = \langle a, b\rangle \equiv C_\infty \times C_\infty$ is a free abelian group of rank two generated by a and b, $I = \{1,2,3,4\}$ and

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & b & b \\ 1 & a & ab & b \\ 1 & a & a & 1 \end{bmatrix}.$$

Let $\mathcal{S} = \mathcal{M}(H, J \times J, Q)$ be the Rees matrix semigroup where $H = \langle b\rangle \equiv C_\infty$ is the infinite cyclic subgroup of G generated by b, $J = \{1,2\}$ and

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & b \end{bmatrix}.$$

Define a homomorphism $R : \mathcal{S} \to M(I)$, with action on the right, motivated by (2), the rule for μ above, using the matrices in Example 2.3, by the following, for any integer n:

$$(1,b^n,1)R = \begin{bmatrix} 0 & b^{-n} & 0 & 0 \\ 0 & b^{-n} & 0 & 0 \\ 0 & 0 & b^{-n} & 0 \\ 0 & 0 & b^{-n} & 0 \end{bmatrix}, \quad (1,b^n,2)R = \begin{bmatrix} b^{-n} & 0 & 0 & 0 \\ b^{-n} & 0 & 0 & 0 \\ 0 & 0 & 0 & b^{-n} \\ 0 & 0 & 0 & b^{-n} \end{bmatrix},$$

$$(2,b^n,1)R = \begin{bmatrix} 0 & b^{-n-1} & 0 & 0 \\ 0 & b^{-n} & 0 & 0 \\ 0 & 0 & b^{-n} & 0 \\ 0 & 0 & b^{-n-1} & 0 \end{bmatrix}, \quad (2,b^n,2)R = \begin{bmatrix} b^{-n-1} & 0 & 0 & 0 \\ b^{-n} & 0 & 0 & 0 \\ 0 & 0 & 0 & b^{-n} \\ 0 & 0 & 0 & b^{-n-1} \end{bmatrix}.$$

Define a homomorphism $L : \mathcal{S} \to M^*(I)$, with action now on the left, again motivated by (2), but instead using the matrices in Example 2.4, by the following, for any integer n:

$$L(1,b^n,1) = \begin{bmatrix} b^{-n} & 0 & 0 & b^{-n+1} \\ 0 & b^{-n} & b^{-n+1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad L(1,b^n,2) = \begin{bmatrix} b^{-n} & 0 & 0 & b^{-n} \\ 0 & b^{-n} & b^{-n} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$L(2,b^n,1) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b^{-n-1} & b^{-n} & 0 \\ b^{-n-1} & 0 & 0 & b^{-n} \end{bmatrix}, \quad L(2,b^n,2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & b^{-n-1} & b^{-n-1} & 0 \\ b^{-n-1} & 0 & 0 & b^{-n-1} \end{bmatrix}.$$

We can now check the adjoint property (3):

$$\big((1,b^n,1)R\big)P \;=\; \begin{bmatrix} b^{-n} & b^{-n} & b^{-n+1} & b^{-n+1} \\ b^{-n} & b^{-n} & b^{-n+1} & b^{-n+1} \\ b^{-n} & ab^{-n} & ab^{-n+1} & b^{-n+1} \\ b^{-n} & ab^{-n} & ab^{-n+1} & b^{-n+1} \end{bmatrix} \;=\; P\big(L(1,b^n,1)\big)\ ,$$

$$\big((1,b^n,2)R\big)P \;=\; \begin{bmatrix} b^{-n} & b^{-n} & b^{-n} & b^{-n} \\ b^{-n} & b^{-n} & b^{-n} & b^{-n} \\ b^{-n} & ab^{-n} & ab^{-n} & b^{-n} \\ b^{-n} & ab^{-n} & ab^{-n} & b^{-n} \end{bmatrix} \;=\; P\big(L(1,b^n,2)\big)\ ,$$

$$\big((2,b^n,1)R\big)P \;=\; \begin{bmatrix} b^{-n-1} & b^{-n-1} & b^{-n} & b^{-n} \\ b^{-n} & b^{-n} & b^{-n+1} & b^{-n+1} \\ b^{-n} & ab^{-n} & ab^{-n+1} & b^{-n+1} \\ b^{-n-1} & ab^{-n-1} & ab^{-n} & b^{-n} \end{bmatrix} \;=\; P\big(L(2,b^n,1)\big)\ ,$$

$$\big((2,b^n,2)R\big)P \;=\; \begin{bmatrix} b^{-n-1} & b^{-n-1} & b^{-n-1} & b^{-n-1} \\ b^{-n} & b^{-n} & b^{-n} & b^{-n} \\ b^{-n} & ab^{-n} & ab^{-n} & b^{-n} \\ b^{-n-1} & ab^{-n-1} & ab^{-n-1} & b^{-n-1} \end{bmatrix} \;=\; P\big(L(2,b^n,2)\big)\ .$$

By Proposition 2.5, $\mathcal{S}\dot\cup\mathcal{M}$ is a semigroup, and an ideal extension of $\mathcal{M}$ by $\mathcal{S}$.

Let E be a partial algebra with a partial multiplication denoted by $*$ with domain D_E. If $(e,f)\in D_E$ then $e*f$ is called a *basic product* in E. It is common to use juxtaposition to denote basic products in partial algebras, but in this context we reserve juxtaposition for concatenation of letters and words in the free semigroup E^+ generated by E. Following the arrow notation introduced in [15,16], define relations $\succ\!\!-$ (the *left arrow*) and $\longrightarrow\!\!\!\!\!>$ (the *right arrow*) on E by

$$e \succ\!\!- f \quad \text{if}\ \ (e,f)\in D_E \ \ \text{and}\ \ e*f=e\ \ ,$$

and

$$e \longrightarrow\!\!\!\!\!> f \quad \text{if}\ \ (e,f)\in D_E \ \ \text{and}\ \ f*e=e\ \ ,$$

together with the following abbreviations:

$$\succ\!\!-\!\!\prec \;=\; \succ\!\!- \cap -\!\!\prec\ , \qquad <\!\!\longrightarrow\!\!\!\!\!> \;=\; <\!\!\!\!\!\longleftarrow \cap \longrightarrow\!\!\!\!\!>\ , \qquad \succ\!\!\longrightarrow\!\!\!\!\!> \;=\; \succ\!\!- \cap \longrightarrow\!\!\!\!\!>\ .$$

Recall that E is a *biordered set* when the left and right arrows are preorders,

$$D_E = \;\succ\!\!- \;\cup \longrightarrow\!\!\!\!\!> \cup <\!\!\!\!\!\longleftarrow \cup -\!\!\prec\ ,$$

and the following axioms hold:

(B2.1), (B(2.1)*, (B2.2), (B(2.2)*, (B3.1), (B(3.1)*, (B3.2), (B(3.2)*, (B4), (B(4)*,

as described in [15,16]. These are a slight reformulation of Nambooripad's original list of biordered set axioms (see [32,33]), avoiding the use of sandwich sets. We do not need to refer directly to these axioms in this article, though (B2.1) and (B2.2) and their duals are invoked implicitly in building arrow diagrams below.

The set $E = E(S)$ of idempotents of a semigroup S forms a biordered set by restricting the domain of the semigroup multiplication to pairs $(e,f)\in E\times E$ such that such that e is a left or right zero for f, or f is a left or right zero for e. As usual, $E(S)$ with this partial multiplication is referred to as the *biordered set of* S.

It is far from routine, however, to show that an arbitrary biordered set E arises in this way, that is, that E can be identified in a natural way with the biordered set of some semigroup. This is in fact true (Theorem 2.9 below), and to make this precise we need the following notions. Let E and F be arbitrary biordered sets. A *biordered set morphism* from E to a boset F is a mapping $\theta\colon E \to F$ such that $(e\theta, f\theta) \in D_F$ and $(ef)\theta = (e\theta)(f\theta)$ for all $(e, f) \in D_E$. A *biordered set isomorphism* from E to F is a bijective mapping $\theta : E \to F$ such that both θ and θ^{-1} are biordered set morphisms, in which case we may write $E \cong F$. Define the semigroup F_E (also denoted by $IG(E)$ in the literature) by the following semigroup presentation:

$$F_E \;=\; \langle E \mid ef = e * f \;\text{ for all basic products } e * f \text{ in } E\rangle\,. \tag{6}$$

Here the biordered set E appears as the alphabet used as the generating set. Elements of F_E are therefore equivalence classes of words over the free semigroup E^+, where two words are equivalent if one can be transformed into the other by some finite sequence of elementary transitions, which have the form of replacing a word ef by a basic product $e * f$ where $(e, f) \in D_E$, or vice versa. We denote equivalence of words in this sense by $\approx$ and the $\approx$-equivalence class of a word $w \in E^+$ by $[[w]]$. Thus $F_E = \big\{[[w]] \;\big|\; w \in E^+\big\}$ and the mapping from E^+ to F_E that sends a word w to its equivalence class $[[w]]$ is a surjective semigroup homomorphism. In fact, the natural mapping from letters in E to their equivalence classes in F_E is a biordered set isomorphism:

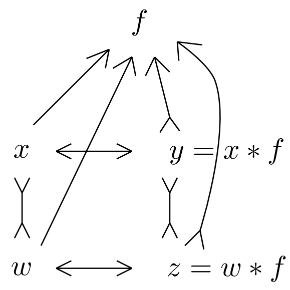
Theorem 2.9. [16] *If E is a biordered set then the natural map: $E \to E(F_E)$, $e \mapsto [[e]]$, for all $e \in E$, is a biordered set isomorphism, so that, as biordered sets, $E \cong E(F_E)$.*

This shows that the biordered set axioms characterise partial algebras of idempotents of semigroups, where multiplication is restricted to pairs of idempotents where one idempotent is a left or right zero of the other. Consequently, due to associativity in the semigroup, arbitrary bracketings in expressions of elements in the biordered set do not alter outcomes, so that brackets may be dispensed with, provided the appropriate basic products are defined. A diagram involving biordered set elements and arrow relations between them is referred to as a *skeleton*, especially when (in small cases) it captures every element of E. Such diagrams may imply information about the positioning of relevant basic products that exist in E.

A basic product $e*f$ in a biordered set E is called *trivial* if $e*f = e$ or $e*f = f$, and *nontrivial* otherwise. Whenever any basic product $e * f$ exists, then the basic product $f * e$ exists and at least one of these will be trivial. A *left-right singular square* in E is any one of the quadruples

$$(x, y, z, w)\,,\ (y, z, w, x)\,,\ (z, w, x, y)\,,\ (w, x, y, z)\,,\ (x, w, z, y)\,,\ (w, z, y, x)\,,\ (z, y, x, w)\,,\ (y, x, w, z)\,,$$

associated with the following diagram in the biordered set E, with basic products $y = x * f$ and $z = w * f$, for some $f \in E$ such that there are right arrows from both x and w to f:



in which case, using the biordered set relations (see (6)), we have, in F_E,

$$[[\,xyzwx\,]] \;=\; [[\,yw\,]] \;=\; [[\,xfw\,]] \;=\; [[\,xw\,]] \;=\; [[\,x\,]]$$

and

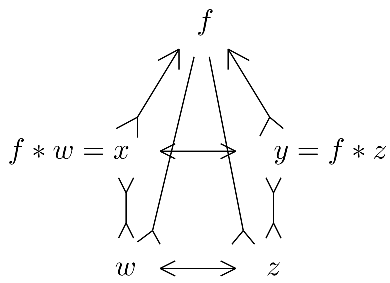
$$[[\,xwzyx\,]] \;=\; [[\,xzx\,]] \;=\; [[xwfx\,]] \;=\; [[\,xfx\,]] \;=\; [[\,x^2\,]] \;=\; [[\,x\,]]\,,$$

and similarly

$$[[\,yzwxy\,]] \;=\; [[\,yxwzy\,]] \;=\; [[\,y\,]]\,, \qquad [[\,zwxyz\,]] \;=\; [[\,zyxwz\,]] \;=\; [[\,z\,]]\,,$$

$$[[\,wzyxw\,]] \;=\; [[\,wxyzw\,]] \;=\; [[\,w\,]]\,.$$

By contrast, an *up-down singular square* in E is again any one of the quadruples listed above, but now associated with the following diagram, with basic products $x = f * w$ and $y = f * z$, for some f such that there are left arrows from both w and z to f:



in which case, by dual use of the biordered set relations,

$$[[\,xyzwx\,]] \;=\; [[\,xwzyx\,]] \;=\; [[\,x\,]]\,, \qquad [[\,yzwxy\,]] \;=\; [[\,yxwzy\,]] \;=\; [[\,y\,]]\,,$$

$$[[\,zwxyz\,]] \;=\; [[\,zyxwz\,]] \;=\; [[\,z\,]]\,, \qquad [[\,wzyxw\,]] \;=\; [[\,wxyzw\,]] \;=\; [[\,w\,]]\,.$$

This proves the following lemma:

Lemma 2.10. *If (x, y, z, w) is a singular square in a biordered set E, of either up-down or left-right type, then $[[xyzwx]] = [[x]]$ in F_E.*

We call a singular square *nontrivial* if all of the four vertices in the quadruple (making up the square in the diagram) are distinct, and *trivial* otherwise.

Suppose that $E = E(S)$ is the biordered set of a semigroup S. Clearly, if $e, f \in E$ then $e \longleftrightarrow f$ if and only if $e\ \mathcal{R}\ f$, and $e \succ\!\!-\!\!\prec f$ if and only if $e\ \mathcal{L}\ f$, so that, when restricted to idempotents of S, we may use Green's relations $\mathcal{R}$ and $\mathcal{L}$ interchangeably with $\longleftrightarrow$ and $\succ\!\!-\!\!\prec$ respectively. Now Green's relation $\mathcal{D}$ on S is the join of $\mathcal{L}$ and $\mathcal{R}$. It is convenient also to use the symbol $\mathcal{D}$ to denote the join of the relations $\longleftrightarrow$ and $\succ\!\!-\!\!\prec$ in an arbitrary biordered set E, which in this context means that, for $e, f \in D$, we have $e\ \mathcal{D}\ f$ if and only if there is some (possibly empty) chain of alternating double left and right arrows that begins with e and ends with f. When this happens we say that e and f are *connected* in the biordered set. These two uses of the symbol $\mathcal{D}$ have to be read in context. Whilst $e\ \mathcal{D}\ f$ in E implies $e\ \mathcal{D}\ f$ in any semigroup S for which $E = E(S)$, the converse may fail: for example in a Brandt semigroup we may have $\mathcal{D}$-related idempotents that are not connected in the underlying biordered set (which, in the case of a Brandt semigroup, is a semilattice). Suppose further that $x, y, z, w \in E = E(S)$ with

$$x\ \mathcal{R}\ y\ \mathcal{L}\ z\ \mathcal{R}\ w\ \mathcal{L}\ x \qquad \text{or} \qquad x\ \mathcal{L}\ y\ \mathcal{R}\ z\ \mathcal{L}\ w\ \mathcal{R}\ x\,.$$

By Green's Lemma, in both cases, the element $xyzwx$ in S lies in H_x, the $\mathcal{H}$-class of x, which is a group with identity element x. In general, there is no reason to expect $xyzwx$ to coincide with x (though it always will if S is a band, or if the four elements are not distinct). However, if the quadruple (x, y, z, w) is a nontrivial singular square in E, then, by Lemma 2.10, the word

$xyzwx$ does represent the same element as x in F_E (and consequently $xyzwx = x$ in S). This observation is crucial in the development below of presentations for maximal subgroups in F_E.

Maximal subgroups of semigroups are, of course, $\mathcal{H}$-classes of idempotents, and are isomorphic wherever they may occur throughout any given $\mathcal{D}$-class of the semigroup. In trying to understand properties of the semigroup F_E, and in particular its maximal subgroups, it is natural then to focus on the $\mathcal{D}$-classes of elements of E, which are the maximal connected subsets of E with respect to the join of $\mathcal{L}$ and $\mathcal{R}$. An important first step in our investigation is to create a certain undirected graph, defined below, associated with a given $\mathcal{D}$-class of E. The idea is to regard the idempotents of the (connected) $\mathcal{D}$-class as vertices, spread out in a certain way according to orderings of the $\mathcal{L}$ and $\mathcal{R}$-classes, and then convert the double left and double right arrows between adjacent idempotents into edges of the graph.

Consider a graph Γ with set of vertices V. We assume that Γ is simple, in the sense that there are no loops at any vertex, and multiple edges between the same pair of vertices are prohibited. A path in Γ will be identified with a nonempty word over the alphabet V with a certain property (see below). The edges of Γ are undirected, but paths are always directed. Note however that an edge, when identified with the path specified by the two vertices that define the edge, in one of two possible orders, becomes directed. A *(directed) path* π in Γ is a sequence

$$\pi \;=\; v_1 v_2 \dots v_k$$

of vertices $v_1, \dots, v_k \in V$, where $k \geq 1$, such that $v_i = v_{i+1}$ or v_i and v_{i+1} define an (undirected) edge in Γ, for $i = 1, \dots, k-1$. Note that we allow repetition of vertices. For the path π above, we say that the vertex v_1 is the *source* and the vertex v_k is the *target*. If $k = 1$ then $\pi = v_1$ becomes a single vertex, called a *trivial path* (and the source and target coincide). We may concatenate paths $\pi_1 = v_1 \dots v_k$ and $\pi_2 = v_k \dots v_\ell$, as words in the free semigroup V^+, to form

$$\pi_1\pi_2 \;=\; v_1 \dots v_k v_k \dots v_\ell \,,$$

provided the source of π_2 coincides with the target of π_1. Note that the vertex v_k is duplicated by this process of concatenation, though we may remove one of the duplicates, using equivalence of paths (defined below). If $\pi = v_1 \dots v_k$ is a path then we define the *inverse path*

$$\pi^{-1} \;=\; v_k \dots v_1 \,,$$

by reversing the order of the vertices (that is, reversing π as a word). In particular, a single edge v_1v_2, regarded as a directed path, has v_2v_1 as its inverse, which uses the same underlying edge, but interchanges its source and target. An *elementary transition* is the following process, or its inverse, which replaces a path $v_1 \dots v_k$ by either

(i) $v_1 \dots v_{i-1} v_i v_i v_{i+1} \dots v_n$

for some i, duplicating one of the vertices next to itself, or

(ii) $v_1 \dots v_{i-1} v_i v v_i v_{i+1} \dots v_n$

for some i and vertex v, where v and v_i are the vertices of some edge in Γ. Alternative (ii) has the effect of inserting into the path an arbitrary edge whose source lies in the path, immediately followed by the edge represented by its inverse. In particular, we may use elementary transitions to insert or remove a duplicate of any given vertex appearing in a path. Note that the result of concatenating $v_1 \dots v_k$ and $v_k \dots v_\ell$ is $v_1 \dots v_k v_k \dots v_\ell$, which may become

$$v_1 \dots v_k \dots v_\ell \;=\; v_1 \dots v_\ell \,,$$

by using an elementary transition to remove one of the duplicates of v_k. Two paths π_1 and π_2 are *equivalent*, and we write $\pi_1 \sim \pi_2$, if there is some (possibly empty) sequence of elementary transitions that takes π_1 to π_2. Denote the $\sim$-equivalence class of a path π by $[\pi]$ and define the multiplication of equivalence classes by

$$[\pi_1][\pi_2] \;=\; [\pi_1\pi_2]$$

whenever the source of π_2 coincides with the target of π_1. This multiplication is well-defined and referred to as *concatenation* of equivalence classes of paths.

A *spanning tree* for Γ is a connected subgraph T of Γ with the same vertex set as Γ such that T contains no cycles. The following facts are well-known:

Proposition 2.11. *Suppose that Γ is a connected graph and e a given vertex of Γ. A spanning tree T always exists (though is not necessarily unique) and the following hold:*

(a) *the graph T is a maximal subtree of the undirected graph Γ;*

(b) *if any other edge of Γ is added to T then a cycle is created (as an undirected graph);*

(c) *for all $v \in V(\Gamma)$ there exists a unique shortest path in T, denoted by τ_v, that starts at e and finishes at v.*

Theorem 2.12. [27] *The set of equivalence classes of paths in Γ forms a groupoid under concatenation, with inversion defined by $[\pi]^{-1} = [\pi^{-1}]$, and maximal subgroups*

$$\{\, [\pi] \;\big|\; \text{source and target of } \pi \text{ equal } v \,\},$$

where v ranges over the vertex set of Γ. The maximal subgroups are free and, when Γ is connected, isomorphic to the fundamental group of Γ, of rank the difference between the number of edges of Γ and the number of edges of a spanning tree of Γ.

We can be more explicit about the free generators in the case that Γ is connected:

Theorem 2.13. [27] *Suppose that Γ is connected with spanning tree T and e is a fixed vertex of Γ. Then the fundamental group F of the graph Γ, with respect to the base point e, is freely generated by equivalence classes of paths*

$$[\tau_x(xy)\tau_y^{-1}]$$

where x —— y is an edge of γ not in T.

Thus we may identify F with the free group G_A with respect to an alphabet A consisting of labels of edges of Γ that do not lie in T, with each such edge given one of two possible orientations (thereby becoming a directed edge).

Consider a finite biordered set E and $e \in E$. We will construct a simple graph Γ associated with the $\mathcal{D}$-class of e. Define the vertex set of Γ to be

$$V = V(\Gamma) = \{\, f \in E \mid f \ \text{ is connected to } e \text{ by a sequence of alternating double arrows}\}\,,$$

which is the set of elements in the $\mathcal{D}$-class of E containing e. We will form edges in Γ in a natural way by identifying Γ with a subset of a rectangle. Let I and J be indexing sets for the $\mathcal{R}$ and $\mathcal{L}$-classes respectively of V, both of which we may assume to be totally ordered by a relation denoted by $<$ in each case (since E is finite). The intersection of an $\mathcal{R}$-class with an $\mathcal{L}$-class is either empty or a singleton set. (All intersections are nonempty if and only if V is rectangular.) Hence we may suppose that

$$V \;\subseteq\; \{\, x_{ij} \mid i \in I\,,\ j \in J\,\}\,,$$

where $x_{ij} \in V$ if and only if $\{x_{ij}\}$ is the nonempty intersection of the ith $\mathcal{R}$-class with the jth $\mathcal{L}$-class. Observe that

$$\begin{array}{lcl} x_{ij}\,,\ x_{i\ell}\ \in V & \quad\text{if and only if}\quad & x_{ij} \longleftrightarrow x_{i\ell}\ \text{ in } E\,, \\ x_{ij}\,,\ x_{kj}\ \in V & \quad\text{if and only if}\quad & x_{ij} \succ\!\!\!-\!\!\!\prec x_{kj}\ \text{ in } E\,. \end{array}$$

We now choose (undirected) edges

$$x_{ij} \;\text{——}\; x_{k\ell}$$

whenever

(i) $i = k$ and j covers ℓ, or ℓ covers j, with respect to the $<$ ordering of J,

or

(ii) $j = \ell$ and i covers k, or k covers i, with respect to the $<$ ordering of I.

Example 2.14. If we have the following $\mathcal{D}$-class of E:

$$\begin{tikzcd}
e = x_{11} \arrow[rr, <->] \arrow[d, >-<] & & x_{13} \arrow[r, <->] \arrow[dd, >-<] & x_{14} \arrow[d, >-<] \\
x_{21} \arrow[r, <->] \arrow[d, >-<] & x_{22} \arrow[rr, <->, crossing over] & & x_{24} \\
x_{31} \arrow[rr, <->] & & x_{33} &
\end{tikzcd}$$

then Γ becomes the following simple graph:

$$\begin{tikzcd}
e = x_{11} \arrow[rr, no head] \arrow[d, no head] & & x_{13} \arrow[r, no head] \arrow[dd, no head] & x_{14} \arrow[d, no head] \\
x_{21} \arrow[r, no head] \arrow[d, no head] & x_{22} \arrow[rr, no head, crossing over] & & x_{24} \\
x_{31} \arrow[rr, no head] & & x_{33} &
\end{tikzcd}$$

We now explain how to create paths in the graph Γ from certain words. Suppose that $f, g \in V = V(\Gamma)$, the $\mathcal{D}$-class of e in the biordered set E, and consider the word fg. Define

$$p(f) \;=\; f \;,$$

the trivial path from f to f (where "p" stands for "path"). If $f = g$ then put

$$p(fg) \;=\; fg \;=\; ff \;,$$

so that $p(fg) = ff \sim f = p(f)$, and also $p(fg) = ff \approx f = p(f)$. Consider the case that $f \neq g$ and $f \longleftrightarrow g$. Then, there is some sequence of edges in Γ:

$$x_{ij} \text{ —— } x_{ij_1} \text{ —— } \ldots \text{ —— } x_{ij_r} \text{ —— } x_{i\ell}$$

where $j < j_1 < \ldots < j_r < \ell$, using coverings with respect to the ordering $<$ of J, with either (i) $f = x_{ij}$ and $g = x_{i\ell}$, or (ii) $f = x_{i\ell}$ and $g = x_{ij}$. Put

$$p(fg) \;=\; \begin{cases} x_{ij}x_{ij_1}\ldots x_{ij_r}x_{i\ell} & \text{in case (i), or} \\ x_{i\ell}x_{ij_r}\ldots x_{ij_1}x_{ij} & \text{in case (ii).} \end{cases}$$

This path moves from left to right, or right to left, along successive edges in a given $\mathcal{R}$-class. Now consider the case that $f \neq g$ and $f \succ\!\!\!-\!\!\!\prec g$, so there is some sequence of edges in Γ:

$$x_{ij} \text{ —— } x_{i_1j} \text{ —— } \ldots \text{ —— } x_{i_sj} \text{ —— } x_{kj}$$

where $i < i_1 < \ldots < i_s < k$, using coverings with respect to the ordering $<$ of I, with either (i) $f = x_{ij}$ and $g = x_{kj}$, or (ii) $f = x_{kj}$ and $g = x_{ij}$. Put

$$p(fg) \;=\; \begin{cases} x_{ij}x_{i_1j}\ldots x_{i_sj}x_{ik} & \text{in case (i), or} \\ x_{ik}x_{i_sj}\ldots x_{i_1j}x_{ij} & \text{in case (ii).} \end{cases}$$

This path moves up or down, along successive edges in a given $\mathcal{L}$-class. As a word in E^+,

$$p(fg) \;\approx\; \begin{cases} f & \text{if } f = g. \\ g & \text{if } f \longleftrightarrow g, \\ f & \text{if } f >\!\!-\!\!< g, \end{cases}$$

using idempotency, the $\mathcal{R}$-relation and the $\mathcal{L}$-relation in F_E respectively, so that, in particular

$$[[\, p(fg)\,]] \;=\; [[\, fg\,]] \;. \tag{7}$$

Write H_e for $H_{[[e]]}$, the $\mathcal{H}$-class of $[[e]]$ in F_E. Suppose that $x_1, \ldots, x_n \in E$ such that

$$[[x_1 \ldots x_n]] \;\in\; H_e \;.$$

We will define a path $p(x_1 \ldots x_n)$ in Γ such that, regarded as a word over the alphabet E,

$$[[\, p(x_1 \ldots x_n)\,]] \;=\; [[\, x_1 \ldots x_n\,]] \;. \tag{8}$$

in F_E. Put $\beta = [[x_1 \ldots x_n]]$ and let γ be the inverse of β in H_e. Using the method of FitzGerald (see [18] and also [22]), put

$$\alpha_i \;=\; [[x_{i+1} \ldots x_n]]\,\gamma\,[[x_1 \ldots x_n]] \tag{9}$$

for $i = 1, \ldots, n-1$. Then for all i we have $\alpha_i^2 = \alpha_i = [[y_i]]$, for some $y_i \in E$, and

$$\beta \;=\; [[e(x_1 * y_1)y_1(y_1 * x_2)y_2(y_2 * x_3)\ldots y_{n-1}(y_{n-1} * x_n)e]] \;,$$

giving the following arrow diagram in E:

$$\begin{tikzcd}[column sep=small]
 & & x_1 \arrow[drr, -|] & & & & x_2 \arrow[drr, -|] & & & & x_3 \arrow[drr, -|] & & \cdots \\
e \arrow[urr] & \longleftrightarrow & e * x_1 \arrow[u, tail] & >\!\!-\!\!< & y_1 \arrow[urr] & \longleftrightarrow & y_1 * x_2 \arrow[u, tail] & >\!\!-\!\!< & y_2 \arrow[urr] & \longleftrightarrow & y_2 * x_3 \arrow[u, tail] & >\!\!-\!\!< & \cdots \\
 & & \| & & & & \| & & & & \| & & \\
 & & x_1 * y_1 & & & & x_2 * y_2 & & & & x_3 * y_3 & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[drr, -|] & & & & x_n \arrow[drr, -|] & & \\
\cdots & >\!\!-\!\!< & y_{n-1} \arrow[urr] & \longleftrightarrow & y_{n-1} * x_n \arrow[u, tail] & >\!\!-\!\!< & e \\
 & & & & \| & & \\
 & & & & x_n * e & &
\end{tikzcd}$$

inducing a path in Γ, denoted by $p(x_1 \ldots x_n)$, that starts and finishes at e:

$$\begin{aligned} p(x_1 \ldots x_n) \;=\; & p\big(e(x_1 * y_1)\big)\, p\big((x_1 * y_1)y_1\big)\, p\big(y_1(y_1 * x_2)\big)\, p\big((y_1 * x_2)y_2\big) p\big(y_2(y_2 * x_3)\big) \\ & \ldots\, p\big(y_{n-1}(y_{n-1} * x_n)\big)\, p\big((y_{n-1} * x_n)e\big) \;, \end{aligned} \tag{10}$$

the result of concatenating paths of the form $p(fg)$ where $f \longleftrightarrow g$ or $f >\!\!-\!\!< g$. Note that if $n = 1$ then $x_1 = e$, and (10) should be interpreted as

$$p(x_1) \;=\; p(e) \;=\; e \;.$$

As an element of F_E, we have

$$\begin{aligned}
[[\,p(x_1 \ldots x_n)\,]] \;&=\; [[\,p\big(e(x_1 * y_1)\big)\, p\big((x_1 * y_1)y_1\big)\, p\big(y_1(y_1 * x_2)\big)\, p\big((y_1 * x_2)y_2\big) \\
&\qquad\qquad p\big(y_2(y_2 * x_3)\big)\, \ldots\, p\big(y_{n-1}(y_{n-1} * x_n)\big)\, p\big((y_{n-1} * x_n)e\big)\,]]\;, \\
&=\; [[\,p\big(e(x_1 * y_1)\big)\,]]\; [[\,p\big((x_1 * y_1)y_1\big)\,]]\; [[\,p\big(y_1(y_1 * x_2)\big)\,]] \\
&\qquad\qquad \ldots [[\,p\big(y_{n-1}(y_{n-1} * x_n)\big)\,]]\; [[\,p\big((y_{n-1} * x_n)e\big)\,]] \\
&=\; [[\,e(x_1 * y_1)\,]]\; [[\,(x_1 * y_1)y_1\,]]\; [[\,y_1(y_1 * x_2)\,]] \\
&\qquad\qquad \ldots [[\,y_{n-1}(y_{n-1} * x_n)\,]]\; [[\,(y_{n-1} * x_n)e\,]] \\
&=\; [[\,e(x_1 * y_1)y_1(y_1 * y_2) \ldots y_{n-1}(y_{n-1} * x_n)e\,]] \;=\; \beta\,,
\end{aligned}$$

so that (8) holds.

3. Presentations of maximal subgroups

Let E be a finite biordered set, $e \in E$ and let Γ be the graph defined in the previous section associated with the $\mathcal{D}$-class of e. Choose some spanning tree T of Γ. Let F be the fundamental group of Γ, with e as the base point, so that, by Theorem 2.13, we may identify F with the free group G_A with respect to the alphabet A consisting of labels of edges of Γ that do not lie in T, with each such edge given one of two possible orientations (thereby becoming directed). We now set up a natural epimorphism from F onto a maximal subgroup of F_E. Recall (see Proposition 2.11) that, for each $v \in V(\Gamma)$, there is a unique shortest path in T, denoted by τ_v, whose source is e and whose target is v. Regarded as a word, τ_v begins with e and ends with v. Define a mapping $\varphi : F \to H_e$ by

$$[\pi] \;\mapsto\; [[\pi]]\;,$$

for all paths π in Γ with source and target e. We claim that φ is well-defined and onto. It is clear that if φ is well-defined then it is a homomorphism. It is easy to check that if π_1 and π_2 differ by an elementary transition then $[[\pi_1]] = [[\pi_2]]$ in F_E. By a simple induction,

$$\pi_1 \;\sim\; \pi_2 \text{ as paths} \quad \text{implies} \quad [[\pi_1]] \;=\; [[\pi_2]] \text{ in } F_E.$$

To verify well-definedness, it remains to show that φ maps into H_e. To see this, we first note the following lemma (where we denote the $\mathcal{R}$-class and $\mathcal{L}$-class of $[[f]] \in F_E$ by R_f and L_f respectively, for any $f \in E$), which follows by Green's lemma and a simple induction on path length:

Lemma 3.1. *For all $v \in V(\Gamma)$, we have $[[\tau_v]] \in R_e \cap L_v$ and $[[\tau_v^{-1}]] \in L_e \cap R_v$.*

Corollary 3.2. *The rule for φ maps elements of F into H_e, the $\mathcal{H}$-class of $[[e]]$ in F_E.*

Proof. It suffices to check that φ maps the equivalence class of a path $\pi = \tau_x(xy)\tau_y^{-1}$ into H_e, where $x \longrightarrow y$ is an edge of Γ not in T. We have $[\pi]\varphi = [[\tau_x x y \tau_y^{-1}]]$. By duality, it suffices to suppose that $x \longleftrightarrow y$. By Lemma 3.1, we have the following egg-box diagram in F_E:

$[[\,\tau_y^{-1}\,]]$		$[[\,x\,]]$		$[[\,y\,]]$
$[[\,e\,]]$		$[[\,\tau_x\,]]$		

By Green's lemma, and properties of idempotents,

$$[[\,\tau_x xy\tau_y^{-1}\,]] \;=\; [[\tau_x]]\,[[x]]\,[[y]]\,[[\tau_y^{-1}]] \;=\; [[\tau_x]]\,[[y]]\,[[\tau_y^{-1}]] \qquad =\; [[\tau_x]]\,[[\tau_y^{-1}]] \;\in\; H_e\,,$$

completing the proof of the corollary. □

So far, we have that φ is a well-defined homomorphism into F_E. In fact, φ is an epimorphism:

Lemma 3.3. *The homomorphism $\varphi : F \to H_e$ is onto.*

Proof. Let $\beta = [[\,x_1 \dots x_n\,]]\ \mathcal{H}\ [[\,e\,]]$ in F_E for some $x_1, \dots, x_n \in E$. By (8), there exists a path $\pi = p(x_1 \dots x_n) \in \Gamma$ with source and target e such that

$$[\pi]\varphi \;=\; [[\,\pi\,]] \;=\; [[\,p(x_1 \dots x_n)\,]] \;=\; [[\,x_1 \dots x_n\,]] \;=\; \beta$$

in F_E, verifying that φ is onto. □

Thus, we have a description of the maximal subgroup in F_E with identity element $[[\,e\,]]$, up to isomorphism:

Corollary 3.4. *The groups H_e and $F/\ker\varphi$ are isomorphic.*

To describe the kernel of φ succinctly, we first say that a singular square (x, y, z, w) is in *standard form* if it is nontrival (so vertices are distinct) and

$$\begin{tikzcd} x = x_{ij} \arrow[r, leftrightarrow] \arrow[d, dash] & x_{i\ell} = y \arrow[d, dash] \\ w = x_{kj} \arrow[r, leftrightarrow] & x_{k\ell} = z \end{tikzcd}$$

for some i, j, k, ℓ such that $i < k$ and $j < \ell$, with respect to the total orderings $<$ of I and J respectively. Observe that we may cyclically permute the vertices of any nontrivial singular square or its inverse to get a singular square in standard form. Consider now any singular square $\mathcal{S} = (x, y, z, w)$ (which may be trivial, or nontrivial and not necessarily in standard form). Let $\pi = \pi_{\mathcal{S}}$ be the following path starting and finishing at e:

$$\pi \;=\; \pi_{\mathcal{S}} \;=\; \tau_x p(xy)p(yz)p(zw)p(wx)\tau_x^{-1}\;. \tag{11}$$

Thus the path π starts at e, enters the square at some vertex x, circumnavigates the square clockwise or anticlockwise, exits at the vertex x, and returns to e. Then, using (7), the facts that $[[\,xyzwx\,]] = [[\,x\,]]$, by Lemma 2.10, and the path τ_x finishes with x, we have

$$\begin{aligned}[\pi]\varphi \;&=\; [[\,\tau_x p(xy)p(yz)p(zw)p(wx)\tau_x^{-1}\,]] \;=\; [[\,\tau_x\,]]\,[[\,p(xy)\,]]\,[[\,p(yz)\,]]\,[[\,p(zw)\,]]\,[[\,p(wx)\,]]\,[[\,\tau_x^{-1}\,]] \\ &=\; [[\,\tau_x\,]]\,[[\,xy\,]]\,[[\,yz\,]]\,[[\,zw\,]]\,[[\,wx\,]]\,[[\,\tau_x^{-1}\,]] \;=\; [[\,\tau_x\,]]\,[[\,xyzwx\,]]\,[[\,\tau_x^{-1}\,]] \;=\; [[\,\tau_x\,]]\,[[\,x\,]]\,[[\,\tau_x^{-1}\,]] \\ &=\; [[\,\tau_x x\tau_x^{-1}\,]] \;=\; [[\,\tau_x\tau_x^{-1}\,]] \;=\; [[\,e\,]]\,.\end{aligned}$$

which shows that $[\pi] \in \ker\varphi$. This verifies the following lemma:

Lemma 3.5. $\big\{\,[\pi_{\mathcal{S}}] \;\big|\; \mathcal{S}$ *is a singular square*$\,\big\} \;\subseteq\; \ker\varphi\,.$

This is the first step towards proving the following theorem:

Theorem 3.6. *The kernel of φ is the normal closure of*

$$\big\{\,[\pi_{\mathcal{S}}] \;\big|\; \mathcal{S}\ \textit{ is a singular square in standard form}\,\big\}\,.$$

If $\mathcal{S} = (x, y, z, w)$ is a singular square then denote by $\mathcal{S}^{-1}$ the singular square (x, w, z, y), which has the opposite orientation to $\mathcal{S}$, so that

$$\begin{aligned}\big[\pi_{\mathcal{S}^{-1}}\big] &= \big[\tau_x p(xw)p(wz)p(zy)p(yx)\tau_x^{-1}\big] = \big[\tau_x p(wx)^{-1}p(zw)^{-1}p(yz)^{-1}p(xy)^{-1}\tau_x^{-1}\big] \\ &= \big[\tau_x p(xy)p(yz)p(zw)p(wx)\tau_x^{-1}\big]^{-1} = \big[\pi_{\mathcal{S}}\big]^{-1},\end{aligned}$$

This proves the following lemma:

Lemma 3.7. *If $\mathcal{S}$ is a singular square then $\big[\pi_{\mathcal{S}^{-1}}\big] = \big[\pi_{\mathcal{S}}\big]^{-1}$ in the fundamental group.*

We also have the following observation about trivial singular squares:

Lemma 3.8. *If $\mathcal{S}$ is a trivial singular square then $[\pi_{\mathcal{S}}] = [e]$ in the fundamental group.*

Proof. By similarity and duality it suffices to suppose that $\mathcal{S} = (x, y, z, w)$ and $x = y \succ\!\!-\!\!\prec z = w$. Then

$$\begin{aligned}\big[\pi_{\mathcal{S}}\big] &= \big[\tau_x p(xy)p(yz)p(zw)p(wx)\tau_x^{-1}\big] = \big[\tau_x p(xx)p(xw)p(ww)p(wx)\tau_x^{-1}\big] \\ &= \big[\tau_x xxp(xw)wwp(wx)\tau_x^{-1}\big] = \big[\tau_x\big]\,\big[p(xw)\big]\,\big[p(xw)\big]^{-1}\big[\tau_x\big]^{-1} = \big[e\big],\end{aligned}$$

completing the proof. □

In the case of nontrivial singular squares, we have the following lemma relating them to singular squares in standard form, using conjugation:

Lemma 3.9. *If $\mathcal{S}$ is a nontrivial singular square then $\big[\pi_{\mathcal{S}}\big]$ is conjugate to $\big[\pi_{\mathcal{S}_0}\big]$ or to $\big[\pi_{\mathcal{S}_0}\big]^{-1}$ for some singular square $\mathcal{S}_0$ in standard form, using the same four vertices.*

Proof. We consider two cases, all other cases being similar. Suppose that $\mathcal{S}_0 = (x, y, z, w)$ is in standard position. If $\mathcal{S} = (y, z, w, x)$ then

$$\begin{aligned}\big[\pi_{\mathcal{S}}\big] &= \big[\tau_y p(yz)p(zw)p(wx)p(xy)\tau_y^{-1}\big] = \big[\tau_y p(yx)\tau_x^{-1}\tau_x p(xy)p(yz)p(zw)p(wx)\tau_x^{-1}\tau_x p(xy)\tau_y^{-1}\big] \\ &= \big[\tau_y p(yx)\tau_x^{-1}\big]\big[\tau_x p(xy)p(yz)p(zw)p(wx)\tau_x^{-1}\big]\big[\tau_x p(xy)\tau_y^{-1}\big] = \gamma^{-1}\big[\pi_{\mathcal{S}_0}\big]\gamma\end{aligned}$$

where $\gamma = \tau_y p(yx)\tau_x^{-1}$, so that $\big[\pi_{\mathcal{S}}\big]$ is a conjugate of $\big[\pi_{\mathcal{S}_0}\big]$. If $\mathcal{S} = (y, x, w, z)$ then then $\mathcal{S}^{-1} = (y, z, w, x)$, so that, by the first case, $\big[\pi_{\mathcal{S}^{-1}}\big] = \gamma^{-1}\big[\pi_{\mathcal{S}_0}\big]\gamma$, for some γ, and hence

$$\big[\pi_{\mathcal{S}}\big] = \big[\pi_{\mathcal{S}^{-1}}\big]^{-1} = \big(\gamma^{-1}\big[\pi_{\mathcal{S}_0}\big]\gamma\big)^{-1} = \gamma^{-1}\big[\pi_{\mathcal{S}_0}\big]^{-1}\gamma,$$

which is a conjugate of $\big[\pi_{\mathcal{S}_0}\big]^{-1}$, completing the proof. □

The following lemma asserts that if we have an existing path π with the vertex e as both source and target, then to form the path $p(\pi)$, we are only inserting duplicates of existing vertices:

Lemma 3.10. *Let π be a path from e to e. Then the paths π and $p(\pi)$ differ only by the insertion of duplicate vertices, whence $\big[p(\pi)\big] = \big[\pi\big]$ in F.*

Proof. We have $\pi = x_1 \dots x_n$ for some sequence of vertices $x_1, \dots, x_n$, where $x_1 = x_n = e$ and, for $i = 1, \dots, n-1$, either $x_i = x_{i+1}$ or there is an edge x_i —— x_{i+1} in Γ, so that, in all cases, either $x_i \longleftrightarrow x_{i+1}$ or $x_i \succ\!\!-\!\!\prec x_{i+1}$ in E. In forming $p(x)$, by definition (9), since $[[e]]$ is the unique inverse of $[[e]]$ in H_e, for $i = 1, \dots, n-1$, there exists $y_i \in E$, such that

$$[[y_i]] = [[x_{i+1} \dots x_n e x_1 \dots x_i]],$$

so that

$$\begin{tikzcd}
x_i \arrow[dr, -|] & & x_{i+1} \\
x_i * y_i \arrow[u, >->] \arrow[r, >-<] & y_i \arrow[ur] \arrow[r, <->] & y_i * x_{i+1} \arrow[u, >->]
\end{tikzcd}$$

whence

$$\begin{tikzcd}
x_i \arrow[dr, |-|] \arrow[d, equal] & & x_{i+1} \arrow[d, equal] \\
x_i * y_i \arrow[r, >-<] & y_i \arrow[ur, <->] \arrow[r, <->] & y_i * x_{i+1}
\end{tikzcd}$$

since the relation $\rightarrowtail$ implies equality in any $\mathcal{D}$-class of E. Because $x_i \longleftrightarrow x_{i+1}$ or $x_i \succ\!\!-\!\!\prec x_{i+1}$, we have $y_i = x_i$ or $y_i = x_{i+1}$. Hence, by definition (10),

$$\begin{aligned}
p(x_1 \dots x_n) \;&=\; p\big(e(x_1 * y_1)\big)\, p\big((x_1 * y_1)y_1\big)\, p\big(y_1(y_1 * x_2)\big)\, p\big((y_1 * x_2)y_2\big) \\
&\qquad\qquad p\big(y_2(y_2 * x_3)\big)\; \dots\; p\big(y_{n-1}(y_{n-1} * x_n)\big)\, p\big((y_{n-1} * x_n)e\big) \\
&=\; p(x_1x_1)p(x_1y_1)p(y_1x_2)\dots p(x_{n-1}y_{n-1})p(y_{n-1}x_n)p(x_nx_n) \\
&=\; x_1x_1x_1y_1y_1x_2 \dots x_{n-1}y_{n-1}y_{n-1}x_nx_n \;,
\end{aligned}$$

the result of adding duplicate vertices to the sequence $x_1 \dots x_n$, and the lemma follows. $\square$

The next lemma is the main idea that leads to the proof of Theorem 3.6. Let N be the normal closure in the fundamental group F of the set

$$\big\{\, [\pi_{\mathcal{S}}] \;\big|\; \mathcal{S} \text{ is a singular square in standard form} \,\big\} \;.$$

Lemma 3.11. *Suppose that $w_1, w_2 \in E^+$ and $f, g \in E$ where $f * g$ is a basic product in the biordered set and*

$$\big[\![w_1fgw_2]\!\big] \;=\; \big[\![e]\!\big]$$

in E. Then

$$\big[p(w_1fgw_2)\big]\,\big[p(w_1(f * g)w_2)\big]^{-1} \;\in\; N \;.$$

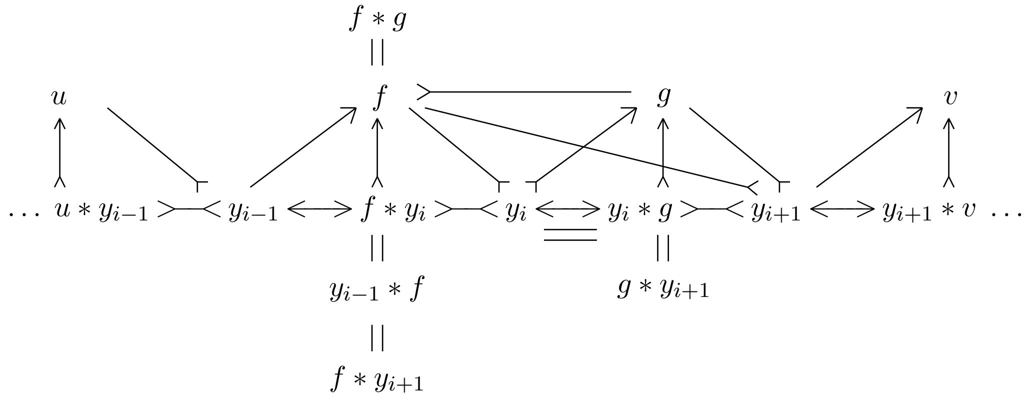
Proof. In what follows we assume that w_1 and w_2 are nonempty, but the argument can be adjusted easily to cover the cases when w_1 or w_2 are empty. Suppose that the last letter of w_1 is u and the first letter of w_2 is v. We have, in the construction of $p(w_1fgw_2)$, using (9) and (10),

$$\begin{tikzcd}
u \arrow[dr, -|] & & f \arrow[dr, -|] & & g \arrow[dr, -|] & & v \\
\dots\; u * y_{i-1} \arrow[u, >->] \arrow[r, >-<] & y_{i-1} \arrow[ur] \arrow[r, <->] & f * y_i \arrow[u, >->] \arrow[r, >-<] \arrow[d, equal] & y_i \arrow[ur] \arrow[r, <->] & y_i * g \arrow[u, >->] \arrow[r, >-<] \arrow[d, equal] & y_{i+1} \arrow[ur] \arrow[r, <->] & y_{i+1} * v \;\dots \arrow[u, >->] \\
& & y_{i-1} * f & & g * y_{i+1} & &
\end{tikzcd}$$

for some connected sequence associated with w_1fgw_2, with, in particular,

$$\big[\![y_{i-1}]\!\big] \;=\; \big[\![fgw_2ew_1]\!\big]\,, \quad \big[\![y_i]\!\big] \;=\; \big[\![gw_2ew_1f]\!\big]\,, \quad \big[\![y_{i+1}]\!\big] \;=\; \big[\![w_2ew_1fg]\!\big]\;. \qquad (12)$$

There are four ways an arrow can exist between f and g. Suppose first that $f \succ\!\!\!\longrightarrow g$, so that $f * g = f$. The previous diagram then becomes



noting that, by (12),

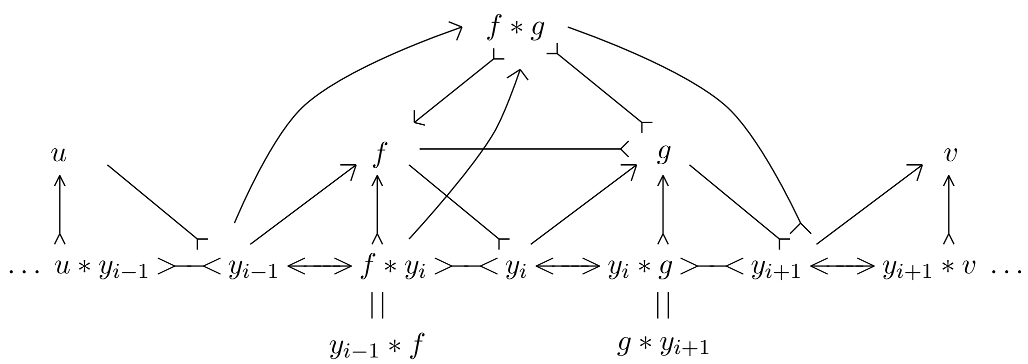
$$\big[\big[y_{i-1} * f\big]\big] \;=\; \big[\big[fgw_2ew_1f\big]\big] \;=\; \big[\big[fw_2ew_1fg\big]\big] \;=\; \big[\big[f * y_{i-1}\big]\big] \,.$$

If follows, from the previous diagram, and since $y_i \mathrel{>\!\!-\!\!<} y_{i+1}$, that

$$\begin{aligned} p(w_1fgw_2) \;&=\; \dots\ p\big((u * y_{i-1})y_{i-1}\big)\, p\big(y_{i-1}(f * y_i)\big)\, p\big((f * y_i)y_i\big) \\ &\qquad\qquad p\big(y_i(y_i * g)\big)\, p\big((y_i * g)y_{i+1}\big)\, p\big(y_{i+1}(y_{i+1} * v)\big)\ \dots \\ &=\; \dots\ p\big((u * y_{i-1})y_{i-1}\big)\, p\big(y_{i-1}(f * y_{i+1})\big)\, p\big((f * y_{i+1})y_i\big) \\ &\qquad\qquad p\big(y_iy_i\big)\, p\big(y_iy_{i+1}\big)\, p\big(y_{i+1}(y_{i+1} * v)\big)\ \dots \\ &\sim\; \dots\ p\big((u * y_{i-1})y_{i-1}\big)\, p\big(y_{i-1}(f * y_{i+1})\big)\, p\big((f * y_{i+1})y_{i+1}\big) p\big(y_{i+1}(y_{i+1} * v)\big)\ \dots \\ &=\; p\big(w_1(f * g)w_2\big)\,. \end{aligned}$$

Hence $\big[p(w_1fgw_2)\big] = \big[p(w_1(f * g)w_2\big]$, so that, trivially, $\big[p(w_1fgw_2)\big]\,\big[p(w_1(f * g)w_2)\big]^{-1} \in N$, completing the analysis for this case. The case $f \longleftarrow g$ is dealt with by a dual argument.

Suppose next that $f \mathrel{-\!\!<} g$, so that the first diagram above may be supplemented by a (possibly) new element $f * g$ and some additional arrows:



By (12),

$$\big[\big[y_{i-1} * (f * g)\big]\big] \;=\; \big[\big[fgw_2ew_1fg\big]\big] \;=\; \big[\big[(f * g)w_2ew_1fg\big]\big] \;=\; \big[\big[(f * g) * y_{i+1}\big]\big] \,.$$

But idempotents are unique in their congruence classes in F_E, so that

$$y_{i-1} * (f * g) \;=\; (f * g) * y_{i+1}\ ,$$

giving the following diagram:

$$\begin{tikzcd}
 & f * y_i \arrow[d, phantom, "||"] & f * (y_i * g) \arrow[d, phantom, "||"] \\
y_{i-1} \arrow[r, leftrightarrow] & y_{i-1} * f \arrow[r, leftrightarrow] \arrow[d, tail] & y_{i-1} * (f * g) = (f * g) * y_{i+1} \arrow[d, tail] \\
 & y_i \arrow[r, leftrightarrow] & y_i * g = g * y_{i+1} \arrow[d, tail] \\
 & & y_{i+1}
\end{tikzcd}$$

Put $a = f * y_i$, $b = f * (y_i * g)$, $c = y_i * g$ and $d = y_i$. Then we obtain the up-down singular square $\mathcal{S} = (a, b, c, d)$ associated with the following diagram:

$$\begin{tikzcd}
 & f \arrow[dl, tail] \arrow[dr, tail] \arrow[ddl, tail] \arrow[ddr, tail] & \\
f * d = a \arrow[rr, leftrightarrow] \arrow[d, tail] \arrow[ur] & & b = f * c \arrow[d, tail] \arrow[ul] \\
d \arrow[rr, leftrightarrow] & & c
\end{tikzcd}$$

We have

$$\begin{aligned}
p\big(w_1 f g w_2\big) \;&=\; \ldots\ p\big((u * y_{i-1})y_{i-1}\big)\, p\big(y_{i-1}(f * y_i)\big)\, p\big((f * y_i)y_i\big) \\
&\qquad\qquad p\big(y_i(y_i * g)\big)\, p\big((y_i * g)y_{i+1}\big)\, p\big(y_{i+1}(y_{i+1} * v)\big)\ \ldots \\
&=\; \pi_1\, p\big(y_{i-1}a\big)\, p\big(ad\big)\, p\big(dc\big)\, p\big(cy_{i+1}\big)\, \pi_2
\end{aligned}$$

for some paths π_1 and π_2, and

$$\begin{aligned}
p\big(w_1(f * g)w_2\big) \;&=\; \ldots\ p\big((u * y_{i-1})y_{i-1}\big)\, p\Big(y_{i-1}\big(y_{i-1} * (f * g)\big)\Big) \\
&\qquad\qquad p\Big(\big((f * g) * y_{i+1}\big)y_{i+1}\Big)\, p\big(y_{i+1}(y_{i+1} * v)\big)\ \ldots \\
&=\; \pi_1\, p\big(y_{i-1}b\big)\, p\big(by_{i+1}\big)\, \pi_2\ ,
\end{aligned}$$

so that

$$\begin{aligned}
&p\big(w_1 fgw_2\big)\, p\big(w_1(f*g)w_2\big)^{-1} \\
&\quad=\ \pi_1\, p\big(y_{i-1}a\big)\, p\big(ad\big)\, p\big(dc\big)\, p\big(cy_{i+1}\big)\, \pi_2\, \pi_2^{-1}\, p\big(y_{i+1}b\big)\, p\big(by_{i-1}\big)\, \pi_1^{-1} \\
&\quad\sim\ \pi_1\, p\big(y_{i-1}a\big)\, p\big(ad\big)\, p\big(dc\big)\, p\big(cb\big)\, p\big(by_{i-1}\big)\, \pi_1^{-1} \\
&\quad\sim\ \pi_1\, p\big(y_{i-1}a\big)\, p\big(ad\big)\, p\big(dc\big)\, p\big(cb\big)\, p\big(ba\big)\, p\big(ay_{i-1}\big)\, \pi_1^{-1} \\
&\quad\sim\ \pi_1\, p\big(y_{i-1}a\big)\, \tau_a^{-1} \Big(\tau_a\, p\big(ad\big)\, p\big(dc\big)\, p\big(cb\big)\, p\big(ba\big)\, \tau_a^{-1}\Big)\, \tau_a\, p\big(ay_{i+1}\big)\, \pi_1^{-1} \\
&\quad=\ \pi_3\, \pi_{\mathcal{S}^{-1}}\, \pi_3^{-1}\,,
\end{aligned}$$

where $\pi_3 = \pi_1\, p\big(y_{i-1}a\big)\, \tau_a^{-1}$. Put

$$\alpha = \big[\, p\big(w_1 fgw_2\big)\big]\, \big[p\big(w_1(f*g)w_2\big)\big]^{-1}\,.$$

The above shows that

$$\alpha \ =\ \big[\, p\big(w_1 fgw_2\big)\, p\big(w_1(f*g)w_2\big)^{-1}\big] \ =\ \big[\pi_3\big]\, \big[\pi_{\mathcal{S}^{-1}}\big]\, \big[\pi_3\big]^{-1}$$

is a conjugate of $\big[\pi_{\mathcal{S}^{-1}}\big]$. If $\mathcal{S}^{-1}$ is trivial, then by Lemma 3.8 we have $\big[\mathcal{S}^{-1}\big] = \big[e\big]$, so that certainly $\alpha = \big[e\big]$, and furthermore α is the trivial element of F, so it lies in N. If $\mathcal{S}^{-1}$ is nontrivial, then, by Lemma 3.9, we have that $\big[\pi_{\mathcal{S}^{-1}}\big]$ is a conjugate of $\big[\pi_{\mathcal{S}_0}\big]$ or $\big[\pi_{\mathcal{S}_0}\big]^{-1}$ for some singular square $\mathcal{S}_0$ in standard form, and it follows that α also is a conjugate of $\big[\pi_{\mathcal{S}_0}\big]$ or $\big[\pi_{\mathcal{S}_0}\big]^{-1}$, and so lies in N, completing the analysis for this case. The case $f \Longrightarrow g$ is dealt with by a dual argument, completing the proof of the lemma. □

We can now prove Theorem 3.6, leading to a succinct description of the kernel of φ.

Proof of Theorem 3.6. Let N be the normal closure of the set

$$\big\{\, [\pi_{\mathcal{S}}] \;\big|\; \mathcal{S} \text{ is a singular square in standard form} \,\big\}\,.$$

By Lemma 3.5, $N \subseteq \ker\varphi$. It remains to prove the reverse set containment. Let π be a path that starts and finishes with e in Γ such that $[\pi] \in \ker\varphi$, that is, $[[\pi]] = [[e]]$ in F_E. We argue by induction on the number of elementary transitions in F_E taking the word π to the word e. If the number of transitions is zero then $\pi = e$ and so $[\pi] = [e] \in N$, which starts an induction. Suppose, as inductive hypothesis, that if $w \in E^+$ and $[[w]] = [[e]]$ in F_E with fewer than n transitions for some positive integer n, then $[p(w)] \in N$. Suppose that we apply one transition to w, so either w becomes w' (a contraction), or w' becomes w (an expansion), where

$$w = w_1 fgw_2 \qquad \text{and} \qquad w' = w_1 fgw_2\,,$$

for some $w_1, w_2 \in E^+$ and $f, g \in E$ such that $f*g$ is a basic product. By Lemma 3.11,

$$\big[p\big(w_1 fgw_2\big)\big]\big[p\big(w_1(f*g)w_2\big)\big]^{-1} \ \in\ N\,.$$

In either case, $\big[p(w')p(w)^{-1}\big] = \big[p(w')\big]\, \big[p(w)\big]^{-1} = \Big(\big[p(w)\big]\, \big[p(w')\big]^{-1}\Big)^{-1} \in N$, so that

$$\big[p(w')\big] \ =\ \big[p(w')p(w)^{-1}p(w)\big] \ =\ \big[p(w')p(w)^{-1}\big]\, \big[p(w)\big] \ \in\ N\,,$$

establishing the inductive step. Hence, by Lemma 3.10, $[\pi] = [p(\pi)] \in N$, completing the proof of the theorem. □

Corollary 3.12. *The maximal subgroup H_e of $[[e]]$ in F_E is isomorphic to F/N where N is the normal closure in F of the set $\big\{\, [\pi_{\mathcal{S}}] \;\big|\; \mathcal{S}$ is a singular square in standard form$\big\}$.*

Proof. By Theorem 3.6, the kernel of $\varphi : F \to H_e$ is N, and, by Lemma 3.3, φ is onto, so $H_e \cong F/\ker\varphi = F/N$, completing the proof of the corollary. $\square$

We can now convert this result into an explicit presentation for the maximal subgroup associated with $e \in E$ in F_E. Recall that the fundamental group F is freely generated by

$$\Sigma \;=\; \left\{ \left[\tau_a(ab)\tau_b^{-1}\right] \;\middle|\; a \text{——} b \text{ is an edge of } \Gamma \text{ not in } T \right\} ,$$

where T is a given chosen spanning tree of Γ. We form an alphabet A in a bijective correspondence with Σ:

$$A \;=\; \left\{ ab \;\middle|\; a \text{——} b \text{ is an edge of } \Gamma \text{ not in } T \right\} .$$

The bijection maps

$$\left[\tau_a(ab)\tau_b^{-1}\right] \;\mapsto\; ab ,$$

where the word ab is regarded as a single directed edge. We use this bijection to form relators, induced by paths that are used to generate the kernel of φ, which is N, the normal closure of the set

$$\left\{ \left[\pi_{\mathcal{S}}\right] \;\middle|\; \mathcal{S} \text{ is a singular square in standard form} \right\} .$$

Consider a singular square $\mathcal{S} = (x, y, z, w)$ in standard form

$$\begin{tikzcd}
x = x_{ij} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow, tail] & x_{i\ell} = y \arrow[d, leftrightarrow, tail] \\
w = x_{kj} \arrow[r, leftrightarrow] & x_{k\ell} = z
\end{tikzcd}$$

We convert $\left[\pi_{\mathcal{S}}\right]$ into a word over the alphabet A by traversing $\mathcal{S}$ clockwise, writing down, in order, the edges of Γ not in T (ignoring edges in T), regarding these edges as directed. Formally, if these directed edges (not in T) in the clockwise traversal of $\mathcal{S}$ are

$$a_1b_1 \,,\; a_2b_2 \,,\; \ldots \,,\; a_mb_m \,,$$

then

$$\pi_{\mathcal{S}} \;=\; \tau_x p(xy)p(yz)p(zw)p(wx)\tau_x^{-1} \;\sim\; \tau_{a_1}(a_1b_1)\tau_{b_1}^{-1}\tau_{a_2}(a_2b_2)\tau_{b_2}^{-1}\ldots\tau_{a_m}(a_mb_m)\tau_{b_m}^{-1}$$

so that $\left[\pi_{\mathcal{S}}\right]$ corresponds, under the above bijection of alphabets, to the word (over A)

$$R_{\mathcal{S}} \;=\; (a_1b_1)(a_2b_2)\ldots(a_mb_m) \,.$$

Put

$$\mathcal{R} \;=\; \left\{ R_{\mathcal{S}} \;\middle|\; \mathcal{S} \text{ is singular in standard form} \right\} .$$

Then, using these identifications, Corollary 3.12 yields finally the following result:

Theorem 3.13. $H_e \;\cong \langle A \mid \mathcal{R}\rangle$.

In examples, it is typical that we can simplify the presentation of Theorem 3.13 using Tietze transformations to recognise this group quickly as some familiar group. It should also be remarked that any given relator that arises by working through the above process where $\mathcal{S}$ is in standard form may be replaced by any of its conjugates, for example, working with the same singular square, but writing down the succession of edges not in T as one traverses the configuration, starting at any of the vertices, and working either clockwise or anticlockwise. The reason why we specify the standard form in the above results is to guarantee uniqueness of the given relator, with respect to any given singular square configuration, and also in case one wishes to automate the processing of information related to the presentation.

4. An example with maximal subgroups that are cyclic of order two

In this section we create a band, and its associated biordered set, with sixteen elements, which is an ideal extension of a 4×3 rectangular band by a four element left zero semigroup. Its biordered set E will have the property that F_E has maximal subgroups that are cyclic of order one or two. We also describe F_E completely as an ideal extension of a Rees matrix semigroup by a left zero semigroup.

Let $I = \{1, 2, 3, 4\}$ and $J = \{1, 2, 3\}$ and put $\mathcal{R} = \{x_{ij} \mid i \in I\,,\quad j \in J\}$, which becomes a rectangular band with formal multiplication

$$x_{ij}x_{k\ell} \;=\; x_{i\ell}\;.$$

To simplify notation, for an element $\alpha = (\lambda, \rho) \in \mathcal{T}^*(I) \times \mathcal{T}(J)$, we write

$$\alpha \;=\; (\lambda 1\; \lambda 2\; \lambda 3\; \lambda 4 \mid 1\rho\; 2\rho\; 3\rho)$$

with the images of λ (with left action) to the left of the vertical line and the images of ρ (with right action) to the right of the vertical line. With this notation, we put

$$e \;=\; (1\,2\,2\,1 \mid 1\,2\,1)\,, \quad f \;=\; (4\,2\,2\,4 \mid 1\,2\,2)\,, \quad g \;=\; (4\,3\,3\,4 \mid 1\,2\,1)\,, \quad h \;=\; (1\,3\,3\,1 \mid 1\,2\,2)\,,$$

and $\mathcal{S} = \{e, f, g, h\}$. Then $\mathcal{S}$ is a left zero semigroup. With multiplication defined in Corollary 2.6, the set $\mathcal{S}\dot{\cup}\mathcal{R}$ becomes a band with two egg-boxes, as depicted below, which becomes an ideal extension of $\mathcal{R}$ by $\mathcal{S}$.

e
f
g
h

x_{11}	x_{12}	x_{13}
x_{21}	x_{22}	x_{23}
x_{31}	x_{32}	x_{33}
x_{43}	x_{42}	x_{41}

We may form the biordered set $E = E(\mathcal{S} \cup \mathcal{R}) = \mathcal{S} \cup \mathcal{R}$. Multiplying elements of $\mathcal{R}$ by elements of $\mathcal{S}$ on the left and right become left-actions on the first subscript and right-actions on the second subscript of each $x_{ij} \in \mathcal{R}$ respectively. For example, using the left and right halves respectively of the actions of $e = (1\ 2\ 2\ 1 \mid 1\ 2\ 1)$, we have

$$ex_{13} \;=\; x_{13}\,, \quad x_{13}e \;=\; x_{11} \;=\; ex_{11} \;=\; x_{11}e\,, \quad x_{31}e \;=\; x_{31}\,, \quad ex_{31} \;=\; x_{21} \;=\; x_{21}e \;=\; ex_{21}\,,$$

yielding the following arrow diagrams and nontrivial basic products in E involving e:

$$\begin{tikzcd} & e \\ x_{13} \arrow[ur] \arrow[r, leftrightarrow] & x_{13} * e = x_{11} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & e \\ x_{31} \arrow[ur, dash] \arrow[r, dash] & e * x_{31} = x_{21} \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} & e \\ x_{23} \arrow[ur] \arrow[r, leftrightarrow] & x_{23} * e = x_{21} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & e \\ x_{32} \arrow[ur, dash] \arrow[r, dash] & e * x_{32} = x_{22} \arrow[u] \end{tikzcd}$$

$$\begin{tikzcd} & e \\ x_{41} \arrow[ur, dash] \arrow[r, dash] & e * x_{41} = x_{11} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & e \\ x_{42} \arrow[ur, dash] \arrow[r, dash] & e * x_{42} = x_{12} \arrow[u] \end{tikzcd}$$

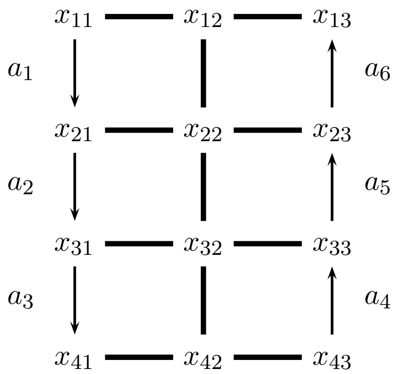
One may check other basic products, obtaining the following skeleton of E, with all nontrivial basic products indicated, with askerisks suppressed, to keep the diagram uncluttered:

$$\begin{tikzcd} e \arrow[d, dash] \\ f \arrow[d, dash] \\ g \arrow[d, dash] \\ h \end{tikzcd}$$

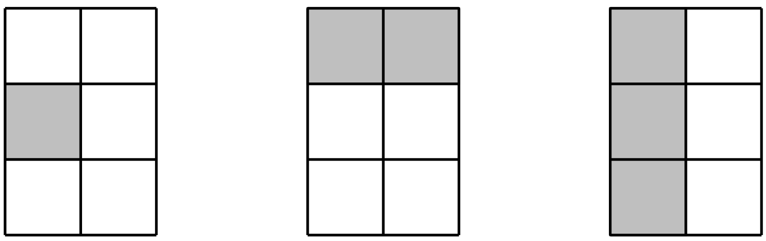
$$\begin{tikzcd}[column sep=small] & & x_{13}h = ex_{42} = hx_{42} \\ hx_{41} = x_{13}e = ex_{41} = x_{11} \arrow[r, leftrightarrow] \arrow[d, dash] & x_{12} \arrow[r, leftrightarrow] \arrow[d, dash] \arrow[ur, equal] & x_{13} \\ & & x_{23}f = ex_{32} = fx_{32} \\ fx_{31} = x_{23}e = ex_{31} = x_{21} \arrow[r, leftrightarrow] \arrow[d, dash] & x_{22} \arrow[r, leftrightarrow] \arrow[d, dash] \arrow[ur, equal] & x_{23} \\ & & gx_{22} = x_{33}h = hx_{22} \\ gx_{21} = x_{33}g = hx_{21} = x_{31} \arrow[r, leftrightarrow] \arrow[d, dash] & x_{32} \arrow[r, leftrightarrow] \arrow[d, dash] \arrow[ur, equal] & x_{33} \\ & & x_{43}f = fx_{12} = gx_{12} \\ gx_{11} = fx_{11} = x_{43}g = x_{41} \arrow[r, leftrightarrow] & x_{42} \arrow[r, leftrightarrow] \arrow[ur, equal] & x_{43} \end{tikzcd}$$

We now apply our apparatus to calculate the maximal subgroups of F_E in the lower $\mathcal{D}$-class of E, which turn out to be cyclic of order two. The lower $\mathcal{D}$-class forms the following simple graph Γ. We have chosen a spanning tree of Γ, consisting of all of the vertices and eleven edges, which have been thickened below. For each edge not in the spanning tree we have added a label and assigned a direction (arbitrarily, but then fixed for what follows).

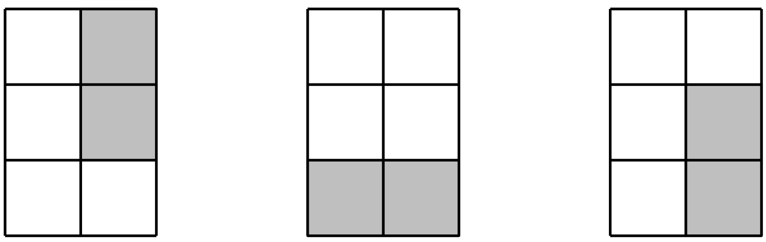
Γ :



Put $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, which, by Theorem 3.13, becomes the generating set in a presentation of the maximal subgroup H_e, for any given element e in Γ The relators in the presentation are traversals, as words over A, of all rectangles associated with nontrivial singular squares in the $\mathcal{D}$-class. One can check that there are exactly six distinct rectangles produced by the nontrivial singular squares, indicated below by shadings of Γ, together with traversals, which can be made using an arbitrary choice of starting vertex and orientation in each case, as words over the alphabet $A \cup A^{-1}$.



a_2 a_1a_6 $a_1a_2a_3$



a_5a_6 a_3a_4 a_4a_5

By Theorem 3.13, the maximal subgroups associated with Γ are isomorphic to the following, which simplifies using Tietze transformations to the cyclic group of order two:

$$\begin{aligned}
&\langle a_1, a_2, a_3, a_4, a_5, a_6 \mid a_2\,,\ a_1a_6\,,\ a_1a_2a_3\,,\ a_5a_6\,,\ a_3a_4\,,\ a_4a_5\rangle \\
&\qquad \cong\ \langle a_1, a_3, a_4, a_5, a_6 \mid a_1a_6\,,\ a_1a_3\,,\ a_5a_6\,,\ a_3a_4\,,\ a_4a_5\rangle \\
&\qquad \cong\ \langle a_1, a_3, a_4, a_5 \mid a_1a_3\,,\ a_5a_1^{-1}\,,\ a_3a_4\,,\ a_4a_5\rangle\ \cong\ \langle a_1, a_3, a_4 \mid a_1a_3\,,\ a_3a_4\,,\ a_4a_1\rangle \\
&\qquad \cong\ \langle a_1, a_3 \mid a_1a_3\,,\ a_3^{-1}a_1\rangle\ \cong\ \langle a_1 \mid a_1^2\rangle\ \cong\ C_2\,.
\end{aligned}$$

Our aim now is to prove that F_E is isomorphic to the ideal extension of a Rees matrix semigroup by a left-zero semigroup described above in Example 2.7. Consider the Rees matrix semigroup

$$\mathcal{M}\ =\ \mathcal{M}(G, I\times J, P)\,,$$

where $G = \langle a\rangle = \{1, a\}$ is a cyclic group of order two, $I = \{1,2,3,4\}$, $J = \{1,2,3\}$, and

$$P\ =\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & a & a & 1 \\ 1 & 1 & a & a \end{bmatrix}\,.$$

Let $L : \mathcal{S}\to M^*(I)$ and $R : \mathcal{S}\to M^(J)$ be the homomorphisms defined in Example 2.7, so that we may form the semigroup $\mathcal{S}\dot\cup\mathcal{M}$. One may check, using the skeleton and nontrivial basic products given above for E that, as biordered sets,

$$E(\mathcal{S}\dot\cup\mathcal{M})\ \cong\ E\ =\ E(\mathcal{S}\dot\cup\mathcal{R})\,.$$

Observe that $\mathcal{S}$ contains 4 elements and $\mathcal{M}$ contains $4\times 3\times 2 = 24$ elements, so that $\mathcal{S}\dot\cup\mathcal{M}$ contains exactly 28 elements. The maximal subgroups of F_E associated with the top $\mathcal{D}$-class are trivial, since $\mathcal{S}$ is a left zero band, so that all singular squares are trivial. Since the maximal subgroups associated with Γ are cyclic or order two, it follows that F_E also has 28 elements. But $\mathcal{S}\dot\cup\mathcal{M}$ is idempotent-generated (because the sandwich matrix P contains the generator a), so that $\mathcal{S}\dot\cup\mathcal{M}$ is a homomorphic image of F_E. Since these semigroups have the same finite size, they must be isomorphic:

$$F_E\ \cong\ \mathcal{S}\dot\cup\mathcal{M}\,.$$

5. An example with maximal subgroups that are free abelian of rank two

We reconstruct an example discovered by Dolinka [10] of a biordered set E with twenty elements, split into a lower $\mathcal{D}$-class with sixteen elements organised as a four by four square, and an upper $\mathcal{D}$-class with four elements organised as a two by two square. In fact E is the biordered set of a band, which may be regarded as an ideal extension of a sixteen element rectangular band by a four element rectangular band. The maximal subgroups in the corresponding $\mathcal{D}$-classes of F_E are free abelian of rank two, for the lower $\mathcal{D}$-class, and infinite cyclic, for the upper $\mathcal{D}$-class. Below, we use the ideal extension machinery of Proposition 2.5 and Corollary 2.6 to construct the entire semigroup F_E up to isomorphism.

Let $I = \{1,2,3,4\}$ and put $\mathcal{R} = \{x_{ij} \mid i,j\in I\}$, with rectangular band multiplication:

$$x_{ij}x_{k\ell}\ =\ x_{i\ell}\,.$$

For an element $\alpha = (\lambda,\rho)\in\mathcal{T}^*(I)\times\mathcal{T}(I)$, we write

$$\alpha\ =\ (\lambda 1\ \lambda 2\ \lambda 3\ \lambda 4 \mid 1\rho\ 2\rho\ 3\rho\ 4\rho)$$

with the images of λ (with left action) to the left of the vertical line and the images of ρ (with right action) to the right of the vertical line. With this notation, put

$$e = (1\,2\,2\,1 \mid 2\,2\,3\,3)\,,\quad f = (1\,2\,2\,1 \mid 1\,1\,4\,4)\,,\quad h = (4\,3\,3\,4 \mid 2\,2\,3\,3)\,,\quad g = (4\,3\,3\,4 \mid 1\,4\,4)\,,$$

and $\mathcal{S} = \{e, f, g, h\}$. Then $\mathcal{S}$ is a rectangular band (and the definitions of e, f, h and g are in fact motivated by the actions that appear in [10]). With the multiplication defined in Corollary 2.6, the set $\mathcal{S}\dot{\cup}\mathcal{R}$ becomes a band with two egg-boxes, the upper egg-box being the $\mathcal{D}$-class consisting of the rectangular band $\mathcal{S}$, and the lower egg-box being the $\mathcal{D}$-class consisting of the elements of the rectangular band $\mathcal{R}$.

e	f
h	g

x_{11}	x_{12}	x_{13}	x_{14}
x_{21}	x_{22}	x_{23}	x_{24}
x_{31}	x_{32}	x_{33}	x_{34}
x_{43}	x_{42}	x_{41}	x_{44}

We may form the biordered set $E = E(\mathcal{S} \cup \mathcal{R}) = \mathcal{S} \cup \mathcal{R}$. Multiplying elements of $\mathcal{R}$ by elements of $\mathcal{S}$ on the left and right become left-actions on the first subscript and right-actions on the second subscript of each $x_{ij} \in \mathcal{R}$ respectively. Using the left and right halves respectively of the actions of $e = (1\ 2\ 2\ 1 \mid 2\ 2\ 3\ 3)$, we have

$$ex_{11} = x_{11}\,, \quad x_{11}e = x_{12} = ex_{12} = x_{12}e\,, \quad x_{43}e = x_{43}\,, \quad ex_{43} = x_{13} = x_{13}e = ex_{13}\,,$$

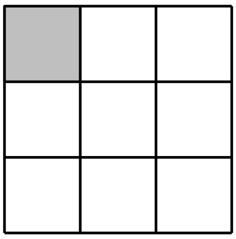
yielding, for example, the following arrow diagrams and basic products:

$$\begin{tikzcd} & e \\ x_{11} \arrow[ur] \arrow[r, leftrightarrow] & x_{11} * e = x_{12} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} & e \\ x_{43} \arrow[ur, dash] \arrow[r, dash] & e * x_{43} = x_{13} \arrow[u] \end{tikzcd}$$

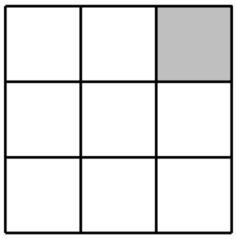
A full description of the biordered set and the singular squares appears in [10], so we will not reproduce them here, but work instead towards applying our machinery for calculating the maximal subgroups associated with the lower $\mathcal{D}$-class, which forms the following undirected simple graph Γ. We have chosen a spanning tree of Γ, consisting of fifteen edges, which have been thickened below: We have added directions (which can be arbitrarily chosen) and labels to the edges not in the spanning tree:

$$\Gamma : \quad \begin{tikzcd} x_{11} \arrow[r, dash, very thick] \arrow[d, dash, very thick] & x_{12} \arrow[r, "a_4"'] \arrow[d, dash, very thick] & x_{13} \arrow[r, dash, very thick] \arrow[d, dash, very thick] & x_{14} \arrow[d, dash, very thick] \\ x_{21} \arrow[r, "a_3"'] \arrow[d, dash, very thick] & x_{22} \arrow[d, dash, very thick] & x_{23} \arrow[l, "a_5"] \arrow[r, "a_7"'] \arrow[d, dash, very thick] & x_{24} \arrow[d, dash, very thick] \\ x_{31} \arrow[d, dash, very thick] & x_{32} \arrow[l, "a_2"] \arrow[r, "a_6"'] \arrow[d, dash, very thick] & x_{33} \arrow[r, "a_8"'] \arrow[d, dash, very thick] & x_{34} \arrow[d, dash, very thick] \\ x_{41} \arrow[r, "a_1"'] & x_{42} \arrow[r, dash, very thick] & x_{43} & x_{44} \arrow[l, "a_9"] \end{tikzcd}$$

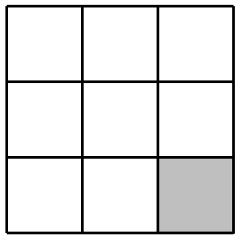
Put $A = \{a_1, a_2, \ldots, a_9\}$. By Theorem 3.13, A becomes the generating set in a presentation of the maximal subgroup H_e, for any given element e in Γ. The relators in the presentation are traversals, as words over the alphabet A, of all of the rectangles associated with nontrivial singular squares associated with Γ. There are exactly eight such nontrivial squares (see [10]), indicated below using shadings, each of which is accompanied by a traversal in standard form.



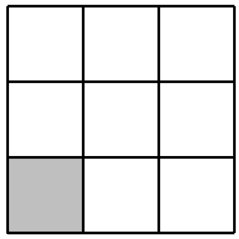
a_3^{-1}



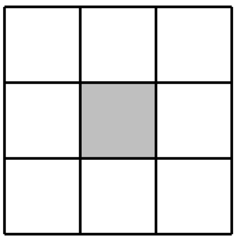
a_7^{-1}



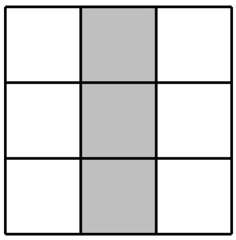
$a_8 a_9$



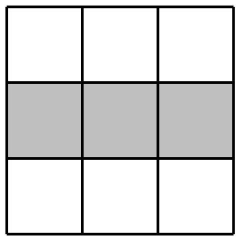
$a_2^{-1} a_1^{-1}$



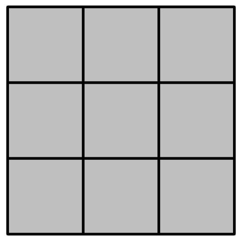
$a_5^{-1} a_6^{-1}$



a_4



$a_3 a_5^{-1} a_7 a_8^{-1} a_6^{-1} a_2$



$a_4 a_9 a_1^{-1}$

Theorem 3.13 then tells us that the maximal subgroups associated with Γ are isomorphic to the following, which simplifies using Tietze transformations to the free abelian group of rank two (in accordance with [10]):

$$
\begin{aligned}
&\langle a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 \mid a_3^{-1}\,,\ a_7^{-1}\,,\ a_8 a_9\,,\ a_2^{-1} a_1^{-1}\,, a_5^{-1} a_6^{-1}\,,\ a_4\,, \\
&\qquad\qquad\qquad\qquad\qquad\qquad a_3 a_5^{-1} a_7 a_8^{-1} a_6^{-1} a_2\,,\ a_4 a_9 a_1^{-1} \rangle \\
&\quad \cong\ \langle a_1, a_2, a_5, a_6, a_8, a_9 \mid a_8 a_9\,,\ a_2^{-1} a_1^{-1}\,,\ a_5^{-1} a_6^{-1}\,,\ a_5^{-1} a_8^{-1} a_6^{-1} a_2\,,\ a_9 a_1^{-1} \rangle \\
&\quad \cong\ \langle a_1, a_2, a_5, a_6, a_8 \mid a_2^{-1} a_1^{-1}\,,\ a_5^{-1} a_6^{-1}\,,\ a_5 a_8^{-1} a_6^{-1} a_2\,,\ a_8^{-1} a_1^{-1} \rangle \\
&\quad \cong\ \langle a_1, a_2, a_5, a_6 \mid a_2^{-1} a_1^{-1}\,,\ a_5^{-1} a_6^{-1}\,,\ a_5 a_1^{-1} a_6^{-1} a_2 \rangle\ \cong\ \langle a_1, a_2, a_5 \mid a_2^{-1} a_1^{-1}\,,\ a_5^{-1} a_1 a_5 a_2 \rangle \\
&\quad \cong\ \langle a_1, a_5 \mid a_5^{-1} a_1 a_5 a_1^{-1} \rangle\ \cong\ C_\infty \times C_\infty\,.
\end{aligned}
$$

Our aim now is to prove that F_E is isomorphic to an ideal extension of one Rees matrix semigroup by another as described above in Example 2.8. Consider the Rees matrix semigroup

$$\mathcal{M}\ =\ \mathcal{M}(G, I \times I, P)\,,$$

where $G = \langle a, b \rangle \cong C_\infty \times C_\infty$ is free abelian of rank two, generated by a and b, $I = \{1, 2, 3, 4\}$, and

$$P\ =\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & b & b \\ 1 & a & ab & a \\ a & a & a & 1 \end{bmatrix}\,.$$

Let $\mathcal{S}$ be the following Rees matrix semigroup

$$\mathcal{S} \;=\; \mathcal{M}(H, J \times J, Q)\ ,$$

where $H = \langle b \rangle \cong C_\infty$ is the infinite cyclic subgroup of G generated by b, $J = \{1, 2\}$ and

$$Q \;=\; \begin{bmatrix} 1 & 1 \\ 1 & b \end{bmatrix}\ .$$

Define homomorphisms $R : \mathcal{S} \to M(I)$ and $L : \mathcal{S} \to M^*(I)$ as in Example 2.8. As explained in that example, the adjoint property holds and, by Proposition 2.5, we may form the semigroup

$$\mathcal{S}\dot{\cup}\mathcal{M}\ .$$

It is routine to check that the following biordered sets are isomorphic

$$E(\mathcal{S}\dot{\cup}\mathcal{M}) \;\cong\; E \;=\; E(\mathcal{S}\dot{\cup}\mathcal{R})\ .$$

But $\mathcal{S}\dot{\cup}\mathcal{M}$ is idempotent-generated (because the sandwich matrices contain generators of the respective groups), so that $\mathcal{S}\dot{\cup}\mathcal{M}$ is a homomorphic image of F_E. If the homomorphism is not onto, since the biordered sets are isomorphic, then there would have to be collapse somewhere involving the maximal subgroups. But this is impossible, because the maximal subgroups in the upper $\mathcal{D}$-class of F_E are infinite cyclic, as are the maximal subgroups of $\mathcal{S}$, and the maximal subgroups in the lower $\mathcal{D}$-class of F_E, we have just shown to be free abelian of rank two, as are the maximal subgroups of $\mathcal{M}$. This proves that the semigroups are isomorphic:

$$F_E \;\cong\; \mathcal{S}\dot{\cup}\mathcal{M}\ .$$

References

[1] Brittenham, M., Margolis, S.W. and Meakin, J. Subgroups of free idempotent generated semigroups need not be free. *J. Algebra.* 321 (2009), 3026–3042.

[2] Clifford, A.H. Extensions of semigroups. *Trans. Am. Math. Soc.* 68 (1950), 165–173.

[3] Clifford, A.H. and Preston, G.B. *The Algebraic Theory of Semigroups, vol. I.* Math. Surveys of the American Math. Soc. 7, Providence, R.I., 1961.

[4] Dandan, Y., Dolinka, I. and Gould, V. Free idempotent generated semigroups and endomorphism monoids of free G-acts. *J. Algebra.* 429 (2015), 133–176.

[5] Dandan, Y., Dolinka, I. and Gould, V. A group-theoretical interpretation of the word problem for free idempotent generated semigroups. *Adv. Math.* 345 (2019), 998–1041.

[6] Dandan, Y. and Gould, V. Every group is a maximal subgroup of a naturally occurring free idempotent generated semigroup. *Semigroup Forum.* 89 (2014), 123–134.

[7] Dandan, Y. and Gould, V. Free idempotent generated semigroups and endomorphism monoids of independence algebras. *Semigroup Forum.* 93 (2016), 535–553.

[8] Dandan, Y. and Gould, V. Free idempotent generated semigroups over bands and biordered sets with trivial products. *Int. J. Algebra Comput.* 26 (2016), 473–507.

[9] Dandan, Y., Gould, V. and Quinn-Gregson, T. Free idempotent generated semigroups: subsemigroups, retracts and maximal subgroups *Commun. Algebra.* 46 (2018), 2264–2277.

[10] Dolinka, I. A note on maximal subgroups of free idempotent generated semigroups over bands. *Periodica Mathematica Hungarica.* 65 (2012), 97–105

[11] Dolinka, I. A note on free idempotent generated semigroups over the full monoid of partial transformations. *Commun. Algebra.* 41 (2013), 565–573.

[12] Dolinka. I. and Gray, R. Maximal subgroups of free idempotent generated semigroups over the full linear monoid. *Trans. Am. Math. Soc.* 366 (2014), 419–455.

[13] Dolinka, I., Gray, R. and Ruskuc, N. On regularity and the word problem for free idempotent generated semigroups *Proc. London Math. Soc.* 114 (2017), 401–432.

[14] Dolinka. I. and Ruskuc, R. Every group is a maximal subgroup of the free idempotent generated semigroup over a band. *Int. J. Algebra Comput.* 23 (2013), 573–581.

[15] Easdown, D. Biordered sets are biordered subsets of idempotents of semigroups. *J. Austral. Math. Soc.* 37 (1984), 258–268.

[16] Easdown, D. Biordered sets come from semigroups. *J. Algebra.* 96 (1985), 581–591.

[17] Easdown, D., Sapir, M. and Volkov, M. Periodic elements of the free idempotent generated semigroup on a biordered set. *Int. J. Algebra and Comp.* 20 (2010), 189–194.

[18] FitzGerald, D.G. On inverses of products of idempotents in regular semigroups. *J. Austral. Math. Soc.* 13 (1972), 335–357.

[19] Graham R.L. On finite 0-simple semigroups and graph theory *Math. Sys. Th.* 2 (1968), 325–339.

[20] Gray, R. and Ruskuc, N. On maximal subgroups of free idempotent generated semigroups. *Israel J. Math.* 189 (2012), 147–176.

[21] Gray, R. and Ruskuc, N. Maximal subgroups of free idempotent-generated semigroups over the full transformation monoid. *Proc. Lond. Math. Soc.* 104 (2012), 997–1018.

[22] Hall, T.E. On regular semigroups. *J. Algebra.* 24 (1973), 1–24.

[23] Higgins, P.M. *Techniques of Semigroup Theory.* Oxford University Press, 1992.

[24] Houghton C.H. Completely 0-simple semigroups and their associated graphs and groups. *Semigroup Forum* 14 (1977), 41–67.

[25] Howie, J.M. The subsemigroup generated by the idempotents of a full transformation semigroup. *J. Lond. Math. Soc.* 41 (1966), 707–716.

[26] Howie, J.M. *An Introduction to Semigroup Theory.* London: Academic Press, 1976.

[27] Lyndon, R. C. and Schupp, P. E. *Combinatorial Group Theory.* Springer Berlin Heidelberg, 2001.

[28] Munn, W.D. Uniform semilattices and bisimple inverse semigroups. *Quart. J. Math. Oxford Ser. (2).* 17 (1966), 151–159.

[29] Munn, W.D. Fundamental inverse semigroups. *Quart. J. Math. Oxford Ser. (2).* 21 (1970), 157–170.

[30] McElwee, B.N. Subgroups of the free semigroup on a biordered set in which principal ideals are singletons. *Comm. Algebra.* 30 (2002), 5513–5519.

[31] McElwee, B.N. Private communication.

[32] Nambooripad, K.S.S. Structure of regular semigroups. I. Fundamental regular semigroups. *Semigroup Forum.* 9 (1975), 354–363.

[33] Nambooripad, K.S.S. Structure of regular semigroups. I. *Mem. Amer. Math. Soc.* 22 (1979).

[34] Nambooripad, K.S.S. and Pastijn, F. Subgroups of free idempotent generated regular semigroups. *Semigroup Forum* 21 (1980), 1–7.

[35] Pastijn, F. Idempotent generated completely 0-simple semigroups. *Semigroup Forum.* 15 (1977), 41–50.

[36] Pastijn, F. The biorder on the partial groupoid of idempotents of a semigroup. *J. Algebra.* 65 (1980), 147–187.

[37] Rusckuc, N. Presentations for subgroups of monoids. *J. Algebra.* 220 (1999), 365–380.

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
E-mail address: david.easdown@sydney.edu.au

School of Mathematics and Statistics, University of New South Wales, NSW 2052, Australia
E-mail address: sean.gardiner@unsw.edu.au

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
E-mail address: brettmcelwee@gmail.com